ON THE EPR PARADOX AND DIRAC EQUATION IN EUCLIDEAN RELAIVITY

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Abstract: Recent experimental results have shown a violation of Bell's inequalities, which are a mathematical formulation of Einstein-Podolsky-Rosen (EPR) paradox. The violation leads to the conclusion that there are no local hidden variable theories that underlie quantum mechanics. However, the Bell's inequalities do not rule out the possibility to construct non-local hidden variable theories that comply with quantum mechanics, in particular, a theory of special and general relativity that permits an instantaneous transmission of interaction. In this work we show that a special relativity with a Euclidean metric that allows not only local interactions but also interactions that can be transmitted instantaneously can be constructed and, furthermore, such special relativity can also be generalised to formulate a general theory of relativity that leads to the same experimental results as Einstein theory of general relativity. We also show that it is possible to formulate Dirac-like relativistic wave equations in this Euclidean relativity with either real mass or imaginary mass, which suggests that the proper mass of a quantum particle may be defined in terms of a differential operator that is associated with a spacetime substructure of the particle.

Even though Einstein general relativity and quantum mechanics are considered as two fundamental theories of modern physics they are formulated within different frameworks that have radically different formalisms for the description of physical reality. Perhaps, the most profound difference between the two theories is the conceptual difficulty that arises from Einstein locality that states that no physical interaction between physical systems can propagate faster than the universal speed, which is assumed to be the speed of light in vacuum. In short, Einstein theory of relativity is a local theory and quantum mechanics is a non-local theory. Einstein theory of relativity is viewed as a local theory because it is formulated on manifolds that are endowed with pseudo-metrics that impose an upper limit on the speed of propagation of interaction [1]. On the other hand, the non-locality of quantum mechanics is due to the principles that underlie its mathematical formalism [2]. Therefore, in order to reconcile the general theory of relativity with the principles of quantum mechanics, and hence to resolve the conceptual difficulties, either local hidden variable theories for quantum mechanics or a theory of relativity that permits an instantaneous transmission of interaction are needed to be devised.

Local hidden variable theories assume that quantum mechanics is an incomplete theory in the sense that there are other variables that may determine a physical system but are not directly observable. These theories postulate that the indeterministic characteristics of quantum mechanics are due to our incomplete knowledge of the hidden substructure of the physical system under investigation. One of the most famous hidden variable theories is de Broglie's interpretation of wave mechanics where the wave functions are considered as pilot waves which are real physical waves that are coupled to associated particles that behave in a classical deterministic manner [3]. Modelling on de Broglie's theory, David Bohm formulated a more complete hidden variable theory to interpret quantum mechanics [4]. However, in order to conform to experimental results, Bohm's theory requires one of the most unacceptable features of local realistic classical physics, the requirement of an action at a distance. Therefore, unlike Bohm's theory, any hidden variable theory that is considered to be a genuine local realistic theory of nature must be constructed so that both determinism and locality are retained. Historically, the question of the existence of a local hidden variable theory emerged from the Einstein-Podolsky-Rosen (EPR) paradox which argued against the completeness of the quantum theory [5]. According to their arguments, a physical theory is complete only when it satisfies the criteria that require that a certainly predicted value of the theory must be corresponded to an element of physical reality and the theory must also satisfy the locality requirement. However, recent results from experiments that are performed to test the Bell's inequalities seem to support the Bell's theorem which rules out local hidden variable theories with the conclusion that a theory that complies with quantum mechanics could not be Lorentz invariant [6,7]. It is worth noting that Bell's conclusion in itself does not provide the ultimate answer to resolve the conceptual difficulties but instead raises the question of whether Lorentz invariance is in fact the cause of the conceptual conflict between classical and quantum mechanics. Our focus in this work will be on the Lorentz invariance and we will show that a special relativity with a Euclidean metric that allows not only local interactions but also instantaneous interactions can be constructed and, furthermore, such special relativity can also be generalised to formulate a general theory of relativity that leads to the same experimental results as Einstein theory of general relativity.

As mentioned above, the local realism in classical physics is a consequence of Einstein theory of special relativity which assumes that, based on the postulates of the principle of relativity and the existence of a universal speed c, the pseudo-Euclidean metric of space and time is invariant under the Lorentz coordinate transformation. In order to associate with Einstein theory of special relativity, Minkowski formulated a unified mathematical spacetime also with a pseudo-Euclidean metric with the assumption that the spacetime interval $-c^2t^2 + x^2 + y^2 + z^2$ is invariant. Consider the coordinate transformation between the inertial frame S with spacetime coordinates (ct, x, y, z) and the inertial frame S' with coordinates (ct', x', y', z') that takes the form of the Lorentz transformation [1,8,9]

$$x' = \gamma(x - \beta ct) \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$ct' = \gamma(-\beta x + ct) \tag{4}$$

where v is the relative translational velocity between the two inertial frames and $\beta = v/c$. From the requirement of the invariance of the spacetime interval $-c^2t^2 + x^2 + y^2 + z^2 = -c^2t'^2 + x'^2 + y'^2 + z'^2$, we obtain

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{5}$$

It is seen from Equation (5) that if we impose that condition $1-\beta^2 > 0$ then $v < c$. Therefore Einstein theory of relativity does not allow interactions at a distance. However, in order to interpret quantum phenomena, such as the quantum entanglement, the transmission of interaction must be assumed to be instantaneous. We now show that it is possible to construct a special relativistic transformation that will endow spacetime with a Euclidean metric rather than a pseudo-Euclidean metric as in the case of the Lorentz transformation and, as a consequence, interaction at a distance is allowed. Consider the following modified Lorentz transformation [10,11]

$$x' = \gamma_E(x - \beta ct) \tag{6}$$

$$y' = y \tag{7}$$

$$z' = z \tag{8}$$

$$ct' = \gamma_E(\beta x + ct) \tag{9}$$

where $\beta = v/c$ and γ_E will be determined from the principle of relativity and the postulate of a universal speed. Instead of assuming the invariance of the Minkowski spacetime interval, if we now assume the invariance of the Euclidean interval $c^2t^2 + x^2 + y^2 + z^2$ then from the modified Lorentz transformation given in Equations (6-9), we obtain the following expression for γ_E

$$\gamma_E = \frac{1}{\sqrt{1+\beta^2}} \tag{10}$$

It is seen from the expression of γ_E given in Equation (10) that there is no upper limit in the relative speed v between inertial frames. The value of γ_E at the universal speed $v = c$ is $\gamma_E = 1/\sqrt{2}$. For the values of $v \ll c$, the modified Lorentz transformation given in Equations (6-9) also reduces to the Galilean transformation. However, it is interesting to observe that when $\beta \to \infty$ we have $\gamma_E \to 0$ and $\beta\gamma_E \to 1$, and in this case from Equations (6) and (9), we obtain $x' \to -ct$ and $ct' \to x$, respectively. This result shows that there is a conversion between space and time when $\beta \to \infty$, therefore in the Euclidean special relativity, not only the concept of motion but the concepts of space and time themselves are also relative. It is also worth mentioning here that the Euclidean relativity of space and time also provides a profound foundation for the temporal dynamics that we have discussed in our other works [12]. In the present situation, if in the inertial frame S with spacetime coordinates (ct, x, y, z) the dynamics of a particle is described by Newton's second law $m\,d^2\mathbf{r}/dt^2 = \mathbf{F}$, then since $x' \to -ct$ and $ct' \to x$ it is seen that the spatial Newton's second law in the inertial frame S is converted to a temporal law of dynamics $D\, d^2\mathbf{t}/dr^2 = \mathbf{F}$ as viewed from the inertial frame S' with spacetime coordinates (ct', x', y', z'). As in the case of the Lorentz transformation given in Equations (1-4), we can also derive the relativistic kinematics and dynamics from the modified Lorentz transformation given in Equations (6-9), such as the transformation of a length, the transformation of a time interval, the transformation of velocities, and the transformation of accelerations. Let L_0 be the proper length then the length transformation can be found as

$$L = \sqrt{1+\beta^2}L_0 \tag{11}$$

It is observed from the length transformation given in Equation (11) that the length of a moving object is expanding rather than contracting as in Einstein theory of special relativity. Now if Δt_0 is the proper time interval then the time interval transformation can also be found to be given by the relation

$$\Delta t = \frac{1}{\sqrt{1+\beta^2}}\Delta t_0 \tag{12}$$

It is also observed from the time interval transformation given in Equation (12) that the proper time interval is longer than the same time interval measured by a moving observer. With the modified Lorentz transformation given in Equations (6-9), the transformation of velocities can be found as follows

$$v_x' = \frac{dx'}{dt'} = \frac{v_x - \beta c}{1 + \frac{\beta v_x}{c}} \tag{13}$$

$$v_y' = \frac{dy'}{dt'} = \frac{v_y}{\gamma_E\left(1 + \frac{\beta v_x}{c}\right)} \tag{14}$$

$$v_z' = \frac{dz'}{dt'} = \frac{v_z}{\gamma_E\left(1 + \frac{\beta v_x}{c}\right)} \tag{15}$$

Form Equation (13), if we let $v_x = c$ then we obtain $v_x' = \left(\frac{c-v}{c+v}\right)c$. Therefore in this case $v_x' = c$ only when the relative speed v between two inertial frames vanishes, $v = 0$. In other words, the universal speed c is not the common speed of any moving physical object or physical field in inertial reference frames. It should be mentioned here that the universal speed c in Einstein theory of special relativity is assumed to be the speed of light in vacuum. It seems that such an assumption was supported by Michelson-Morley experiment [13]. However, as shown in Appendix 1, by using a relativistic transformation, the Michelson-Morley experiment can be used to identify the universal speed as the speed of light only when the shift of the fringe pattern is absolute zero [14]. In order to specify the nature of the assumed universal speed we observe that in Einstein theory of special relativity it is assumed that the spatial space of an inertial frame remains static and this assumption is contradicted to

Einstein theory of general relativity that shows that the spatial space is actually expanding. Therefore it seems reasonable to suggest that the universal speed c in the modified Lorentz transformation given in Equations (6-9) is the universal speed of expansion of the spatial spaces of all inertial frames. The transformations of accelerations can be derived from the modified Lorentz transformation and the transformations of velocities given in Equations (13-15). The transformation of the accelerations can be found as

$$\frac{dv_x^{'}}{dt^{'}} = \frac{1}{\gamma_E{}^3\left(1+\frac{\beta v_x}{c}\right)^3}\frac{dv_x}{dt} \tag{16}$$

$$\frac{dy_x^{'}}{dt^{'}} = \frac{1}{\gamma_E{}^2\left(1+\frac{\beta v_x}{c}\right)^2}\frac{dv_y}{dt} - \frac{\beta v_y}{c\gamma_E{}^3\left(1+\frac{\beta v_x}{c}\right)^3}\frac{dv_x}{dt} \tag{17}$$

$$\frac{dz_x^{'}}{dt^{'}} = \frac{1}{\gamma_E{}^2\left(1+\frac{\beta v_x}{c}\right)^2}\frac{dv_z}{dt} - \frac{\beta v_z}{c\gamma_E{}^3\left(1+\frac{\beta v_x}{c}\right)^3}\frac{dv_x}{dt} \tag{18}$$

By carrying out the thought experiment of the collision of two identical masses in two inertial frames that are moving relative to each other, we can derive the following relationship between the rest mass m_0 observed in the rest frame and the mass m observed from other frame as [8,9]

$$m = \frac{m_0}{\sqrt{1+\beta^2}} \tag{19}$$

It is seen from Equation (19) that $m \to 0$ when $\beta \to \infty$. However, when $\beta \to \infty$ we also have the conversion between space and time $x^{'} \to -ct$, therefore we may speculate that there may also be a conversion between the spatial mass m and the temporal mass D of a particle when $\beta \to \infty$ [12]. Form Equation (19) we obtain

$$m^2c^2 + m^2v^2 = m_0^2c^2 \tag{20}$$

Since both m and $\mathbf{v}$ are variables, we obtain the following relation by differentiation

$$c^2dm = -v^2dm - m\mathbf{v}.\mathrm{d}\mathbf{v} \tag{21}$$

On the other hand, from Newton's second law $\mathbf{F} = d(m\mathbf{v})/dt$, we have

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} \tag{22}$$

Using Equations (22), the change of kinetic energy $dT = \mathbf{F}.d\mathbf{s} = \mathbf{F}.\mathbf{v}dt$ can be obtained as

$$dT = v^2dm + m\mathbf{v}.\mathrm{d}\mathbf{v} \tag{23}$$

From Equations (21) and (23) we arrive at

$$dT = -c^2 dm \tag{24}$$

Since $dm < 0$, therefore $dT > 0$. By integrating both sides of Equation (24)

$$\int_{v=0}^{v} dT = -\int_{m_0}^{m} c^2 dm \tag{25}$$

we obtain the following expression for the kinetic energy

$$T = (m_0 - m)c^2 = (1 - \gamma_E)m_0 c^2 \tag{26}$$

For $v \ll c$, we have $\gamma_E \sim 1 - \beta^2/2$ and Equation (26) reduces to $T \sim m_0 v^2/2$. However, we have $T \to m_0 c^2$ when $\beta \to \infty$. The relativistic momentum $\mathbf{p}$ of a particle of mass m with velocity $\mathbf{v}$ can also be defined by the following relation

$$\mathbf{p} = m\mathbf{v} = \gamma_E m_0 \mathbf{v} \tag{27}$$

In magnitudes, $pc = mvc = \beta mc^2 = \beta E$, where the total energy E is defined by the relation $E = mc^2 = m_0 c^2 - T$. From this definition, we obtain $E \to 0$ when $\beta \to \infty$. Using the relations $E = mc^2$ and $pc = \beta E$, we also obtain the following Euclidean relativistic energy-momentum relationship

$$E^2 = (m_0 c^2)^2 - (pc)^2 \tag{28}$$

The energy-momentum relation given in Equation (28) is different from that in the pseudo-Euclidean Minkowski spacetime by the negative sign of the momentum term. However, we now show that it is still possible to formulate relativistic wave equations by using Dirac's mathematical method [15]. By applying the canonical method of quantisation in quantum mechanics, we replace the energy E, the momentum p and the mass m_0 by operators as follows

$$E \to i\frac{\partial}{\partial t}, \quad p_x \to -i\frac{\partial}{\partial x}, \quad p_y \to -i\frac{\partial}{\partial y}, \quad p_z \to -i\frac{\partial}{\partial z} \quad \text{and} \quad m_0 \to m_0 \tag{29}$$

For simplicity, we use the mathematical units in which $\hbar = c = 1$. The Dirac's first order relativistic partial differential equation is written in the form

$$E\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0\right)\psi \tag{30}$$

where the unknown operators α_i and β are assumed to be independent of the momentum p and the mass m_0. From Equation (30), we obtain

$$E^2\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0\right)^2\psi \tag{31}$$

By expanding Equation (31), and due the fact that all linear momentum operators commute mutually, in order to reduce to the form of the relationship given in Equation (28), the operators α_i and β must satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{32}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{33}$$

$$\alpha_i^2 = -1 \tag{34}$$

$$\beta^2 = 1 \tag{35}$$

As shown in Appendix 2, to satisfy the conditions given in Equations (32-35), the operators α_i and β can be represented as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{36}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{37}$$

where σ_i are Pauli matrices given by $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If we multiply Equation (30) by the operator β, then it can be rewritten in a covariant form as

$$\left(i\gamma^\mu\partial_\mu - m_0\right)\psi \tag{38}$$

where $\partial_\mu = \left(\partial_t, \partial_x, \partial_y, \partial_z\right)$, $\gamma^i = \beta\alpha_i$ and $\gamma^0 = \beta$. As in the case of the Dirac equation in the pseud-Euclidean Minkowski spacetime, the solutions to particles at rest can be found as

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}, \quad \psi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imt}, \quad \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imt} \tag{39}$$

Now, it is observed that the canonical quantisation given in Equation (29) treats the proper mass m_0 differently from the energy E and the momentum $\mathbf{p}$. If the energy E and the momentum $\mathbf{p}$ are replaced by operators then, due to the equivalence between mass and energy, the proper mass m_0 should also replaced by a differential operator, which may be related to the substructure of a quantum particle. Therefore, instead of the quantisation given in Equation (29), we assume the following alternative quantisation in which the proper mass m_0 is replaced by im_0

$$E \to i\frac{\partial}{\partial t}, \quad p_x \to -i\frac{\partial}{\partial x}, \quad p_y \to -i\frac{\partial}{\partial y}, \quad p_z \to -i\frac{\partial}{\partial z} \quad \text{and} \quad m_0 \to im_0 \tag{40}$$

With $m_0 \to im_0$ and using the units $\hbar = c = 1$, Equation (28) can be rewritten in the form

$$(iE)^2 = p^2 + {m_0}^2 \tag{41}$$

The Dirac equation now becomes

$$iE\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0\right)\psi \tag{42}$$

From Equation (42), we also obtain

$$(iE)^2\psi = \left(\alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta m_0\right)^2\psi \tag{43}$$

By expanding Equation (43), and also due the fact that all linear momentum operators commute mutually, in order to reduce to the form of the relationship given in Equation (41), the operators α_i and β now must satisfy the following relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \text{for } i \neq j \tag{44}$$

$$\beta\alpha_i + \alpha_i\beta = 0 \tag{45}$$

$$\alpha_i^2 = 1 \tag{46}$$

$$\beta^2 = 1 \tag{47}$$

As shown in Appendix 2, to satisfy the conditions given in Equations (44-47), the operators α_i and β can be represented as follows

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{48}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{49}$$

where σ_i are Pauli matrices given as $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If we multiply Equation (42) by the operator β, then it can also be rewritten in a covariant form as

$$\left(i\gamma^\mu\partial_\mu - m_0\right)\psi \tag{50}$$

where $\partial_\mu = \left(\partial_t, \partial_x, \partial_y, \partial_z\right)$, $\gamma^i = \beta\alpha_i$ and $\gamma^0 = i\beta$. However, the solutions to particles at rest in this case are real and can be found as

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-mt}, \quad \psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-mt}, \quad \psi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{mt}, \quad \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{mt} \tag{51}$$

In the following we will extend our presentation of the Euclidean relativity and show that the special Euclidean relativity can also be generalised to formulate a general theory of Euclidean relativity that also leads to the same experimental results as Einstein theory of general relativity. We assume that Einstein field equations of general relativity can also be applied to Riemannian spacetime manifolds which are endowed with positive definite metrics. In the original Einstein theory of general relativity, the field equations of the gravitational field are proposed to take the form [1]

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa T_{\alpha\beta} \tag{52}$$

where $T_{\alpha\beta}$ is the covariant form of the energy-momentum tensor, $R_{\alpha\beta}$ is the Ricci tensor defined by the relation

$$R_{\mu\nu} = \frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} \tag{53}$$

and the metric connection $\Gamma^{\sigma}_{\mu\nu}$ is defined in terms of the metric tensor $g_{\alpha\beta}$ as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \tag{54}$$

and $R = g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar curvature. As shown in Appendix 3, the Schwarzschild vacuum solution can be obtained with a Riemannian positive definite metric for a centrally symmetric field given in the form

$$ds^2 = e^{2\nu(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{55}$$

The Schwarzschild vacuum solution is found as

$$ds^2 = \left(1 + \frac{C}{r}\right)c^2dt^2 + \left(1 + \frac{C}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{56}$$

where C is a constant of integration that can be identified with the mass of the source of a physical field. In order to investigate the nature of the constant C we examine in this spacetime the motion that is described by the geodesic equation

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda}\frac{dx^{\nu}}{ds}\frac{dx^{\lambda}}{ds} = 0 \tag{57}$$

With $x^0 = ct$ and $ds = cd\tau$, the geodesic equation for $\mu = 0$ can be found to satisfy the relation [16]

$$\left(1 + \frac{C}{r}\right)\frac{dt}{d\tau} = C_1 \tag{58}$$

where C_1 is a constant of integration. For $\mu = 1, 2, 3$ we obtain following the relations

$$\frac{d^2r}{d\tau^2} + \frac{C}{2r^2}\left(1 + \frac{C}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{C}{2r^2}\left(1 + \frac{C}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r\left(1 + \frac{C}{r}\right)\left(\frac{d\theta}{d\tau}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{59}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{60}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{61}$$

On the other hand, if we divide the line element given in Equation (56) by $ds^2 = c^2d\tau^2$, we obtain the equation

$$1 = \left(1 + \frac{C}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2}\left(1 + \frac{C}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{c^2}r^2\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right) \quad (62)$$

For a planar motion with $\theta = \pi/2$, Equation (60) reduces to

$$r^2\frac{d\phi}{d\tau} = C_2 \quad (63)$$

where C_2 is a constant of integration. Using Equations (58) and (63), Equation (62) is reduced to the equation

$$1 = \left(1 + \frac{C}{r}\right)^{-1} C_1^2 + \frac{C_2^2}{c^2}\left(1 + \frac{C}{r}\right)^{-1}\frac{1}{r^4}\left(\frac{d\phi}{dr}\right)^2 + \frac{C_2^2}{c^2 r^2} \quad (64)$$

Using the identity $\frac{1}{r^4}\left(\frac{d\phi}{dr}\right)^2 = \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2$, Equation (64) is simplified to

$$\left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} = \frac{c^2}{C_2^2}(1 - C_1^2) + \frac{c^2 C}{rC_2^2} - \frac{C}{r^3} \quad (65)$$

By differentiating Equation (65) with respect to ϕ, we have

$$\frac{d}{d\phi}\left(\frac{1}{r}\right)\left(\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = \frac{d}{d\phi}\left(\frac{1}{r}\right)\left(\frac{c^2 C}{2C_2^2} - \frac{3C}{2r^2}\right) \quad (66)$$

From Equation (66), we obtain the following two equations

$$\frac{d}{d\phi}\left(\frac{1}{r}\right) = 0 \quad (67)$$

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{c^2 C}{2C_2^2} - \frac{3C}{2r^2} \quad (68)$$

It is seen that as in the case of Schwarzschild solution with the Minkowski pseudo-Riemannian metric, Equation (67) describes a circle and Equation (68) can be used to describe the precession of planetary orbits around a gravitational mass if the constant C is identified with the gravitational mass M as $C = -2GM/c^2$ and the constant C_2 is defined in terms of the semi-latus rectum p of an ellipse as $C_2^2 = pGM$.

It is noted that if the field endowed with the Riemannian metric given in Equation (45) is still spherically symmetric but now time-dependent then the metric can be shown to be written in the form [16]

$$ds^2 = e^{2\nu(r,t)}c^2dt^2 + e^{2\lambda(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (69)$$

Similar to the case of time-independent spherically symmetric metric as shown in Appendix 3, the time-dependent metric given in Equation (69) can be reduced to the time-independent Schwarzschild metric given in Equation (55) if the following condition is assumed

$$e^{-2\lambda} = 1 + \frac{C}{r} \tag{70}$$

where λ can be shown to be time-independent from the condition $R_{01} = (2/rc)\, d\lambda/dt = 0$. For the case of a gravitational field, the constant of integration C can be identified as $C = -2MG/c^2$. Therefore, if λ is time-independent then the mass M of a gravitational source must be constant. This is the content of Birkhoff's theorem which states that any spherically symmetric vacuum solution of the field equations of general relativity is necessary static. It can be observed that even though Birkhoff's theorem is a perfect mathematical theorem, it cannot practically be applied to physical reality because there is no physical object which has a constant mass can be a physical star. We would like to give a further remark here on the formulation of Robertson-Walker metric to describe the dynamical structure of the observable universe in modern cosmology. The Robertson-Walker pseudo-metric can be written in the following form

$$ds^2 = -c^2dt^2 + S^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{71}$$

In the cosmological line element given in Equation (71), the time t is a universal time and the factor $S^2(t)$ is an expansion factor. However, since the metric is conformally flat in order for the spatial section of spacetime to be described as a curved space it must be embedded into a four-dimensional Euclidean space R^4. Since a flat space R^4 does not exist in Einstein general relativity, a fictitious flat space R^4 must be introduced so that a three-dimensional hypersurface can be embedded. However, as has been discussed above, within the framework of the Euclidean relativity, the flat space R^4 exists naturally and in this case the Robertson-Walker pseudo-metric is modified to take the following form

$$ds^2 = c^2dt^2 + S^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{72}$$

Similar to the case when the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ are introduced to describe a circle in the three-dimensional Euclidean space R^3, the three-dimensional spatial section can be described by introducing the spherical coordinates [8,16]

$$x^1 = a\sin\chi\sin\theta\cos\phi \tag{73}$$

$$x^2 = a\sin\chi\sin\theta\sin\phi \tag{74}$$

$$x^3 = a\sin\chi\cos\theta \tag{75}$$

$$x^4 = a\cos\chi \tag{76}$$

With the spherical coordinates given in Equations (73-76), the line element given in Equation (72) can be expressed in the form

$$ds^2 = c^2dt^2 + R^2(t)\left(\frac{1}{1-Kr^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{77}$$

where the Gaussian curvature K can take values $K = 0, 1/a^2, -1/a^2$.

Appendix 1

The postulate of the invariance of the speed of light has been re-examined recently by many authors, and it has been shown that the special theory of relativity can be developed using only the principle of relativity, which postulates the invariance of physical laws in any inertial frame of reference [17-19]. Even though relativistic transformations can be derived from the principle of relativity alone, these formulations do not specify or determine the universal speed that must be accompanied the special relativity for any further development or application of the theory. Furthermore, it should be mentioned here that even within Einstein's theory of general relativity, whether the constancy of the speed of light has a global character is a question that has also been discussed [20,21]. In this work, we will use the relativistic transformations that are derived only from the principle of relativity to show that the Michelson-Morley experiment can be used to verify the fact that the speed of light is not universal as postulated in Einstein's theory of special relativity. For the clarity for our discussions in the following, first we recapture the necessary procedure to calculate a possible shift of the interference pattern in the Michelson-Morley experiment. In the Michelson-Morley experiment, light rays are made to travel along two optical paths l_1 and l_2 which are perpendicular to each other. In this work we assume the length of all optical paths to be kept constant. If the whole apparatus is moving in the direction of l_1 at speed v then by using the Galilean law of composition of velocities the times t_1 and t_2 taken for light to travel along l_1 and l_2 can be calculated, respectively, as follows [22]

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{2l_1/c}{1-v^2/c^2} \tag{1}$$

$$t_2 = \frac{2l_2/c}{(1-v^2/c^2)^{1/2}} \tag{2}$$

where v is the velocity of the earth in its orbit. From Equations (1) and (2), a time difference $\Delta_1 = t_1 - t_2$ is obtained

$$\Delta_1 = \frac{2l_1/c}{1-v^2/c^2} - \frac{2l_2/c}{(1-v^2/c^2)^{1/2}} \tag{3}$$

If $v \ll c$ then, using the relation $(1+x)^k \approx 1 + kx$, where k is a real number, the time difference Δ_1 is approximated

$$\Delta_1 \approx \frac{2(l_1 - l_2)}{c} + \frac{2l_1v^2}{c^3} - \frac{l_2v^2}{c^3} \tag{4}$$

Now, when the whole apparatus is turned so that its direction of motion is parallel to l_2 then a new time difference $\Delta_2 = t_1 - t_2$ is obtained

$$\Delta_2 = \frac{2l_1/c}{(1 - v^2/c^2)^{1/2}} - \frac{2l_2/c}{1 - v^2/c^2} \tag{5}$$

If $v \ll c$ then the time difference Δ_2 is approximated

$$\Delta_2 \approx \frac{2(l_1 - l_2)}{c} + \frac{l_1 v^2}{c^3} - \frac{2l_2 v^2}{c^3} \tag{6}$$

From the time differences given in Equations (3) and (5), the interference pattern would shift by an amount $\delta = c(\Delta_1 - \Delta_2)/\lambda$ as follows

$$\delta = \frac{c}{\lambda}\left(\frac{2l_1/c}{1 - v^2/c^2} - \frac{2l_2/c}{(1 - v^2/c^2)^{1/2}} - \frac{2l_1/c}{(1 - v^2/c^2)^{1/2}} + \frac{2l_2/c}{1 - v^2/c^2}\right) \tag{7}$$

From Equation (7), an approximate amount of the shift of the interference pattern for the case $v \ll c$ is found as

$$\delta = \frac{c(\Delta_1 - \Delta_2)}{\lambda} \approx \frac{(l_1 + l_2)v^2}{\lambda c^2} \tag{8}$$

And when $l_1 = l_2 = l$, then

$$\delta \approx \frac{2lv^2}{\lambda c^2} \tag{9}$$

With $v \approx 30\ \mathrm{km/sec}$, $\lambda = 6 \times 10^{-7}\mathrm{m}$ and $l = 1.2\ \mathrm{m}$, the relation (9) gives $\delta \approx 0.04$ fringe. Michelson and Morley reported to observe only a small shift of the fringe pattern of at most 0.005 fringe [23]. This has been considered as a null result. The null result obtained from the Michelson-Morley experiment has been considered to be consistent with Einstein's postulate of the invariance of the speed of light in empty space, which results in the following transformation of velocities

$$u_x = \frac{u_x' + v}{1 + u_x' v/c^2}, \tag{10}$$

$$u_y = \frac{u_y'(1 - u_x v/c^2)}{\sqrt{1 - v^2/c^2}} \tag{11}$$

It should be emphasized here that the shift of the interference pattern given by the relation (9) is derived from the Galilean transformation. However, the use of the Galilean transformation in the Michelson-Morley experiment is probably not appropriate to determine whether the speed of light in vacuum is universal. In fact, as will be argued in the following, when only the principle of relativity is used to formulate the special relativity then even the smallest fringe shift obtained from the Michelson-Morley experiment can be used to verify that the speed of light in vacuum is not universal.

As shown in the above-mentioned references [17-19], without postulating the constancy of the velocity of light in vacuum, the principle of relativity alone can be used to derive the relativistic addition law for parallel velocities as follows

$$u_x = \frac{u_x' + v}{1 + K u_x' v} \tag{12}$$

where K is a universal constant. If the optical path l_1 is along the direction of u_x and if $u_x' = c$ then the time t_1 for light to travel along l_1 is given by

$$t_1 = \frac{l_1}{\dfrac{c - v}{1 - Kcv}} + \frac{l_1}{\dfrac{c + v}{1 + Kcv}} = \frac{2l_1}{c}\left(\frac{1 - Kv^2}{1 - v^2/c^2}\right) \tag{13}$$

To calculate the time t_2 for light to travel along l_2 in the direction perpendicular to the direction of v, we note that in order to be consistent with the transformation of the perpendicular component in Einstein's theory of special relativity given in Equation (11), the perpendicular component of velocity in the special relativity that is derived only from the principle of relativity should be transformed as

$$u_y = \frac{u_y'(1 - K u_x v)}{\sqrt{1 - Kv^2}} \tag{14}$$

In the case when $u_x = 0$ and $u_y' = c\sqrt{1 - v^2/c^2}$, then we obtain

$$u_y = \frac{c\sqrt{1 - v^2/c^2}}{\sqrt{1 - Kv^2}} \tag{15}$$

The time $t_2 = 2l_2/u_y$ for light to travel along l_2 is calculated as

$$t_2 = \frac{2l_2}{c}\left(\frac{1 - Kv^2}{1 - v^2/c^2}\right)^{1/2} \tag{16}$$

Using the relativistic transformations given in Equations (13) and (16), the interference pattern would shift by an amount $\delta = c(\Delta_1 - \Delta_2)/\lambda$ given by

$$\delta = \frac{c}{\lambda}\left(\frac{2l_1}{c}\left(\frac{1-Kv^2}{1-v^2/c^2}\right) - \frac{2l_2}{c}\left(\frac{1-Kv^2}{1-v^2/c^2}\right)^{1/2} - \frac{2l_1}{c}\left(\frac{1-Kv^2}{1-v^2/c^2}\right)^{1/2} + \frac{2l_2}{c}\left(\frac{1-Kv^2}{1-v^2/c^2}\right)\right) \tag{17}$$

For the case when $l_1 = l_2 = l$, Equation (17) is reduced to

$$\delta = \frac{4l}{\lambda}\left(\left(\frac{1 - Kv^2}{1 - v^2/c^2}\right) - \left(\frac{1 - Kv^2}{1 - v^2/c^2}\right)^{1/2}\right) \tag{18}$$

From Equation (18), an approximate amount of the shift of the interference pattern for the case $v \ll c$ is found as

$$\delta = \frac{c(\Delta_1 - \Delta_2)}{\lambda} \approx \frac{2lv^2}{\lambda}\left(\frac{1}{c^2} - K\right) \tag{19}$$

This result show that if an absolute null result is obtained from the Michelson-Morley experiment, $\delta \equiv 0$, then $K = 1/c^2$. In this case light rays would move at the same speed in all reference frames as postulated in Einstein's theory of special relativity. However, even with the so-called null result of $\delta \approx 0.005$ fringe, as obtained from Michelson-Morley experiment, the speed of light in vacuum is not universal. If we let $c_g = 1/\sqrt{K}$ then it is seen from Equation (19) that $c_g > c$. The speed c_g is a universal speed.

Appendix 2

Assume the operators α_i are represented in terms of the operators σ_i in the forms

$$\alpha_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{1}$$

Then we obtain

$$\alpha_i^2 = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} \tag{2}$$

If $\sigma_i^2 = 1$ then $\alpha_i^2 = 1$. On the other hand, if $\sigma_i^2 = -1$ then $\alpha_i^2 = -1$. Now, if the operators α_i are given in the forms

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3}$$

Then in this case we obtain

$$\alpha_i^2 = \begin{pmatrix} -\sigma_i^2 & 0 \\ 0 & -\sigma_i^2 \end{pmatrix} \tag{4}$$

If $\sigma_i^2 = 1$ then $\alpha_i^2 = -1$. On the other hand, if $\sigma_i^2 = -1$ and then $\alpha_i^2 = 1$.

If we write the operator σ_i as a two by two matrix in the form

$$\sigma_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

then from the requirement $\sigma_i^2 = 1$, we arrive at the following system of equations for the unknown quantities a, b, c and d

$$a^2 + bc = 1 \tag{6}$$

$$b(a + d) = 0 \tag{7}$$

$$c(a + d) = 0 \tag{8}$$

$$d^2 + bc = 1 \tag{9}$$

From Equations (6) and (9) we obtain $d = \pm a$. If $d = a$ then $b = c = 0$ and the operator σ_i can take the values $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\sigma_i = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. If $d = -a$ and if $b = c = 0$, then the operator σ_i can be as $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If $d = -a$ but $b \neq 0$ and $c \neq 0$, then the operator σ_i can be written in the form $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\sigma_i = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. These are only a few standard representations of the operators σ_i. It is also seen from the representations of the operators α_i given in Equations (1) that there are many different combinations that can be chosen for the operators α_i and β to satisfy the following relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \qquad \text{for } i \neq j \tag{10}$$

$$\beta \alpha_i + \alpha_i \beta = 0 \tag{11}$$

$$\alpha_i^2 = 1 \tag{12}$$

$$\beta^2 = 1 \tag{13}$$

The most common use of the forms of the operators α_i is defined in terms of Pauli matrices $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$. In this case the operators α_i are found as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{14}$$

In addition, if the operator β is defined in terms of the operators σ_i as $\beta = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$ then with $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the operator β takes the form

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

Now, for the operators α_i and β to satisfy the Euclidean relations

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \qquad \text{for } i \neq j \tag{16}$$

$$\beta \alpha_i + \alpha_i \beta = 0 \tag{17}$$

$$\alpha_i^2 = -1 \tag{18}$$

$$\beta^2 = 1 \qquad (19)$$

the operators α_i can still be defined in terms of Pauli matrices $\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by using the representation $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$. In this case the operators α_i are found as follows

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (20)$$

However, the operator β is still defined in terms of the operators σ_i as $\beta = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$ with $\sigma_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as above.

Appendix 3

With the line element given in Equation (55), the tensor metric $g_{\alpha\beta}$ and its inverse are given as

$$g_{\alpha\beta} = \begin{pmatrix} e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g^{\alpha\beta} = \begin{pmatrix} e^{-2\nu} & 0 & 0 & 0 \\ 0 & e^{-2\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \qquad (1)$$

The non-zero components of the affine connections are

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{d\nu}{dr}$$

$$\Gamma^1_{00} = -e^{2\nu-2\lambda}\frac{d\nu}{dr}, \qquad \Gamma^1_{11} = \frac{d\lambda}{dr}, \qquad \Gamma^1_{22} = -re^{-2\lambda}, \qquad \Gamma^1_{33} = -r\sin^2\theta e^{-2\lambda}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \qquad (2)$$

The non-zero components of the Ricci curvature tensor are

$$R_{00} = e^{2\nu-2\lambda}\left(-\frac{d^2\nu}{dr^2} - \left(\frac{d\nu}{dr}\right)^2 + \frac{d\nu}{dr}\frac{d\lambda}{dr} - \frac{2}{r}\frac{d\nu}{dr}\right)$$

$$R_{11} = -\frac{d^2\nu}{dr^2} - \left(\frac{d\nu}{dr}\right)^2 + \frac{d\nu}{dr}\frac{d\lambda}{dr} + \frac{2}{r}\frac{d\nu}{dr}$$

$$R_{22} = \left(-1 + r\frac{d\lambda}{dr} - r\frac{d\nu}{dr}\right)e^{-2\lambda} + 1$$

$$R_{33} = R_{22}\sin^2\theta \tag{3}$$

For the vacuum solution, from $R_{00} = 0$ and $R_{11} = 0$, we obtain the identity

$$\frac{d\lambda}{dr} + \frac{d\nu}{dr} = 0 \tag{4}$$

On integration Equation (4) we have

$$\lambda(r) + \nu(r) = C \tag{5}$$

where C is an undetermined constant. However, with the assumption that the metric given in Equation (1) will approach the Euclidean metric as $r \to \infty$, we have $C = 0$. Therefore we have

$$\lambda(r) = -\nu(r) \tag{6}$$

With the condition given in Equation (6), the component R_{22} can be rewritten as

$$\frac{d(re^{2\nu})}{dr} = 1 \tag{7}$$

From Equation (7) by integration we obtain

$$e^{2\nu} = 1 + \frac{C}{r} \tag{8}$$

where C is a constant of integration.

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