The Squared Case of π^n is Irrational Gives π is Transcendental

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This article proves π^2 is irrational with annotations that show the steps used yield a proof of the general π^n case and, in turn, π's transcendence.

This article may be viewed as part of continuum of ever evolving proofs of the transcendence of e followed by π. e was proven to be transcendental in 1882 by Hermite [6]. Ten years later, using a key idea from Hermite's proof, Lindemann proved π is transcendental [13]. Hilbert and others shortened both proofs in 1893 [3]. Niven [14] in his 1936 proof of the transcendence of π credits Hurwitz's treatment of e's transcendence [8] for his use of $e^{-z}F(z)$ to shorten π's transcendence proof.

At some point these transcendence proofs yielded sufficiently short and simple proofs that their techniques could reasonably be used for irrationality proofs. Niven's famously short π is irrational proof [15] is an example. Its perverse terseness was arguably improved by Jones some years later [9]. Two recent articles also use transcendence techniques more directly to prove e^n and π are irrational [1, 10]. In these last two articles the mean value theorem is used for e and complex integration for π to give equivalents to Lemma 2 below; both use Hurwitz's $e^{-z}F(z)$ idea. As Lemma 2 drops the necessity of separate real and complex cases, as well as Hurwitz's innovation, some slight gain of simplicity and efficiency is achieved. In addition, the natural number powers of π and e can be proven irrational with the transcendence of these constants an easy generalization.

This article is designed to parallel [11] which gives a proof that e^n is irrational and e transcendental. This article uses the same lemmas. The only essential modification is the real x for the e case is changed to the complex z for the π. The lemmas given here could be used, intact, for the e article, the real case needed for e being a special case of the complex needed for π. But we think simplest is best and, given a chance to repeat the e to π

connections and evolution, two articles seems, though a little redundant, an interesting idea.

In what follows, z is complex number, all polynomials are integer polynomials, and p is a prime.

Definition 1. Given a polynomial $f(z)$, lowercase, the sum of all its derivatives is designated with $F(z)$, uppercase.

Definition 2. For non-negative integers n, let $\epsilon_n(z)$ denote the infinite series

$$\frac{z}{n+1} + \frac{z^2}{(n+1)(n+2)} + \cdots + \frac{z^j}{(n+1)(n+2)\dots(n+j)} + \dots.$$

Lemma 1. *If $f(z) = cz^n$, then*

$$F(0)e^z = F(z) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n.

Proof. As $F(x) = c(x^n + nx^{n-1} + \cdots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^x &= cn!(1 + x/1 + x^2/2! + \cdots + x^n/n! + \dots) \\ &= cx^n + cnx^{(n-1)} + \cdots + cn! + cx^{n+1}/(n+1)! + \dots \\ &= F(x) + cx^n(x/(n+1) + x^2/(n+1)(n+2) + \dots) \\ &= F(x) + f(x)\epsilon_n(x). \end{aligned}$$

Now $f(x)$ has polynomial growth in n and $\epsilon_n(x) \leq e^x$, so the product has polynomial growth in n. □

Lemma 2. *If F is the sum of the derivatives of the polynomial $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ of degree n, then*

$$e^z F(0) = F(z) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f.

Proof. Let $f_j(x) = c_j x^j$, for $0 \leq j \leq n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^{n} (f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n,$$

where F_j is the sum of the derivatives of f_j. Using Lemma 1,

$$e^x F_k(0) = F_k(x) + \epsilon \tag{3}$$

and summing (3) from $k = 0$ to n, gives

$$e^x F(0) = F(x) + n\epsilon.$$

As the finite sum of functions with polynomial growth in n also has polynomial growth in n, we arrive at (2). $\square$

Lemma 3. *If $f(z) = (z-r)^m(c_{n-m}z^{n-m} + \cdots + c_0)$, then $f^{(q)}(r) = 0$ for $0 \leq q \leq m-1$, $f^{(m+j)}(r) = (m+j)!c_j$ for $0 \leq j \leq n-m$.*

Proof. If $r = 0$, the term with the least power of z is $c_0 z^m$. Its 0 through $m-1$ derivatives are 0 at 0. At the mth derivative this term is $m!c_0$ and all other terms are 0. Similarly the $m+j$ derivative $j > m$ yields constants of the form $(m+j)!c_j$.

If $r \neq 0$, then $f(z) = (z-r)^m Q(z)$, where $Q(z)$ is a polynomial of degree $n-m$. Define $g(z) = f(z+r) = z^m Q(z+r)$. Then $g^{(k)}(0) = f^{(k)}(r)$ for all $k \geq 0$, where k superscripts give derivatives. As $Q(z+r)$ is of degree $n-m$, the same argument used for the $r = 0$ case applies. $\square$

Lemma 4. *Let polynomial $f(z)$ have root $r = 0$ of multiplicity $p-1$ then, for large enough p, $p \nmid |F(0)|$.*

Proof. We can write $f(x) = x^{p-1}(b_j x^j + \cdots + b_0)$. The $p-1$ derivative is $(p-1)!b_0$ and all subsequent derivatives have $p!$ in all their terms. Now if $p > b_0$, then $p \nmid F(0)$. $\square$

Lemma 5. *If a and b are Gaussian integers and p is a prime, $p > |a|$, then $|a(p-1)! + bp!|$ is a non-zero integer divisible by $(p-1)!$.*

Proof. As $a(p-1)! + bp!$ is of the form $A - B + (C-D)i$ with $A - B \neq 0$ or $C - D \neq 0$ the result follows. $\square$

Theorem 1. *π^2 is irrational.*

Proof. Suppose $\pi^2 = a/b$, with a and b natural numbers, $a > b$. Let $a_2(z) = z^2 - (\pi i)^2$, then a_2 has two roots: $r_1 = \pi i$ and $r_2 = -\pi i$, one of which is πi.① As one root is πi, we have

$$0 = (1 + e^{r_1})(1 + e^{r_2}) = 1 + e^{r_1} + e^{r_2} + e^{r_1 + r_2} = 2 + e^{r_1} + e^{r_2}.② \tag{4}$$

Form a polynomial for roots 0 with multiplicity $p-1$ and the non-zero exponents in (4) with multiplicity p. Multiply it by a power of b that makes it an integer polynomial:

$$f_2(z) = b^{2p-1}z^{p-1}[(z-\pi i)(z+\pi i)]^p = (bz)^{p-1}(bz+a)^p. ③ \tag{5}$$

We then have, using (4) with (5),

$$0 = F(0)(1+e^{r_1})(1+e^{r_2}) = 2F(0) + F(r_1) + F(r_2) + \epsilon.$$

Using 4, for $p > \max\{2, b\}$, $p \nmid 2F(0)$ and $(p-1)!|2F(0)$. Now, per Lemma 3, the coefficients of $F(r_1)$ and $F(r_2)$ will be of the form $(p+j)!c_j$. We can observe the sum of the powers of the non-zero roots involved in $F(r_1)+F(r_2)$ will be integers as well: odd powers cancel to zero and even powers are under the rationality assumption of π^2. For example,

$$(b\pi i)^{2n} + (-b\pi i)^{2n} = (bi)^{2n}(a/b)^n + (bi)^{2n}(a/b)^n = 2(i)^{2n}a^n b^n,$$

a power of i times an integer.④

Finally,

$$0 = \frac{2F(0) + F(r_1) + F(r_2) + \epsilon}{(p-1)!}$$

gives a contradiction for large enough p. □

① In general, $a_n(z) = z^n - (\pi i)^n$ will have n roots, r_j, one of which is πi.
② In general, the exponents will consist of sums of r_j roots taken one through n at a time, with some adding to 0 and being absorbed in the A value.
③ In general, the fundamental theorem of symmetric functions insures that the sum of roots polynomial will have coefficients that are integer polynomials of the *by assumption* polynomial, $a_n(z)$; that is the sum of the roots, as they are symmetric, generate a polynomial with coefficients that are integer polynomials of the coefficients of $a_n(z)$. Consequently, as the only coefficient of $a_n(z)$ is a/b, a power of b will work. Making the power of b the maximum exponent of z works for this purpose.
④ In general, Newton's identities show that the sum of the powers of the roots are symmetric functions and as such can be expressed as integer polynomials of the coefficients of the polynomial they are roots of. So, the pattern is coefficients of $a_n(z)$ form the coefficients of $f_n(z)$ and thus the sums of the powers of the roots of $f_n(z)$ are, in turn, integer polynomials of a/b, the only coefficient in $a_n(z)$.

The annotations of the square case of the irrationality of π shows the general π^n case. With a slight adjustment of this π^n case, π is proven transcendental.

Theorem 2. *π is transcendental.*

Proof. A number is transcendental if it doesn't solve an integer polynomial. Suppose πi solves an nth degree integer polynomial $a_t(z)$ with roots r_j, then the roots in the proof of the irrationality of π^n are replaced with these roots: all the steps are the same and lead, as in the irrationality case, to a contradiction. □

Readers might be a little miffed at the brevity of the main results in this article. We are seeking clarity of concepts over precision of details. For the latter see [5, 7]. In that regard, the terseness and style of Lemmas 3, 4, and 5 may show inconsistency with this goal of clarity. These results are covered in other articles more leisurely; [10] gives a tutorial on Leibniz tables that can visually quickly give the ideas hidden in these lemmas; [2] also presents a more leisurely, meaning longer, development of the ideas, including the real version of Lemma 2 used for e. Finally, [12] provides case studies for the first, second, third, and fourth powers of π. Interestingly regular polygons, via complex roots of $a_n(z)$ for these values, appear and suggest the validity of both the fundamental theorem of symmetric functions and Newton's identities. For elementary treatments of these results see [4].

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