Question 408: A Trigonometric Formula

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Abstract. This note presents a simple formula for pi.

1. Introduction: Formula

Notation:

$\mathbb{Z}=\{...,-3,-2,-1,0,1,2,3,...\}$, set of all integers

$\mathbb{R}$ real numbers

$\mathbb{C}$ complex numbers

$i=\sqrt{-1}$, imaginary unit

$|\ |$, absolute value (modulus)

$\operatorname{Re} z$, real part of z

$\operatorname{Im} z$, imaginary part of z

Trigonometric formula:

For $n\in\mathbb{Z}, z=x+iy\in\mathbb{C}, x,y\in\mathbb{R}, |z|=1$, we have

$$\pi=2\sin^{-1}\left(\left|\operatorname{Re}\left(z^n\right)\right|\right)+2\sin^{-1}\left(\left|\operatorname{Im}\left(z^n\right)\right|\right) \tag{1}$$

$$\pi=2\sin^{-1}\left(\left|\operatorname{Re}\left(\left(x+iy\right)^n\right)\right|\right)+2\sin^{-1}\left(\left|\operatorname{Im}\left(\left(x+iy\right)^n\right)\right|\right) \tag{2}$$

2. Examples

Example 1: $z=\frac{3+4i}{5}$

$$\pi=2\sin^{-1}\left(\left|\operatorname{Re}\left(\left(\frac{3+4i}{5}\right)^n\right)\right|\right)+2\sin^{-1}\left(\left|\operatorname{Im}\left(\left(\frac{3+4i}{5}\right)^n\right)\right|\right) \quad ,n\in\mathbb{Z} \tag{3}$$

$$n=1\Rightarrow\pi=2\sin^{-1}\left(\frac{3}{5}\right)+2\sin^{-1}\left(\frac{4}{5}\right) \tag{4}$$

$$n=2\Rightarrow\pi=2\sin^{-1}\left(\frac{7}{5^2}\right)+2\sin^{-1}\left(\frac{24}{5^2}\right) \tag{5}$$

$$n=6 \Rightarrow \pi = 2\sin^{-1}\left(\frac{10296}{5^6}\right)+2\sin^{-1}\left(\frac{11753}{5^6}\right) \tag{6}$$

$$n=11 \Rightarrow \pi = 2\sin^{-1}\left(\frac{34182196}{5^{11}}\right)+2\sin^{-1}\left(\frac{34867797}{5^{11}}\right) \tag{7}$$

Example 2: $z=\frac{5+12i}{13}$

$$\pi = 2\sin^{-1}\left(\left|\mathrm{Re}\left(\left(\frac{5+12i}{13}\right)^n\right)\right|\right)+2\sin^{-1}\left(\left|\mathrm{Im}\left(\left(\frac{5+12i}{13}\right)^n\right)\right|\right) \quad ,n\in\mathbb{Z} \tag{8}$$

$$n=2 \Rightarrow \pi = 2\sin^{-1}\left(\frac{119}{13^2}\right)+2\sin^{-1}\left(\frac{120}{13^2}\right) \tag{9}$$

$$n=3 \Rightarrow \pi = 2\sin^{-1}\left(\frac{828}{13^3}\right)+2\sin^{-1}\left(\frac{2035}{13^3}\right) \tag{10}$$

$$n=4 \Rightarrow \pi = 2\sin^{-1}\left(\frac{239}{13^4}\right)+2\sin^{-1}\left(\frac{28560}{13^4}\right) \tag{11}$$

Example 3: $z=\frac{2+i\sqrt{5}}{3}$

$$\pi = 2\sin^{-1}\left(\left|\mathrm{Re}\left(\left(\frac{2+i\sqrt{5}}{3}\right)^n\right)\right|\right)+2\sin^{-1}\left(\left|\mathrm{Im}\left(\left(\frac{2+i\sqrt{5}}{3}\right)^n\right)\right|\right) \quad ,n\in\mathbb{Z} \tag{12}$$

$$n=1 \Rightarrow \pi = 2\sin^{-1}\left(\frac{2}{3}\right)+2\sin^{-1}\left(\frac{\sqrt{5}}{3}\right) \tag{13}$$

$$n=2 \Rightarrow \pi = 2\sin^{-1}\left(\frac{1}{3^2}\right)+2\sin^{-1}\left(\frac{4\sqrt{5}}{3^2}\right) \tag{14}$$

$$n=3 \Rightarrow \pi = 2\sin^{-1}\left(\frac{22}{3^3}\right)+2\sin^{-1}\left(\frac{7\sqrt{5}}{3^3}\right) \tag{15}$$

References

1. Olver, F.W.J., Lozier, D.W., Boisvert, R.F., and Clark, C.W.: NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.