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New solution of Maxwell's equations for spherical wave in the far zone

Contents

Annotation

It is noted that the known solution for a spherical electromagnetic wave in the far zone does not satisfy the law of conservation of energy (it is retained only on the average), the electric and magnetic intensities of the same name (in coordinate) are in phase, only one of Maxwell's equations is satisfied. A solution is offered that is free from these shortcomings.

1. Introduction

In [1], a cylindrical electromagnetic wave is considered. Below we consider a spherical electromagnetic wave far from the vibrator - in the so-called the far zone, where the longitudinal (radial-directed) electric and magnetic intensities can be neglected. The main drawbacks of the known solution (see Appendix 1) are that

1. the law of conservation of energy is fulfilled only on the average (in time),
2. the magnetic and electrical components are in phase,
3. in the Maxwell equations system, in the known solution, only one equation of eight is satisfied.

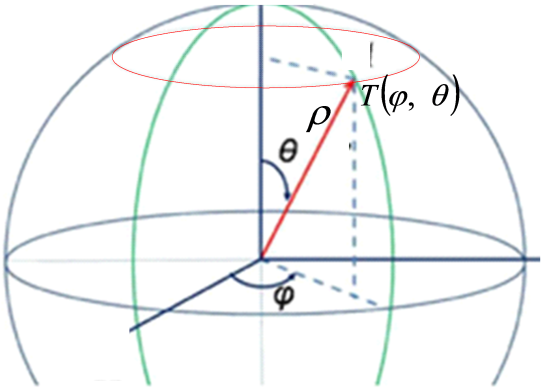

Fig. 1.

2. Solution of the Maxwell's equations

Fig. 1 shows the spherical coordinate system (ρ, θ, φ). Expressions for the rotor and the divergence of vector E in these coordinates are given in Table 1 [2]. The following notation is used:

E - electrical intensities,

H - magnetic intensities,

μ - absolute magnetic permeability,

ε - absolute dielectric constant.

The Maxwell's equations in spherical coordinates in the absence of charges and currents have the form given in Table. 2. Next, we will seek a solution for $E_\rho = 0,\ H_\rho = 0$ and in the form of the functions $E,\ H$ presented in Table 3, where the function $g(\theta)$ and functions of the species $E_{\varphi\rho}(\rho)$ are to be calculated. We assume that the intensities $E,\ H$ do not depend on the argument φ. Under these conditions, we transform Table 1 in Table 3a. Further we substitute functions from Table 3 in Table 3a. Then we get Table 4.

Substituting the expressions for the rotors and divergences from Table 4 into the Maxwell's equations (see Table 2), differentiating with respect to time and reducing the common factors, we obtain a new form of the Maxwell's equations - see Table 5.

Consider the Table 5. From line 2 it follows:

$$\frac{H_{\varphi\rho}}{\rho}+\frac{\partial H_{\varphi\rho}}{\partial\rho}=0, \tag{2}$$

$$\chi H_{\varphi\rho}+\frac{\omega\varepsilon}{c}E_{\theta\rho}=0. \tag{3}$$

Consequently,

$$H_{\varphi\rho}=\frac{h_{\varphi\rho}}{\rho}, \tag{4}$$

$$H_{\varphi\rho}=-\frac{\omega\varepsilon}{\chi c}E_{\theta\rho}, \tag{5}$$

where $h_{\varphi\rho}$ is some constant. Likewise, from lines 3, 5, 5 should be correspondingly:

$$H_{\theta\rho}=\frac{h_{\theta\rho}}{\rho}, \tag{6}$$

$$H_{\theta\rho}=\frac{\omega\varepsilon}{\chi c}E_{\varphi\rho}, \tag{7}$$

$$E_{\varphi\rho}=\frac{e_{\varphi\rho}}{\rho}, \tag{8}$$

$$E_{\varphi\rho}=\frac{\omega\mu}{\chi c}H_{\theta\rho}, \tag{9}$$

$$E_{\theta\rho}=\frac{e_{\theta\rho}}{\rho}, \tag{10}$$

$$E_{\theta\rho}=-\frac{\omega\mu}{\chi c}H_{\varphi\rho}. \tag{11}$$

It follows from (5) that

$$E_{\theta\rho}=-\frac{\chi c}{\omega\varepsilon}H_{\varphi\rho}, \tag{12}$$

and from a comparison of (11) and (12) it follows that

$$\frac{\omega\mu}{\chi c}=\frac{\chi c}{\omega\varepsilon}$$

or

$$\chi=\frac{\omega}{c}\sqrt{\varepsilon\mu}. \tag{13}$$

The same formula follows from a comparison of (7) and (9).
It follows from (5, 13) that

$$H_{\varphi\rho} = -\sqrt{\frac{\varepsilon}{\mu}} E_{\theta\rho}, \tag{14}$$

and it follows from (14, 4, 11, 12) that

$$h_{\varphi\rho} = -e_{\theta\rho}\sqrt{\frac{\varepsilon}{\mu}}, \tag{15}$$

Similarly, it follows from (7, 13) that

$$H_{\theta\rho} = -\sqrt{\frac{\varepsilon}{\mu}} E_{\varphi\rho}, \tag{16}$$

and it follows from (16, 6, 8, 12) that

$$h_{\theta\rho} = -e_{\varphi\rho}\sqrt{\frac{\varepsilon}{\mu}}. \tag{17}$$

From a comparison of (15) and (17) it follows that

$$\frac{h_{\varphi\rho}}{h_{\theta\rho}} = \frac{e_{\theta\rho}}{e_{\varphi\rho}} = q, \tag{18}$$

$$\frac{h_{\varphi\rho}}{e_{\theta\rho}} = \frac{h_{\theta\rho}}{e_{\varphi\rho}} = -\sqrt{\frac{\varepsilon}{\mu}}. \tag{19}$$

Further we notice that lines 1, 4, 7 and 8 coincide, from which it follows that the function $g(\theta)$ is a solution of the differential equation

$$\frac{g(\theta)}{\operatorname{tg}(\theta)} + \frac{\partial(g(\theta))}{\partial\theta} = 0. \tag{20}$$

In Appendix 2 it is shown that the solution of this equation is the function

$$g(\theta) = \frac{1}{A \cdot |\sin(\theta)|}, \tag{20a}$$

where A is a constant. We note that in the well-known solution $g(\theta) = \sin(\theta)$ - see Appendix 1. It is easy to see that such a function does not satisfy equation (20). Consequently,

> **in the known solution 4 Maxwell's equations with expressions $\operatorname{rot}_\rho(E)$, $\operatorname{rot}_\rho(H)$, $\operatorname{div}(E)$, $\operatorname{div}(H)$ are not satisfied.**

Thus, the solution of the Maxwell's equations for a spherical wave in the far zone has the form of the intensities presented in Table 3, where

$$H_{\varphi\rho} = \frac{h_{\varphi\rho}}{\rho},\ H_{\theta\rho} = \frac{h_{\theta\rho}}{\rho},\ E_{\varphi\rho} = \frac{e_{\varphi\rho}}{\rho},\ E_{\theta\rho} = \frac{e_{\theta\rho}}{\rho} \qquad (21)$$

$$\chi = \frac{\omega}{c}\sqrt{\varepsilon\mu} \quad \text{(see 13)}, \quad \frac{g(\theta)}{\mathrm{tg}(\theta)} + \frac{\partial(g(\theta))}{\partial\theta} = 0 \quad \text{(see 20)}$$

and the constants $h_{\varphi\rho}$, $h_{\theta\rho}$, $e_{\theta\rho}$, $e_{\varphi\rho}$ satisfy conditions

$$\frac{h_{\varphi\rho}}{h_{\theta\rho}} = \frac{e_{\theta\rho}}{e_{\varphi\rho}} = q \quad \text{(см. 18)}, \quad \frac{h_{\varphi\rho}}{e_{\theta\rho}} = \frac{h_{\theta\rho}}{e_{\varphi\rho}} = -\sqrt{\frac{\varepsilon}{\mu}}. \quad \text{(см. 19)}$$

From Table. 3 it follows that

> **the same (with respect to the coordinates φ and θ) electric and magnetic intensities are shifted in phase by a quarter of the period.**

This corresponds to experimental electrical engineering. In Fig. 2 shows the intensities vectors in a spherical coordinate system.

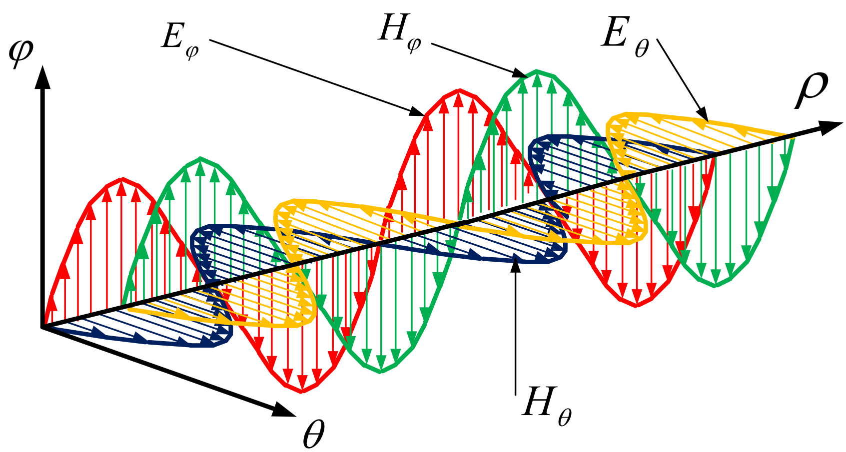

Fig. 2.

3. Energy Flows

Also, as in [1], the flow density of electromagnetic energy - the Poynting vector is

$$S = \eta E \times H, \qquad (1)$$

where

$$\eta = c/4\pi. \qquad (2)$$

In spherical coordinates φ, θ, ρ the flow density of electromagnetic energy has three components S_φ, S_θ, S_ρ directed along the radius, along the circumference, along the axis, respectively. They are determined by the formula

$$S=\begin{bmatrix} S_\varphi \\ S_\theta \\ S_\rho \end{bmatrix}=\eta(E\times H)=\eta\begin{bmatrix} E_\theta H_\rho - E_\rho H_\theta \\ E_\rho H_\varphi - E_\varphi H_\rho \\ E_\varphi H_\theta - E_\theta H_\varphi \end{bmatrix}. \tag{4}$$

From here and from Table 3 it follows that

$$\begin{aligned} & S_\varphi = 0 \\ & S_\theta = 0 \\ & S_\rho = \eta\begin{pmatrix} E_{\varphi\rho}H_{\theta\rho}\left(g(\theta)\sin(\chi\rho+\omega t)\right)^2 - \\ - E_{\theta\rho}H_{\varphi\rho}\left(g(\theta)\cos(\chi\rho+\omega t)\right)^2 \end{pmatrix} \end{aligned} . \tag{5}$$

It follows from (2.9, 2.11) that

$$E_{\varphi\rho}H_{\theta\rho}=\frac{\omega\mu}{\chi c}\left(H_{\theta\rho}\right)^2, \tag{6}$$

$$E_{\theta\rho}H_{\varphi\rho}=-\frac{\omega\mu}{\chi c}\left(H_{\varphi\rho}\right)^2. \tag{7}$$

Further from (6, 7, 2.4, 2.6) it follows that

$$E_{\varphi\rho}H_{\theta\rho}=\frac{\omega\mu}{\chi c}\left(h_{\theta\rho}\right)^2\frac{1}{\rho^2}, \tag{8}$$

$$E_{\theta\rho}H_{\varphi\rho}=-\frac{\omega\mu}{\chi c}\left(h_{\varphi\rho}\right)^2\frac{1}{\rho^2}. \tag{9}$$

From (5, 8, 9) we obtain:

$$S_\rho=\eta\cdot g^2(\theta)\frac{\omega\mu}{\chi c}\frac{1}{\rho^2}\begin{pmatrix} \left(h_{\theta\rho}\right)^2\left(\sin(\chi\rho+\omega t)\right)^2 + \\ + \left(h_{\varphi\rho}\right)^2\left(\cos(\chi\rho+\omega t)\right)^2 \end{pmatrix}. \tag{9}$$

Further from (9, 2.13, 2.18) it follows that

$$S_\rho=\eta\cdot g^2(\theta)\omega\sqrt{\frac{\mu}{\varepsilon}}\frac{1}{\rho^2}\begin{pmatrix} \left(h_{\theta\rho}\right)^2\left(\sin(\chi\rho+\omega t)\right)^2 + \\ + \left(qh_{\theta\rho}\right)^2\left(\cos(\chi\rho+\omega t)\right)^2 \end{pmatrix}. \tag{10}$$

where q is a previously undefined constant. If we take

$$q=1, \tag{10a}$$

then we get

$$S_\rho=\eta\cdot g^2(\theta)\omega\sqrt{\frac{\mu}{\varepsilon}}\frac{h_{\theta\rho}^2}{\rho^2}. \tag{11}$$

We also note that the surface area of a sphere with a radius ρ is equal to $4\pi\rho^2$. Then the flow of energy passing through a sphere with a radius ρ is

$$\overline{S_\rho} = 4\pi\eta\omega \cdot g^2(\theta) h_{\theta\rho}^2 \sqrt{\frac{\mu}{\varepsilon}} . \qquad (12)$$

It follows from (12) that

> **in a spherical electromagnetic wave, the energy flux passing through the spheres along the radius remains constant with increasing radius and does not change with time.**

This strictly corresponds to the law of conservation of energy.

It follows from (12) that the energy flow density varies along the meridian in accordance with the law $g^2(\theta)$.

4. Conclusion

An exact solution of the Maxwell equations for the far zone, which is presented in the table 3 is obtained, where

$H_{\varphi\rho}(\rho)$, $H_{\theta\rho} = (\rho)$, $E_{\varphi\rho} = (\rho)$, $E_{\theta\rho} = (\rho)$ are functions defined by (2.21, 2.18, 2.19),

$g(\theta)$ is a function defined by (2.20a),

χ is the constant determined by (2.13).

- The electric and magnetic intensities of the same name (with respect to the coordinates φ and θ) are phase shifted by a quarter of a period.
- In a spherical electromagnetic wave, the energy flux passing through the spheres along the radius remains constant with increasing radius and does NOT change with time and this strictly corresponds to the law of conservation of energy.
- The energy density varies along the meridian according to the law $g^2(\theta)$.

Thus, we obtained a rigorous solution of the Maxwell equations in the far zone, free from the drawbacks indicated above. At the same time, it should be noted that in the near zone, where radial electric and magnetic intensities are present, the known solution has an even greater list of disadvantages, in particular [3],

1. the energy conservation law is satisfied only on the average,
2. The solution is inhomogeneous and it is practically necessary to divide it into separate zones (as a rule, near, middle and far), in which the solutions turn out to be completely different,
3. In the near zone there is no flow of energy with the real value
4. The magnetic and electrical components are in phase,
5. In the near zone, the solution is not wave (i.s. the distance is not an argument of the trigonometric function),
6. The known solution does not satisfy Maxwell's system of equations (a solution that satisfies a single equation of the system can not be considered a solution of the system of equations).

In practice, these drawbacks of the known solution mean that they (mathematical solutions) do not strictly describe the real radiation of technical devices. A more rigorous solution, when applied in the design systems of such devices, must certainly improve their quality.

The solution of the Maxwell equations for spherical coordinates in the general case obtained and the author seeks for cooperation with organization interested in the practical application of this solution.

Appendix 1

The known solution has the form [3]:

$$E_\theta = e_\theta \frac{1}{\rho} \sin(\theta) \sin(\omega t - \chi\rho), \tag{1}$$

$$H_\varphi = h_\varphi \frac{1}{\rho} \sin(\theta) \sin(\omega t - \chi\rho), \tag{2}$$

$k_{e\theta} = \dfrac{\chi^2 lI}{4\pi\omega\varepsilon\varepsilon_o}$, $k_{h\varphi} = \dfrac{\chi lI}{4\pi}$, where l, I - length and current of the vibrator. We notice, that

$$\frac{e_\theta}{h_\varphi} = \frac{\chi}{\omega\varepsilon} \tag{3}$$

It should be noted that these tensions <u>are in phase</u>, which contradicts practical electrical engineering.

Let us consider how equations (1, 2) relate to Maxwell's system of equations - see Table 2. The intensities (1, 2) enter only in equation (6) from Table 2, which has the form

$$\mathrm{rot}_{\varphi} E + \frac{\mu}{c}\frac{\partial H_{\varphi}}{\partial t} = 0 \tag{4}$$

or

$$\frac{E_{\theta}}{\rho} + \frac{\partial E_{\theta}}{\partial \rho} + \frac{\mu}{c}\frac{\partial H_{\varphi}}{\partial t} = 0\,. \tag{5}$$

We substitute (1, 2) into (5) and obtain:

$$\begin{gathered} -e_{\theta}\frac{\chi}{\rho}\sin(\theta)\cos(\omega t - \chi\rho) - \\ -h_{\varphi}\frac{\chi}{\rho}\frac{\mu}{c}\sin(\theta)\cos(\omega t - \chi\rho) = 0 \end{gathered} \tag{6}$$

or

$$\frac{e_{\theta}}{h_{\varphi}} + \frac{\mu}{c} = 0\,. \tag{7}$$

From a comparison of (3) and (7) it follows that the intensities (1, 2) satisfy equation (4). The remaining 7 Maxwell equations are violated. In the equations (2, 3, 5) from Table 2 one of the terms differs from zero, and the other is equal to zero. The violation of equations (1, 4, 7, 8) from Table. 2 is shown above in Section 2. So,

> **the known solution does not satisfy Maxwell's system of equations.**

Appendix 1

We consider (2.20):

$$\frac{g(\theta)}{\mathrm{tg}(\theta)} + \frac{\partial(g(\theta))}{\partial \theta} = 0 \tag{1}$$

or

$$\frac{\partial(g(\theta))}{\partial \theta} = -\mathrm{ctg}(\theta)\cdot g(\theta) \tag{2}$$

or

$$\ln(g(\theta)) = -\int_{\theta} \mathrm{ctg}(\theta)\partial\theta\,. \tag{4}$$

It is known that

$$\int_{\theta} \mathrm{ctg}(\theta)\partial\theta = \ln(A\cdot|\sin(\theta)|). \tag{5}$$

where A is a constant. From (4, 5) we obtain:

$$\ln(g(\theta)) = -\ln(A\cdot|\sin(\theta)|) \tag{6}$$

or

$$g(\theta)=\frac{1}{A\cdot|\sin(\theta)|}. \qquad (8)$$

Tables

Table 1.

1	2	3
1	$\mathrm{rot}_{\rho}(E)$	$\frac{E_{\varphi}}{\rho \mathrm{tg}(\theta)}+\frac{\partial E_{\varphi}}{\rho \partial \theta}-\frac{\partial E_{\theta}}{\rho \sin(\theta)\partial \varphi}$
2	$\mathrm{rot}_{\theta}(E)$	$\frac{\partial E_{\rho}}{\rho \sin(\theta)\partial \varphi}-\frac{E_{\varphi}}{\rho}-\frac{\partial E_{\varphi}}{\partial \rho}$
3	$\mathrm{rot}_{\varphi}(E)$	$\frac{E_{\theta}}{\rho}+\frac{\partial E_{\theta}}{\partial \rho}-\frac{\partial E_{\rho}}{\rho \partial \varphi}$
4	$\mathrm{div}(E)$	$\frac{E_{\rho}}{\rho}+\frac{\partial E_{\rho}}{\partial \rho}+\frac{E_{\theta}}{\rho \mathrm{tg}(\theta)}+\frac{\partial E_{\theta}}{\rho \partial \theta}+\frac{\partial E_{\varphi}}{\rho \sin(\theta)\partial \varphi}$

Table 2.

1	2
1.	$\mathrm{rot}_{\rho} H-\frac{\varepsilon}{c}\frac{\partial E_{\rho}}{\partial t}=0$
2.	$\mathrm{rot}_{\theta} H-\frac{\varepsilon}{c}\frac{\partial E_{\theta}}{\partial t}=0$
3.	$\mathrm{rot}_{\varphi} H-\frac{\varepsilon}{c}\frac{\partial E_{\varphi}}{\partial t}=0$
4.	$\mathrm{rot}_{\rho} E+\frac{\mu}{c}\frac{\partial H_{\rho}}{\partial t}=0$
5.	$\mathrm{rot}_{\theta} E+\frac{\mu}{c}\frac{\partial H_{\theta}}{\partial t}=0$
6.	$\mathrm{rot}_{\varphi} E+\frac{\mu}{c}\frac{\partial H_{\varphi}}{\partial t}=0$
7.	$\mathrm{div}(E)=0$
8.	$\mathrm{div}(H)=0$

Table 3.

1	**2**
	$E_\theta = E_{\theta\rho}(\rho)g(\theta)\cos(\chi\rho + \omega t)$
	$E_\varphi = E_{\varphi\rho}(\rho)g(\theta)\sin(\chi\rho + \omega t)$
	$E_\rho = 0$
	$H_\theta = H_{\theta\rho}(\rho)g(\theta)\sin(\chi\rho + \omega t)$
	$H_\varphi = H_{\varphi\rho}(\rho)g(\theta)\cos(\chi\rho + \omega t)$
	$H_\rho = 0$

Table 3a.

1	**2**	**3**
1	$\mathrm{rot}_\rho(E)$	$\frac{E_\varphi}{\rho \mathrm{tg}(\theta)} + \frac{\partial E_\varphi}{\rho \partial \theta}$
2	$\mathrm{rot}_\theta(E)$	$-\frac{E_\varphi}{\rho} - \frac{\partial E_\varphi}{\partial \rho}$
3	$\mathrm{rot}_\varphi(E)$	$\frac{E_\theta}{\rho} + \frac{\partial E_\theta}{\partial \rho}$
4	$\mathrm{div}(E)$	$\frac{E_\theta}{\rho \mathrm{tg}(\theta)} + \frac{\partial E_\theta}{\rho \partial \theta}$

Table 4.

1	**2**	**3**
1	$\mathrm{rot}_\rho(E)$	$\frac{E_\varphi}{\rho \mathrm{tg}(\theta)} + \frac{\partial E_\varphi}{\rho \partial \theta}$
2	$\mathrm{rot}_\theta(E)$	$-\left(\frac{E_{\varphi\rho}}{\rho}\sin(...) + \frac{\partial E_{\varphi\rho}}{\partial \rho}\sin(...) + \chi E_{\varphi\rho}\cos(...)\right)g(\theta)$
3	$\mathrm{rot}_\varphi(E)$	$\left(\frac{E_{\theta\rho}}{\rho}\cos(...) + \frac{\partial E_{\theta\rho}}{\partial \rho}\cos(...) - \chi E_{\theta\rho}\sin(...)\right)g(\theta)$
4	$\mathrm{div}(E)$	$\frac{E_\theta}{\rho \mathrm{tg}(\theta)} + \frac{\partial E_\theta}{\rho \partial \theta}$
5	$\mathrm{rot}_\rho(H)$	$\frac{H_\varphi}{\rho \mathrm{tg}(\theta)} + \frac{\partial H_\varphi}{\rho \partial \theta}$

6	$\mathrm{rot}_\theta(H)$	$-\left(\frac{H_{\varphi\rho}}{\rho}\cos(...)+\frac{\partial H_{\varphi\rho}}{\partial\rho}\cos(...)-\chi H_{\varphi\rho}\sin(...)\right)g(\theta)$
7	$\mathrm{rot}_\varphi H$	$\left(\frac{H_{\theta\rho}}{\rho}\sin(...)+\frac{\partial H_{\theta\rho}}{\partial\rho}\sin(...)+\chi H_{\theta\rho}\cos(...)\right)g(\theta)$
8	$\mathrm{div}(H)$	$\frac{H_\theta}{\rho\,\mathrm{tg}(\theta)}+\frac{\partial H_\theta}{\rho\partial\theta}$

Table 5.

1	2
1.	$\frac{g(\theta)}{\mathrm{tg}(\theta)}+\frac{\partial(g(\theta))}{\partial\theta}=0$
2.	$-\frac{H_{\varphi\rho}}{\rho}\cos(...)-\frac{\partial H_{\varphi\rho}}{\partial\rho}\cos(...)+\chi H_{\varphi\rho}\sin(...)+\frac{\omega\varepsilon}{c}E_{\theta\rho}\sin(...)=0$
3.	$\frac{H_{\theta\rho}}{\rho}\sin(...)+\frac{\partial H_{\theta\rho}}{\partial\rho}\sin(...)+\chi H_{\theta\rho}\cos(...)-\frac{\omega\varepsilon}{c}E_{\varphi\rho}\cos(...)=0$
4.	$\frac{g(\theta)}{\mathrm{tg}(\theta)}+\frac{\partial(g(\theta))}{\partial\theta}=0$
5.	$-\frac{E_{\varphi\rho}}{\rho}\sin(...)-\frac{\partial E_{\varphi\rho}}{\partial\rho}\sin(...)-\chi E_{\varphi\rho}\cos(...)+\frac{\omega\mu}{c}H_{\theta\rho}\sin(...)=0$
6.	$\frac{E_{\theta\rho}}{\rho}\cos(...)+\frac{\partial E_{\theta\rho}}{\partial\rho}\cos(...)-\chi E_{\theta\rho}\sin(...)-\frac{\omega\mu}{c}H_{\varphi\rho}\sin(...)=0$
7.	$\frac{g(\theta)}{\mathrm{tg}(\theta)}+\frac{\partial(g(\theta))}{\partial\theta}=0$
8.	$\frac{g(\theta)}{\mathrm{tg}(\theta)}+\frac{\partial(g(\theta))}{\partial\theta}=0$

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