FUNDAMENTAL SET THEORY

ANDERS LINDMAN

Abstract. The fundamental set theory (FST) is defined as an axiomatic set theory using nonclassical three-valued logic in the foundation and classical two-valued logic in its applications. In this way the nonclassical logic becomes encapsulated and is only used for resolving inconsistencies such as Russell's paradox.

1. Introduction

Axiomatic set theories, including Zermelo-Fraenkel set theory with the axiom of choice (ZFC), have been introduced in order to resolve problems with unrestricted comprehension, e.g. Russell's paradox. This paper defines an axiomatic set theory that eliminates Russell's paradox while including the universal set that is missing in ZFC.

In order to start with a foundation, the fundamental properties of sets are defined:

Definition 1.1. Object An *object* represents any thing that is either a collection that can contain other objects or an atomic object which cannot contain other objects.

Definition 1.2. Element An *element* is an atomic object.

Definition 1.3. Element operator The operator *element(o)* has the value true iff o is an element.

Definition 1.4. Set A *set* is a collection that can contain unordered unique objects.

Definition 1.5. Set operator The operator *set(o)* has the value true iff o is a set.

2. Basic definitions for the Fundamental Set Theory

First the operations equality and member of are defined together with their logical complements. In order to avoid an endless regression of definitions these operators are defined by text descriptions.

Definition 2.1. Equality operator $\forall x \forall y((x = y \leftrightarrow y = x) \leftrightarrow$ x and y are the same object)

Definition 2.2. Not equal operator $\forall x(x \neq y \leftrightarrow \neg(x = y))$

Definition 2.3. Member of operator $\forall x(x \in S \leftrightarrow$ S contains x)

Definition 2.4. Not member of operator $\forall x(x \notin S \leftrightarrow \neg(x \in S))$

2010 *Mathematics Subject Classification.* Primary 03E70.
Key words and phrases. Set theory, logic, foundations.

Then the operations union and intersection are defined:

Definition 2.5. Union operator $\forall x \forall y(z = x \cup y \leftrightarrow \forall a(a \in x \vee a \in y \leftrightarrow a \in z))$

Definition 2.6. Intersection operator $\forall x \forall y(z = x \cap y \leftrightarrow \forall a(a \in x \wedge a \in y \leftrightarrow a \in z))$

The subset operator is defined as any set B being a subset of A iff it contains only elements in A or no elements:

Definition 2.7. Subset operator $\forall A \forall B(A \subseteq B \leftrightarrow \forall x \in B(x \in A))$

A proper subset B of A is a subset of A different from A:

Definition 2.8. Proper subset operator $\forall A \forall B(A \subset B \leftrightarrow A \subseteq B \wedge A \neq B)$

The power set operator P is defined as P(S) being the set of all possible subsets of S:

Definition 2.9. Power set operator $\forall S(A = P(S) \leftrightarrow \forall x \in A(x \subseteq S) \wedge \forall x(x \subseteq S \rightarrow x \in A))$

Definition 2.10. Cardinality of set The cardinality of a set S, denoted by $\mid S \mid$, is the number of elements in S, countably infinite or uncountably infinite.

Definition 2.11. Finite set A finite set is a set that can have at most only one subset with the same cardinality as the set itself.

Definition 2.12. Infinite set An infinite set is a set that can have more than one subsets with the same cardinality as the set itself.

The empty set is a set without members:

Definition 2.13. Empty set $\exists \emptyset(set(\emptyset) \wedge \forall x(x \notin \emptyset))$

The universal set is a set that contains all objects including itself:

Definition 2.14. Universal set $\exists U \forall x(x \in U)$

3. Nonclassical logic foundation

In order to eliminate the problem with Russell's paradox in unrestricted comprehension, nonclassical three-valued logic is used in FST, here called set logic which is only used in an intermediate stage. FST encapsulates the nonclassical logic and hides it so that when using FST as a foundation, only classical two-valued logic is needed.

Set logic has three values *true*, *false* and *null* and is the same as classical two-valued logic with an equality operator == added.

Definition 3.1. Three-valued equality operator The operator == converts two three-valued values to a two-valued value as follows:
False == False = True
False == True = False
False == Null = False
True == True = True
True == Null = False
Null == Null = True
$\forall x \forall y(x == y \leftrightarrow y == x)$

Definition 3.2. Predicate P(x) P(x) ≡ Any classical two-valued predicate for a set S = {x | P(x)}.

Definition 3.3. Predicate Q(x) Q(x) ≡ A three-valued predicate: $\exists S \forall x(x \in S \leftrightarrow P(x))$ with the value *null* when there is a contradiction.

The predicate Q(x) is the same as unrestricted comprehension except that it is a three-valued predicate with the possible values *true*, *false* or *null*.

In order to hide the three-valued logic, another two-valued predicate P'(x) is defined as:

Definition 3.4. Predicate P'(x) P'(x) ≡ Q(x) == null

Predicate P'(x) can according to definition 3.1 only have the value *true* or *false* and is in FST used in the following two-valued classical logic axiom schema:

Axiom of Consistency $\neg\exists x(P'(x) \rightarrow \exists S \forall x(x \in S \leftrightarrow P(x)))$

The axiom of consistency says that if there is no x satisfying P' then there exists a set S such that all members of S are precisely those sets that satisfy P.

For example in a case of Russell's paradox, Q(x) is *null* for some x when P(x) = x ∉ x which means that there is an x satisfying P' and therefore there is no set $\{x \mid x \notin x\}$ in FST according to the axiom of consistency.

Theorem 3.5. *All sets constructed with the fundamental set theory (FST) are consistent.*

Proof. All inconsistent set predicates P(x) result in Q(x) having the value *null* and P'(x) having the value *true* which results in the axiom of consistency excluding the inconsistent set predicates P(x). □