The Irrationality and Transcendence of e Connected

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Abstract

Using just the derivative of the sum is the sum of the derivatives and simple undergraduate mathematics a proof is given showing e^n is irrational. The proof of e's transcendence is a simple generalization from this result.

Using the techniques of a proof of e's transcendence given in Herstein's Topics in Algebra [2], Beatty gave a proof of the irrationality of e^n, n a positive integer [1]. In this article we show how the mean value theorem, used in both Herstein and Beatty's proofs, can be avoided in favor of a simpler approach that yields a nice path from the irrationality of e^n to e's transcendence.

In what follows, x is a real number, all polynomials are integer polynomials, and p is a prime.

Definition 1. Given a polynomial $f(x)$, lowercase, the sum of all its derivatives is designated with $F(x)$, uppercase.

Definition 2. For non-negative integers n, let $\epsilon_n(x)$ denote the infinite series

$$\frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \cdots + \frac{x^j}{(n+1)(n+2)\dots(n+j)} + \dots.$$

Lemma 1. *If $f(x) = cx^n$, then*

$$F(0)e^x = F(x) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n.

Proof. As $F(x) = c(x^n + nx^{n-1} + \dots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^x &= cn!(1 + x/1 + x^2/2! + \dots + x^n/n! + \dots) \\ &= cx^n + cnx^{(n-1)} + \dots + cn! + cx^{n+1}/(n+1)! + \dots \\ &= F(x) + cx^n(x/(n+1) + x^2/(n+1)(n+2) + \dots) \\ &= F(x) + f(x)\epsilon_n(x). \end{aligned}$$

Now $f(x)$ has polynomial growth in n and $\epsilon_n(x) \le e^x$, so the product has polynomial growth in n. □

Lemma 2. *If $f(x) = c_0 + c_1x + \dots + c_nx^n$, then*

$$e^x F(0) = F(x) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f.

Proof. Let $f_j(x) = c_jx^j$, for $0 \le j \le n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^{n} (f_0 + f_1 + \dots + f_n)^{(k)} = F_0 + F_1 + \dots + F_n,$$

where F_j is the sum of the derivatives of f_j. Using Lemma 1,

$$e^x F_k(0) = F_k(x) + \epsilon \tag{3}$$

and summing (3) from $k = 0$ to n, gives

$$e^x F(0) = F(x) + n\epsilon.$$

As the finite sum of functions with polynomial growth in n also has polynomial growth in n, we arrive at (2). □

Lemma 3. *If the polynomial $f(x)$ has a non-zero root r of multiplicity p, $p!|F(r)$.*

Proof. We can write $f(x) = (x - r)^pQ(x)$, where $Q(x)$ is a polynomial. The sum of the derivatives of $f(x)$ are given by the Leibniz table, Table 1. When $x = r$ only the last column remains non-zero and the value in each of its cells is multiplied by $p!$. □

	$(x-r)^p$	$p(x-r)^{p-1}$	$\dots$	$p!$
$Q(z)$				
$Q'(z)$				
$\vdots$				
$Q^{(k)}(x)$				

Table 1: Leibniz table showing $p!|F(r)$, where $F(x) = (x-r)^p Q(x)$.

	x^{p-1}	px^{p-1}	$\dots$	$(p-1)!$
$[\prod(x-r_i)]^p$				
$p\dots$				
$\vdots$				
$p\dots$				

Table 2: Leibniz table showing $(p-1)!|F(0)$, where $F(x) = x^{p-1}[\prod(x-r_i)]^p$.

Lemma 4. *Let polynomial $f(x)$ have root $r = 0$ of multiplicity $p-1$ and n other roots r_i of multiplicity p, then, for large enough p,*

$$F(0) + F(r_1) + \cdots + F(r_n) \tag{4}$$

is a non-zero integer divisible by $(p-1)!$.

Proof. Using Lemma 3, $p!|F(r_i)$ for each i, $1 \le i \le n$, and, referring to Table 2, we see $(p-1)!|F(0)$, but $p \nmid F(0)$ when $p > r_1 r_2 \dots r_n$; (4) follows. □

Theorem 1. *For positive, non-zero rational r, e^r is irrational.*

Proof. It is sufficient to prove that e^n, n a natural number is irrational. Suppose not, suppose $e^n = a/b$ with a, b natural numbers $a > b$. Define $f(x) = x^{p-1}(x-n)^p$. Then, using Lemma 2, $e^n F(0) = F(n) + \epsilon$ and this implies $aF(0) - bF(n) = b\epsilon$. Dividing by $(p-1)!$ gives

$$\frac{aF(0) - bF(n)}{(p-1)!} = \frac{b\epsilon}{(p-1)!}. \tag{5}$$

If p is sufficiently large, (5), using Lemmas 3 and 4, gives an absolute value of the left hand side that is at least 1 while the absolute value of the right hand side is less than 1, a contradiction. □

Theorem 2. *e is transcendental.*

Proof. A number is transcendental if it doesn't solve an integer polynomial. Suppose e solves an nth degree integer polynomial, then

$$0 = c_n e^n + c_{n-1} e^{n-1} + \cdots + c_0.$$

Define $f_n(x) = x^{p-1}[(x-1)(x-2)\cdots(x-n)]^p$. Using Lemma 2, we have

$$0 = F_n(0)(c_n e^n + c_{n-1} e^{n-1} + \cdots + c_0) = c_0 F_n(0) + \sum_{k=1}^{n} c_k F_n(k) + \epsilon. \qquad (6)$$

Now using Lemma 4, when (6) is divided by $(p-1)!$, $c_0 F_n(0) + \sum_{k=1}^{n} c_k F_n(k)$ is a non-zero integer. As $\epsilon/(p-1)!$ can be made as small as we please with increasing primes p, the sum of the two can't be zero. We have a contradiction of the right hand side of (6). □

References

[1] T. Beatty and T.W. Jones, A Simple Proof that $e^{p/q}$ is Irrational, *Math. Magazine*, **87**, (2014) 50–51.

[2] I. N. Herstein, *Topics in Algebra*, 2nd ed., John Wiley, New York, 1975.