EUCLIDEAN RELATIVITY

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Abstract: In this work, we present in more details the formulation of Euclidean relativity. We show that even though there are profound differences between Einstein special relativity and Euclidean special relativity, general relativity with both pseudo-Euclidean metric and Euclidean metric have many common features. For example, both forms of metric can be used to describe the precession of planetary orbits around a gravitational mass and the cosmological evolution.

In our work on the motion of quantum particles, when they are viewed as three-dimensional Riemannian manifolds, we suggested that their motion could be described by extending the isometric transformations in classical physics to the isometric embedding between smooth manifolds [1]. According to the Whitney embedding theorem, in order to smoothly embed three-dimensional Riemannian manifolds we would need an ambient six-dimensional Euclidean space [2,3]. It has also been shown in our works on the temporal dynamics that a six-dimensional Minkowski pseudo-Euclidean spacetime is obtained by extending one-dimensional temporal continuum to three-dimensional temporal manifold [4]. While the question of whether it is possible to smoothly embed three-dimensional Riemannian manifolds in six-dimensional pseudo-Euclidean spacetime remains, we showed that it is possible to apply the principle of relativity and the postulate of a universal speed to formulate a special theory of relativity in which the geometry of spacetime has a positive definite metric by modifying the Lorentz transformation. The modified Lorentz transformation gives rise to new interesting features, such as there is no upper limit for the relative speed between inertial reference frames, the assumed universal speed is not the speed of any physical object or physical field but rather the common speed of expansion of the spatial space of all inertial frames. Furthermore, we also showed that when the ratio of the relative speed and the universal speed approaches infinite values, there will be a conversion between space and time, therefore not only the concept of motion but the concepts of space and time themselves are also relative [1]. In this work we will develop and present in more details the Euclidean special relativity and extend it to the Euclidean general relativity.

In classical physics, the concept of a pseudo-Euclidean spacetime was introduced by Minkowski in order to accommodate Einstein theory of special relativity in which the coordinate transformation between the inertial frame S with spacetime coordinates (ct, x, y, z) and the inertial frame S' with coordinates (ct', x', y', z') are derived from the principle of relativity and the postulate of a universal speed c. The coordinate transformation is the Lorentz transformation given by

$$x^{'} = \gamma(x - \beta ct) \tag{1}$$

$$y^{'} = y \tag{2}$$

$$z^{'} = z \tag{3}$$

$$ct^{'} = \gamma(-\beta x + ct) \tag{4}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$ [5]. It is shown that the Lorentz transformation given in Equations (1-4) leaves the Minkowski spacetime interval $-c^2t^2 + x^2 + y^2 + z^2$ invariant. Spacetime with this metric is a pseudo-Euclidean space. We now show that it is possible to construct a special relativistic transformation that will make spacetime a Euclidean space rather than a pseudo-Euclidean space as in the case of the Lorentz transformation. Consider the following modified Lorentz transformation

$$x^{'} = \gamma_E(x - \beta ct) \tag{5}$$

$$y^{'} = y \tag{6}$$

$$z^{'} = z \tag{7}$$

$$ct^{'} = \gamma_E(\beta x + ct) \tag{8}$$

where $\beta = v/c$ and γ_E will be determined from the principle of relativity and the postulate of a universal speed. Instead of assuming the invariance of the Minkowski spacetime interval, if we now assume the invariance of the Euclidean interval $c^2t^2 + x^2 + y^2 + z^2$ then from the modified Lorentz transformation given in Equations (5-8), we obtain the following expression for γ_E

$$\gamma_E = \frac{1}{\sqrt{1+\beta^2}} \tag{9}$$

It is seen from the expression of γ_E given in Equation (9) that there is no upper limit in the relative speed v between inertial frames. The value of γ_E at the universal speed $v = c$ is $\gamma_E = 1/\sqrt{2}$. For the values of $v \ll c$, the modified Lorentz transformation given in Equations (5-8) also reduces to the Galilean transformation. However, it is interesting to observe that when $\beta \to \infty$ we have $\gamma_E \to 0$ and $\beta\gamma_E \to 1$, and in this case from Equations (5) and (8), we obtain $x^{'} \to -ct$ and $ct^{'} \to x$, respectively. This result shows that there is a conversion between space and time when $\beta \to \infty$, therefore in Euclidean special relativity, not only the concept of motion but the concepts of space and time themselves are also relative. It is also worth mentioning here that the Euclidean relativity of space and time also provides a profound foundation for the temporal dynamics that we have discussed in our other works [6]. In the present situation, if in the inertial frame S with spacetime coordinates (ct, x, y, z) the dynamics of a particle is described by Newton's second law $m\, d^2\mathbf{r}/dt^2 = \mathbf{F}$, then since

$x^{'} \rightarrow -ct$ and $ct^{'} \rightarrow x$ it is seen that the spatial Newton's second law in the inertial frame S is converted to a temporal law of dynamics $D\, d^2\mathbf{t}/dr^2 = \mathbf{F}$ in the inertial frame S' with spacetime coordinates $(ct^{'}, x^{'}, y^{'}, z^{'})$.

As in the case of the Lorentz transformation given in Equations (1-4), we can also derive the relativistic kinematics and dynamics from the modified Lorentz transformation given in Equations (5-8), such as the transformation of a length, the transformation of a time interval, the transformation of velocities, and the transformation of accelerations. Let L_0 be the proper length then the length transformation can be found as

$$L = \sqrt{1+\beta^2} L_0 \tag{10}$$

It is observed from the length transformation given in Equation (10) that the length of a moving object is expanding rather than contracting as in Einstein theory of special relativity. Now if Δt_0 is the proper time interval then the time interval transformation can also be found to be given by the relation

$$\Delta t = \frac{1}{\sqrt{1+\beta^2}} \Delta t_0 \tag{11}$$

It is also observed from the time interval transformation given in Equation (11) that the proper time interval is longer than the same time interval measured by a moving observer. With the modified Lorentz transformation given in Equations (5-8), the transformation of velocities can be found as follows

$$v_x^{'} = \frac{dx^{'}}{dt^{'}} = \frac{v_x - \beta c}{1 + \frac{\beta v_x}{c}} \tag{12}$$

$$v_y^{'} = \frac{dy^{'}}{dt^{'}} = \frac{v_y}{\gamma_E \left(1 + \frac{\beta v_x}{c}\right)} \tag{13}$$

$$v_z^{'} = \frac{dz^{'}}{dt^{'}} = \frac{v_z}{\gamma_E \left(1 + \frac{\beta v_x}{c}\right)} \tag{14}$$

Form Equation (12), if we let $v_x = c$ then we obtain $v_x^{'} = \left(\frac{c-v}{c+v}\right)c$. Therefore in this case $v_x^{'} = c$ only when the relative speed v between two inertial frames vanishes, $v = 0$. In other words, the universal speed c is not the common speed of any moving physical object or physical field in inertial reference frames. In order to specify the nature of the assumed universal speed we observe that in Einstein theory of special relativity it is assumed that spatial space of an inertial frame remains static and this assumption is contradicted to Einstein theory of general relativity that shows that spatial space is actually expanding. Therefore it seems reasonable to suggest that the universal speed c in the modified Lorentz transformation given in Equations (5-8) is the universal speed of expansion of the spatial space of all inertial frames. The transformations of accelerations can be derived from the modified Lorentz transformation and the transformations of velocities given in Equations (12-14). The transformation of the accelerations can be found as

$$\frac{dv'_x}{dt'} = \frac{1}{\gamma_E{}^3\left(1+\frac{\beta v_x}{c}\right)^3}\frac{dv_x}{dt} \tag{15}$$

$$\frac{dy'_x}{dt'} = \frac{1}{\gamma_E{}^2\left(1+\frac{\beta v_x}{c}\right)^2}\frac{dv_y}{dt} - \frac{\beta v_y}{c\gamma_E{}^3\left(1+\frac{\beta v_x}{c}\right)^3}\frac{dv_x}{dt} \tag{16}$$

$$\frac{dz'_x}{dt'} = \frac{1}{\gamma_E{}^2\left(1+\frac{\beta v_x}{c}\right)^2}\frac{dv_z}{dt} - \frac{\beta v_z}{c\gamma_E{}^3\left(1+\frac{\beta v_x}{c}\right)^3}\frac{dv_x}{dt} \tag{17}$$

By carrying out the thought experiment of the collision of two identical masses in two inertial frames that are moving relative to each other, we can derive the following relationship between the rest mass m_0 observed in the rest frame and the mass m observed from other frame as [7,8]

$$m = \frac{m_0}{\sqrt{1+\beta^2}} \tag{18}$$

It is seen from Equation (18) that $m \to 0$ when $\beta \to \infty$. However, when $\beta \to \infty$ we also have the conversion between space and time $x' \to -ct$, therefore we may speculate that there may also be a conversion between the spatial mass m and the temporal mass D of a particle when $\beta \to \infty$ [4]. Form Equation (18) we obtain

$$m^2c^2 + m^2v^2 = m_0^2c^2 \tag{19}$$

Since both m and $\mathbf{v}$ are variables, we obtain the following relation by differentiation

$$c^2dm = -v^2dm - m\mathbf{v}.\mathrm{d}\mathbf{v} \tag{20}$$

On the other hand, from Newton's second law $\mathbf{F} = d(m\mathbf{v})/dt$, we have

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} \tag{21}$$

Using Equations (21), the change of kinetic energy $dT = \mathbf{F}.d\mathbf{s} = \mathbf{F}.\mathbf{v}dt$ can be obtained as

$$dT = v^2dm + m\mathbf{v}.\mathrm{d}\mathbf{v} \tag{22}$$

Using Equations (21) and (22) we arrive at

$$dT = -c^2dm \tag{23}$$

Since $dm < 0$, therefore we have $dT > 0$. By integrating both sides of Equation (23)

$$\int_{v=0}^{v} dT = -\int_{m_0}^{m} c^2 dm \tag{24}$$

we obtain the following expression for the kinetic energy

$$T = (m_0 - m)c^2 = (1 - \gamma_E)m_0c^2 \tag{25}$$

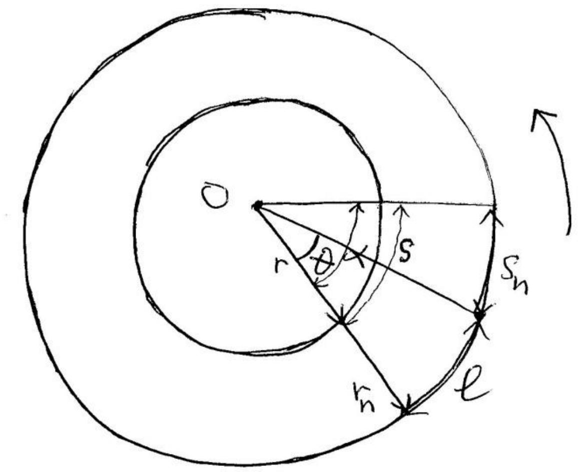
For $v \ll c$, we have $\gamma_E \sim 1 - \beta^2/2$ and Equation (25) reduces to $T \sim m_0v^2/2$. However, we have $T \rightarrow m_0c^2$ when $\beta \rightarrow \infty$. The relativistic momentum **p** of a particle of mass m with velocity **v** can also be defined by the following relation

$$\mathbf{p} = m\mathbf{v} = \gamma_E m_0 \mathbf{v} \tag{26}$$

Then we have $p \rightarrow m_0c$ when $\beta \rightarrow \infty$. In magnitudes, $pc = mvc = \beta mc^2 = \beta E$, where the total energy E is defined by the relation $E = mc^2 = m_0c^2 - T$. From this definition, we obtain $E \rightarrow 0$ when $\beta \rightarrow \infty$. Using the relations $E = mc^2$ and $pc = \beta E$, we also obtain the following Euclidean relativistic energy-momentum relationship

$$E^2 = (m_0c^2)^2 - (pc)^2 \tag{27}$$

Now consider the rotating frames of reference in the form of concentric circles as shown in the figure below [9]



Let r and t be the radius and time of a circular frame which is regarded as being stationary. Let r_n and t_n be the radius and time of a circular frame which is rotating with respect to the (r,t)-frame with a constant angular speed ω about the common centre O. Denote s and s_n the arc-length positions of a particle in the (r,t)-frame and (r_n,t_n)-frame respectively. If we assume $t_n = t$, then from the figure above we obtain the following relations

$$s = r\alpha \tag{28}$$

$$l = r_n\theta = r_n\omega t \tag{29}$$

$$s_n + r_n\omega t = r_n\alpha \tag{30}$$

From Equations (28) and (30), we obtain

$$s=\frac{r}{r_n}(s_n+r_n\omega t) \tag{31}$$

Together with $t_n=t$, Equation (31) can be seen as a form of kinematical Galilean transformations of circular reference frames. In order to formulate a Euclidean special relativity for circular reference frames, we assume that the relativistic transformations for the rotating frames take the following forms

$$s=\frac{Rr}{r_n}(s_n+r_n\omega t_n) \tag{32}$$

$$ct=\frac{Rr}{r_n}\left(-\frac{r_n\omega s_n}{c}+ct_n\right) \tag{33}$$

where R is a quantity that will be determined and c is an undetermined universal speed. The quantity R can be determined if we simply assume the following Euclidean identity

$$s_n^2+c^2t_n^2=s^2+c^2t^2 \tag{34}$$

With the assumed relation given in Equation (34), we obtain

$$R=\frac{r_n}{r\sqrt{1+\frac{r_n^2\omega^2}{c^2}}} \tag{35}$$

We finally obtain the following Euclidean special relativistic transformations for circular reference frames

$$s=\frac{1}{\sqrt{1+\frac{r_n^2\omega^2}{c^2}}}(s_n+r_n\omega t_n) \tag{36}$$

$$ct=\frac{1}{\sqrt{1+\frac{r_n^2\omega^2}{c^2}}}\left(-\frac{r_n\omega s_n}{c}+ct_n\right) \tag{37}$$

As in the case of linear inertial reference frames, there are new features considering the special relativity of circular reference frames that have a Euclidean metric. For example, we have $R\to 0$ and $(r_n\omega/c)R\to 1$ when $\omega\to\infty$, and from Equations (36) and (37) we have $s\to ct_n$ and $ct\to -s_n$, respectively. There is also a conversion between space and time.

In the following we will extend our presentation of the Euclidean relativity to general relativity. It is obvious that we can still assume that Einstein theory of general relativity can also be applied to Riemannian spacetime manifolds which are endowed with positive definite metrics. In the original Einstein theory of general relativity, the field equations of the gravitational field are proposed to take the form [5]

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta} \quad (38)$$

where $T_{\alpha\beta}$ is the covariant form of the energy-momentum tensor, $R_{\alpha\beta}$ is the Ricci tensor defined by the relation

$$R_{\mu\nu} = \frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} - \frac{\partial \Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} \quad (39)$$

and the metric connection $\Gamma^{\sigma}_{\mu\nu}$ is defined in terms of the metric tensor $g_{\alpha\beta}$ as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \quad (40)$$

and $R = g^{\alpha\beta} R_{\alpha\beta}$ is the Ricci scalar curvature. However, as discussed in our previous works on the field equations of general relativity [10], if we rewrite Einstein field equations in the following form

$$T_{\alpha\beta} = \frac{1}{\kappa} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \quad (41)$$

then Einstein field equations can be interpreted as a definition of an energy-momentum tensor as that of Maxwell theory of the electromagnetic field. In this case, the basic equations of the gravitational field can be proposed using the contracted Bianchi identities

$$\nabla_{\beta} R^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R \quad (42)$$

Even though Equation (42) is purely geometrical, it has a form of Maxwell field equations of the electromagnetic tensor, $\partial_{\alpha} F^{\alpha\beta} = \mu j^{\beta}$. If the quantity $\frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R$ can be perceived as a physical entity, such as a four-current of gravitational matter, then Equation (42) has the status of a dynamical law of a physical theory. With the assumption that the quantity $\frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R$ to be identified with a four-current of gravitational matter then a four-current $j^{\alpha} = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as follows

$$j^{\alpha} = \frac{1}{2} g^{\alpha\beta} \nabla_{\beta} R \quad (43)$$

For a purely gravitational field, Equation (42) reduces to

$$\nabla_{\beta} R^{\alpha\beta} = 0 \quad (44)$$

Using the identity $\nabla_{\mu} g^{\alpha\beta} \equiv 0$, Equation (44) implies

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \quad (45)$$

where Λ is an undetermined constant. Using the identities $g_{\alpha\beta}g^{\alpha\beta} = 4$ and $g_{\alpha\beta}R^{\alpha\beta} = R$, we obtain $\Lambda = R/4$, and the energy-momentum tensor given in Equation (41) reduces to

$$T_{\alpha\beta} = -\frac{\Lambda}{\kappa} g_{\alpha\beta} \tag{46}$$

As shown in the appendix 1, the Schwarzschild vacuum solution with $\Lambda = 0$ can be obtained with a Riemannian positive definite metric for a centrally symmetric field given in the form

$$ds^2 = e^{2\nu(r)}c^2dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{47}$$

The Schwarzschild vacuum solution is found as

$$ds^2 = \left(1 + \frac{C}{r}\right)c^2dt^2 + \left(1 + \frac{C}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{48}$$

where C is a constant of integration that can be identified with the mass of the source of a physical field. It should be mentioned here that if the line element given in Equation (47) is the description of the physical field of a three-dimensional physical object which is isometrically embedded in a six-dimensional Euclidean space R^6 then the time t can be assumed to be the temporal arclength of a temporal curve. In order to investigate the nature of the constant C we examine the motion in this spacetime that is described by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0 \tag{49}$$

With $x^0 = ct$ and $ds = cd\tau$, the geodesic equation for $\mu = 0$ can be found to satisfy the relation [11]

$$\left(1 + \frac{C}{r}\right)\frac{dt}{d\tau} = C_1 \tag{50}$$

where C_1 is a constant of integration. For $\mu = 1, 2, 3$ we obtain following the relations

$$\frac{d^2r}{d\tau^2} + \frac{C}{2r^2}\left(1 + \frac{C}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{C}{2r^2}\left(1 + \frac{C}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r\left(1 + \frac{C}{r}\right)\left(\frac{d\theta}{d\tau}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{51}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{52}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{53}$$

On the other hand, if we divide the line element given in Equation (48) by $ds^2 = c^2d\tau^2$, we obtain the equation

$$1 = \left(1 + \frac{C}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2}\left(1 + \frac{C}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{c^2}r^2\left(\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right) \tag{54}$$

For a planar motion with $\theta = \pi/2$, Equation (53) reduces to

$$r^2 \frac{d\phi}{d\tau} = C_2 \tag{55}$$

where C_2 is a constant of integration. Using Equations (50) and (55), Equation (54) is reduced to the equation

$$1 = \left(1 + \frac{C}{r}\right)^{-1} C_1^2 + \frac{C_2^2}{c^2}\left(1 + \frac{C}{r}\right)^{-1} \frac{1}{r^4}\left(\frac{d\phi}{dr}\right)^2 + \frac{C_2^2}{c^2 r^2} \tag{56}$$

Using the identity $\frac{1}{r^4}\left(\frac{d\phi}{dr}\right)^2 = \left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2$, Equation (56) is simplified to

$$\left(\frac{d}{d\phi}\left(\frac{1}{r}\right)\right)^2 + \frac{1}{r^2} = \frac{c^2}{C_2^2}(1 - C_1^2) + \frac{c^2 C}{r C_2^2} - \frac{C}{r^3} \tag{57}$$

By differentiating Equation (57) with respect to ϕ, we have

$$\frac{d}{d\phi}\left(\frac{1}{r}\right)\left(\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) = \frac{d}{d\phi}\left(\frac{1}{r}\right)\left(\frac{c^2 C}{2C_2^2} - \frac{3C}{2r^2}\right) \tag{58}$$

From Equation (58), we obtain the following two equations

$$\frac{d}{d\phi}\left(\frac{1}{r}\right) = 0 \tag{59}$$

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{c^2 C}{2C_2^2} - \frac{3C}{2r^2} \tag{60}$$

It is seen that as in the case of Schwarzschild solution with the Minkowski pseudo-Riemannian metric, Equation (59) describes a circle and Equation (60) can be used to describe the precession of planetary orbits around a gravitational mass if the constant C is identified with the gravitational mass M as $C = -2GM/c^2$ and the constant C_2 is defined in terms of the semi-latus rectum p of an ellipse as $C_2^2 = pGM$.

It is noted that if the field endowed with the Riemannian metric given in Equation (47) is still spherically symmetric but now time-dependent then the metric can be shown to be written in the form [11]

$$ds^2 = e^{2\nu(r,t)} c^2 dt^2 + e^{2\lambda(r,t)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{61}$$

Similar to the case of time-independent spherically symmetric metric as shown in the appendix 1, the time-dependent metric given in Equation (61) can be reduced to the time-independent Schwarzschild metric given in Equation (48) if the following condition is assumed

$$e^{-2\lambda} = 1 + \frac{C}{r} \tag{62}$$

where λ can be shown to be time-independent from the condition $R_{01} = (2/rc)\, d\lambda/dt = 0$. For the case of a gravitational field, the constant of integration C can be identified as $C = -2MG/c^2$. Therefore, if λ is time-independent then the mass M of a gravitational source must be constant. This is the content of Birkhoff's theorem which states that any spherically symmetric vacuum solution of the field equations of general relativity is necessary static. It can be observed that even though Birkhoff's theorem is a perfect mathematical theorem, it cannot practically applied to physical reality because there is no physical object which has a constant mass can be a physical star. In fact, most of the stars that we are observing at the moment had already turned into other forms of energy many billion years ago. Therefore, the Birkhoff's mathematical stars can shine brightly in a mathematical universe but it must completely stay dim in the physical universe that we are living in. And definitely the Birkhoff's theorem cannot be applied to spacetime structures of quantum particles even though for convenience a spherically symmetric spacetime line element may be assumed. However, it may be speculated that due to the conversion between space and time as well as the conversion between the spatial mass and the temporal mass, the Birkhoff's theorem could be applied instead to the total conservation of spatial-temporal mass of a physical system that is defined entirely in terms of geometrical objects. This total symmetry of spacetime needs a sophisticated mathematical formulation that requires further investigation. In the mean time, as has been shown in our previous works that we can derive equations that can be used to construct line elements to describe the spacetime structures of quantum particles for given Ricci scalar curvatures [10,12]. For example, if we assume a quantum particle to have a Ricci scalar given by the equation of the form

$$R = \frac{M}{\left(\sqrt{4\pi kt}\right)^3} e^{-\frac{x^2+y^2+z^2}{4kt}} \tag{63}$$

then, as shown in the appendix 2, by seeking a line element of the form

$$ds^2 = D(cdt)^2 + A(x,y,z,t)[(dx)^2 + (dy)^2 + (dx)^2] \tag{64}$$

where $D > 0$ is constant, the quantity $A(x,y,z,t) > 0$ satisfies the following differential equation

$$-\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2} - \frac{3}{4c^2DA^2}\left(\frac{\partial A}{\partial t}\right)^2 - \frac{2}{A^2}\nabla^2 A + \frac{3}{2A^3}(\boldsymbol{\nabla} A)^2 = \frac{M}{\left(\sqrt{4\pi kt}\right)^3} e^{-\frac{x^2+y^2+z^2}{4kt}} \tag{65}$$

Similarly, the spacetime structure of a quantum particle can be described by the equation

$$-\frac{3}{c^2DA}\frac{\partial^2A}{\partial t^2}-\frac{3}{4c^2DA^2}\left(\frac{\partial A}{\partial t}\right)^2-\frac{2}{A^2}\nabla^2A+\frac{3}{2A^3}(\nabla A)^2$$
$$=k\left(m\sum_{\mu=1}^{3}\left(\frac{dx^\mu}{dt}\right)^2-\hbar\frac{\partial_t\psi+\sum_{\mu=1}^{3}\partial_\mu\psi\left(\frac{dx^\mu}{dt}\right)}{\psi}\right) \quad (66)$$

where ψ is a wavefunction that satisfies the Schrödinger wave equation

$$\nabla^2\psi+\frac{2m}{\hbar^2}(E-V)\psi=0 \quad (67)$$

We can extend the above discussions to the case when we consider not only space but time to be a three-dimensional manifold. The infinitesimal distance ds between two neighbouring points x^μ and $x^\mu+dx^\mu$ is defined by the relation $ds^2=g_{\mu\nu}dx^\mu dx^\nu$, where $x^\mu=(t^1,\ t^2,\ t^3,\ x^1,\ x^2,\ x^3)$. If we consider the case when both spatial and temporal manifolds are centrally symmetric then a general spacetime line element of the six-dimensional spacetime manifold endowed with positive definite metric can be written as

$$ds^2=e^\psi c^2dt^2+c^2t^2\left(d\theta_T{}^2+\sin^2\theta_Td\phi_T{}^2\right)+e^\chi dr^2+r^2\left(d\theta_S{}^2+\sin^2\theta_Sd\phi_S{}^2\right) \quad (68)$$

where the infinitesimal distance has been chosen to have a spatial dimension. We now consider the case when we can arrange the (θ,ϕ) directions of both the spatial manifold and the temporal manifold so that

$$\theta_S=\theta_T=\theta,\quad \text{and}\quad \phi_S=\phi_T=\phi \quad (69)$$

then the line element given in Equation (68) becomes

$$ds^2=e^\psi c^2dt^2+e^\chi dr^2+(r^2+c^2t^2)(d\theta^2+\sin^2\theta d\phi^2) \quad (70)$$

In the following, we show that there are profound differences in the structure of space-time that arise from the line element given in Equation (70). First, we show that the line element given in Equation (70) can lead to the conventional structure of space-time in which, effectively, space has three dimensions and time has one dimension. The line element in Equation (70) can be re-written in the form

$$ds^2=e^\psi c^2dt^2+e^\chi dr^2+r^2\left(1+\frac{c^2}{v^2}\right)(d\theta^2+\sin^2\theta d\phi^2) \quad (71)$$

where we have defined the new quantity that has the dimension of speed as $v=r/t$. The meaning of the speed v can be interpreted as follows. As discussed in our previous works [13], spacetime can be considered as being composed of spatial-temporal quanta that have a very short lifespan. Each of these quanta of spacetime has its own spacetime structure after having been created, which can be described by the line element given by Equation (70). In order for a quantum of spacetime to disintegrate it simply expands rapidly. Therefore the speed v in the line element given by Equation (71) is the speed at which a quantum of spacetime is expanding. Overall, the structure of spacetime at any given moment is a collection of expanding spatial-temporal quanta. When $v \to \infty$, then the only observable structure of spacetime is that of the form of the positive definite Schwarzschild metric $ds^2 = e^{\psi}c^2dt^2 + e^{\chi}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Instead of the form given in Equation (71), the line element given in Equation (70) can also be re-written in a different form as follows

$$ds^2 = e^{\psi}c^2dt^2 + c^2t^2\left(1 + \frac{v^2}{c^2}\right)(d\theta^2 + \sin^2\theta d\phi^2) + e^{\chi}dr^2 \tag{72}$$

When $v \to 0$, then the line element given in Equation (72) is reduced to the line element for a spacetime manifold in which, effectively, time has three dimensions and space has one dimension, namely, $ds^2 = e^{\psi}c^2dt^2 + c^2t^2(d\theta^2 + \sin^2\theta d\phi^2) + e^{\chi}dr^2$. This line element is that of a quantum cell of spacetime and this gives the reason why a three-dimensional time could not be observed at the macroscopic scale and the microscopic objects that occupy these quantum elements of spacetime can be described as string-like objects. It is aslo seen from Equations (71) and (72) that there is a conversion between space-like quantum cell and time-like quantum cell when $v \to c$.

We would like to give a remark here on the formulation of Robertson-Walker metric to describe the dynamical structure of the observable universe in modern cosmology. The Robertson-Walker metric can be written in the following form

$$ds^2 = -c^2dt^2 + S^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{73}$$

In the cosmological line element given in Equation (73), the time t is a universal time and the factor $S^2(t)$ is an expansion factor. However, since the metric is conformally flat in order for the spatial section of spacetime to be described as a curved space it must be embedded into a four-dimensional Euclidean space R^4. Since a flat space R^4 does not exist in Einstein general relativity, a fictitious flat space R^4 must be introduced so that a three-dimensional hypersurface can be embedded. However, as has been discussed above, within the framework of the Euclidean relativity, a six-dimensional Euclidean space R^6, which is regarded as a natural setting, must exist in order to isometrically embed physical objects, which are considered as three-dimensional Riemannian manifolds, including the observable universe as a whole. For example, if we describe the dynamics of a physical object which is viewed as a three-dimensional spatial manifold with respect to the time t, which can be taken as the temporal arclength of a temporal curve in the three-dimensional temporal manifold, then the Robertson-Walker metric can be modified to take the following form

$$ds^2 = c^2dt^2 + S^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{74}$$

Similar to the case when the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ are introduced to describe a circle in the three-dimensional Euclidean space R^3, the three-dimensional spatial section can be described by introducing the spherical coordinates [7,11]

$$x^1 = a\sin\chi\sin\theta\cos\phi \tag{75}$$

$$x^2 = a\sin\chi\sin\theta\sin\phi \tag{76}$$

$$x^3 = a\sin\chi\cos\theta \tag{77}$$

$$x^4 = a\cos\chi \tag{78}$$

With the spherical coordinates given in Equations (75-78), the line element given in Equation (74) can be expressed in the form

$$ds^2 = c^2dt^2 + R^2(t)\left(\frac{1}{1-Kr^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) \tag{79}$$

where the Gaussian curvature K can take values $K = 0, 1/a^2, -1/a^2$.

Appendix 1

With the line element given in Equation (47), the tensor metric $g_{\alpha\beta}$ and its inverse are given as

$$g_{\alpha\beta} = \begin{pmatrix} e^{2\nu} & 0 & 0 & 0 \\ 0 & e^{2\lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g^{\alpha\beta} = \begin{pmatrix} e^{-2\nu} & 0 & 0 & 0 \\ 0 & e^{-2\lambda} & 0 & 0 \\ 0 & 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 0 & \dfrac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{1}$$

The non-zero components of the affine connections are

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{d\nu}{dr}$$

$$\Gamma^1_{00} = -e^{2\nu-2\lambda}\frac{d\nu}{dr}, \qquad \Gamma^1_{11} = \frac{d\lambda}{dr}, \qquad \Gamma^1_{22} = -re^{-2\lambda}, \qquad \Gamma^1_{33} = -r\sin^2\theta e^{-2\lambda}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \tag{2}$$

The non-zero components of the Ricci curvature tensor are

$$R_{00} = e^{2\nu-2\lambda}\left(-\frac{d^2\nu}{dr^2} - \left(\frac{d\nu}{dr}\right)^2 + \frac{d\nu}{dr}\frac{d\lambda}{dr} - \frac{2}{r}\frac{d\nu}{dr}\right)$$

$$R_{11} = -\frac{d^2\nu}{dr^2} - \left(\frac{d\nu}{dr}\right)^2 + \frac{d\nu}{dr}\frac{d\lambda}{dr} + \frac{2}{r}\frac{d\nu}{dr}$$

$$R_{22} = \left(-1 + r\frac{d\lambda}{dr} - r\frac{d\nu}{dr}\right)e^{-2\lambda} + 1$$

$$R_{33} = R_{22}\sin^2\theta \tag{3}$$

For the vacuum solution, from $R_{00} = 0$ and $R_{11} = 0$, we obtain the identity

$$\frac{d\lambda}{dr} + \frac{d\nu}{dr} = 0 \tag{4}$$

On integration Equation (5) we have

$$\lambda(r) + \nu(r) = C \tag{5}$$

where C is an undetermined constant. However, with the assumption that the metric given in Equation (1) will approach the Euclidean metric as $r \to \infty$, we have $C = 0$. Therefore we have

$$\lambda(r) = -\nu(r) \tag{6}$$

With the condition given in Equation (7), the component R_{22} can be rewritten as

$$\frac{d(re^{2\nu})}{dr} = 1 \tag{7}$$

From Equation (8) by integration we obtain

$$e^{2\nu} = 1 + \frac{C}{r} \tag{8}$$

where C is a constant of integration.

Appendix 2

With the line element given in Equation (64), we obtain the following non-zero components of the affine connection

$$\Gamma_{01}^{1} = \Gamma_{10}^{1} = \frac{1}{2cA}\frac{\partial A}{\partial t}, \qquad \Gamma_{02}^{2} = \Gamma_{20}^{2} = \frac{1}{2cA}\frac{\partial A}{\partial t}, \qquad \Gamma_{03}^{3} = \Gamma_{30}^{3} = \frac{1}{2cA}\frac{\partial A}{\partial t}$$

$$\Gamma_{11}^{0} = -\frac{1}{2cD}\frac{\partial A}{\partial t}, \qquad \Gamma_{11}^{1} = \frac{1}{2A}\frac{\partial A}{\partial x}, \qquad \Gamma_{11}^{2} = -\frac{1}{2A}\frac{\partial A}{\partial y}, \qquad \Gamma_{11}^{3} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma_{12}^{1} = \Gamma_{21}^{1} = \frac{1}{2A}\frac{\partial A}{\partial y}, \quad \Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{2A}\frac{\partial A}{\partial x}, \quad \Gamma_{13}^{1} = \Gamma_{31}^{1} = \frac{1}{2A}\frac{\partial A}{\partial z}, \quad \Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{2A}\frac{\partial A}{\partial x}$$

$$\Gamma^0_{22} = -\frac{1}{2cD}\frac{\partial A}{\partial t}, \quad \Gamma^1_{22} = \frac{1}{2A}\frac{\partial A}{\partial x}, \quad \Gamma^2_{22} = \frac{1}{2A}\frac{\partial A}{\partial y}, \quad \Gamma^3_{22} = -\frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^0_{33} = -\frac{1}{2cD}\frac{\partial A}{\partial t}, \quad \Gamma^1_{33} = -\frac{1}{2A}\frac{\partial A}{\partial x}, \quad \Gamma^2_{33} = -\frac{1}{2A}\frac{\partial A}{\partial y}, \quad \Gamma^3_{33} = \frac{1}{2A}\frac{\partial A}{\partial z}$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{2A}\frac{\partial A}{\partial z}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2A}\frac{\partial A}{\partial y} \tag{1}$$

From the components of the affine connection given in Equation (5), we obtain

$$R_{11} = -\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} - \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{22} = -\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial z^2} - \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{33} = -\frac{1}{2c^2D}\frac{\partial^2 A}{\partial t^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial x^2} - \frac{1}{2A}\frac{\partial^2 A}{\partial y^2} - \frac{1}{A}\frac{\partial^2 A}{\partial z^2} - \frac{3}{4c^2AD}\left(\frac{\partial A}{\partial t}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial x}\right)^2 + \frac{1}{4A^2}\left(\frac{\partial A}{\partial y}\right)^2 + \frac{1}{A^2}\left(\frac{\partial A}{\partial z}\right)^2$$

$$R_{00} = -\frac{3}{2c^2A}\frac{\partial^2 A}{\partial t^2} + \frac{3}{4c^2A^2}\left(\frac{\partial A}{\partial t}\right)^2 \tag{2}$$

Using the relation $R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$ we obtain

$$R = -\frac{3}{c^2DA}\frac{\partial^2 A}{\partial t^2} - \frac{3}{4c^2DA^2}\left(\frac{\partial A}{\partial t}\right)^2 - \frac{2}{A^2}\nabla^2 A + \frac{3}{2A^3}(\boldsymbol{\nabla} A)^2 \tag{3}$$

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