Elementary Formulas for Pi

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abstract

This note presents some formulas for $\pi = 3.141592...$

1. Introduction

The sequence t_n , $n = 0,1,2,3,...$:

$$t_{n+2} = \frac{2p}{q} t_{n+1} - t_n \quad , n = 0,1,2,3,... \tag{1}$$

$$p, q \in \mathbb{N} = \{1,2,3,...\}, \, p < q \, , t_0 = 1, t_1 = \frac{p}{q} \tag{2}$$

Explicit formula:

$$t_n = \frac{1}{2}\left\{ \left(\frac{p + i\sqrt{q^2 - p^2}}{q} \right)^n + \left(\frac{p - i\sqrt{q^2 - p^2}}{q} \right)^n \right\} \quad , n \in \mathbb{N} \cup \{0\}, i = \sqrt{-1} \tag{3}$$

This note presents some formulas for π .

2. Pi Formulas

Let

- $p, q \in \mathbb{N}, \, p < q$
- $N \in \mathbb{N} - \{1\} = \{2,3,4,5,...\}$
- $t_N := t_N(p,q)$

- $m = floor\left(\frac{N\cos^{-1}(p/q)}{\pi}\right) = \left[\frac{N\cos^{-1}(p/q)}{\pi}\right]$
- $k = floor\left(\frac{m}{2}\right) = \left[\frac{m}{2}\right]$

Then:

Case 1: $m = 2k \wedge t_N > 0$

- If $N - 4k - 1 = 0$ then:

$$N\sin^{-1}\left(\frac{p}{q}\right) = \sin^{-1}(t_N) \tag{4}$$

$$\pi = 2N\sin^{-1}\left(\frac{p}{q}\right) + 2\sin^{-1}\sqrt{1-t_N^2} \tag{5}$$

- If $N - 4k - 1 \neq 0$ then:

$$\pi = \frac{2}{N-4k-1}\left(N\sin^{-1}\left(\frac{p}{q}\right) - \sin^{-1}(t_N)\right) \tag{6}$$

Case 2: $m = 2k \wedge t_N < 0$

$$\pi = \frac{2}{N-4k-1}\left(N\sin^{-1}\left(\frac{p}{q}\right) + \sin^{-1}(-t_N)\right) \tag{7}$$

Case 3: $m = 2k + 1 \wedge t_N > 0$

$$\pi = \frac{2}{N+1-4(k+1)}\left(N\sin^{-1}\left(\frac{p}{q}\right) + \sin^{-1}(t_N)\right) \tag{8}$$

Case 4: $m = 2k + 1 \wedge t_N < 0$

- If $N + 1 - 4(k+1) = 0$ then:

$$N\sin^{-1}\left(\frac{p}{q}\right) = \sin^{-1}(-t_N) \tag{9}$$

$$\pi = 2N\sin^{-1}\left(\frac{p}{q}\right) + 2\sin^{-1}\sqrt{1-t_N^2} \tag{10}$$

- If $N + 1 - 4(k+1) \neq 0$ then:

$$\pi = \frac{2}{N+1-4(k+1)}\left(N\sin^{-1}\left(\frac{p}{q}\right) - \sin^{-1}(-t_N) \right) \tag{11}$$

3. Examples

Example 1: $p = 1, q = 3$,

$$t_n = \left\{1, \frac{1}{3}, -\frac{7}{9}, -\frac{23}{27}, \frac{17}{81}, \frac{241}{243}, \frac{329}{729}, \ldots\right\} \tag{12}$$

- $N = 2, t_2 = -7/9, m = 0, k = 0$:

$$\pi = 4\sin^{-1}\left(\frac{1}{3}\right) + 2\sin^{-1}\left(\frac{7}{9}\right) \tag{13}$$

- $N = 3, t_3 = -23/27, m = 1, k = 0$:

$$\pi = 6\sin^{-1}\left(\frac{1}{3}\right) + 2\sin^{-1}\left(\frac{10\sqrt{2}}{27}\right) \tag{14}$$

- $N = 4, t_4 = 17/81, m = 1, k = 0$:

$$\pi = 8\sin^{-1}\left(\frac{1}{3}\right) + 2\sin^{-1}\left(\frac{17}{81}\right) \tag{15}$$

- $N = 5, t_5 = 241/243, m = 1, k = 0$:

$$\pi = 5\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{241}{243}\right) \tag{16}$$

$$\pi = 10\sin^{-1}\left(\frac{1}{3}\right) - 2\sin^{-1}\left(\frac{22\sqrt{2}}{243}\right) \tag{17}$$

- $N = 6, t_6 = 324/729, m = 2, k = 1$:

$$\pi = 12\sin^{-1}\left(\frac{1}{3}\right) - 2\sin^{-1}\left(\frac{329}{729}\right) \tag{18}$$

Example 2: $p = 1, q = 4$,

$$t_n = \left\{1, \frac{1}{4}, -\frac{7}{8}, -\frac{11}{16}, \frac{17}{32}, \frac{61}{64}, -\frac{7}{128}, -\frac{251}{256}, \ldots\right\} \tag{19}$$

- $N = 2, t_2 = -7/8, m = 0, k = 0$:

$$\pi = 4\sin^{-1}\left(\frac{1}{4}\right) + 2\sin^{-1}\left(\frac{7}{8}\right) \tag{20}$$

- $N = 3, t_3 = -11/16, m = 1, k = 0$:

$$\pi = 6\sin^{-1}\left(\frac{1}{4}\right) + 2\sin^{-1}\left(\frac{3\sqrt{15}}{16}\right) \tag{21}$$

- $N = 4, t_4 = 17/32, m = 1, k = 0$:

$$\pi = 8\sin^{-1}\left(\frac{1}{4}\right) + 2\sin^{-1}\left(\frac{17}{32}\right) \tag{22}$$

- $N = 5, t_5 = 61/64, m = 2, k = 1$:

$$\pi = 10\sin^{-1}\left(\frac{1}{4}\right) + 2\sin^{-1}\left(\frac{5\sqrt{15}}{64}\right) \tag{23}$$

- $N = 6, t_6 = -7/128, m = 2, k = 1$:

$$\pi = 12\sin^{-1}\left(\frac{1}{4}\right) + 2\sin^{-1}\left(\frac{7}{128}\right) \tag{24}$$

Example 3: $p = 1, q = 5$,

$$\pi = 4\sin^{-1}\left(\frac{1}{5}\right) + 2\sin^{-1}\left(\frac{23}{25}\right) \tag{25}$$

$$\pi = 5\sin^{-1}\left(\frac{1}{5}\right) + 2\sin^{-1}\left(\frac{42\sqrt{6}}{125}\right) \tag{26}$$

$$\pi = 16\sin^{-1}\left(\frac{1}{5}\right) - 2\sin^{-1}\left(\frac{15647}{5^8}\right) \tag{27}$$

Example 4: $p = 1, q = 6$,

$$\pi = 16\sin^{-1}\left(\frac{1}{6}\right) + 2\sin^{-1}\left(\frac{3007}{2\cdot 3^8}\right) \tag{28}$$

$$\pi = 20\sin^{-1}\left(\frac{1}{6}\right) - 2\sin^{-1}\left(\frac{12223}{2\cdot 3^{10}}\right) \tag{29}$$

Example 5: $p = 1, q = 7$,

$$\pi = 20\sin^{-1}\left(\frac{1}{7}\right) + 2\sin^{-1}\left(\frac{38667887}{7^{10}}\right) \tag{30}$$

Example 6: $p=1, q=8$

$$\pi = 24\sin^{-1}\left(\frac{1}{8}\right) + 2\sin^{-1}\left(\frac{2241857}{2^{25}}\right) \tag{31}$$

Example 7: $p=1, q=10$

$$\pi = 24\sin^{-1}\left(\frac{1}{10}\right) + 2\sin^{-1}\left(\frac{88008913}{5^{12}}\right) \tag{32}$$

$$\pi = 28\sin^{-1}\left(\frac{1}{10}\right) + 2\sin^{-1}\left(\frac{2046592849}{2\cdot 5^{14}}\right) \tag{33}$$

$$\pi = 32\sin^{-1}\left(\frac{1}{10}\right) - 2\sin^{-1}\left(\frac{9728091649}{2\cdot 5^{16}}\right) \tag{34}$$

Example 8: $p=1, q=11$

$$\pi = 32\sin^{-1}\left(\frac{1}{11}\right) + 2\sin^{-1}\left(\frac{5237881125969601}{11^{16}}\right) \tag{35}$$

Example 9: $p=1, q=12$

$$\pi = 40\sin^{-1}\left(\frac{1}{12}\right) - 2\sin^{-1}\left(\frac{714044432603951}{2^{21}\cdot 3^{20}}\right) \tag{36}$$

Example 10: $p=1, q=20$

$$\pi = 60\sin^{-1}\left(\frac{1}{20}\right) + 2\sin^{-1}\left(\frac{1402261021713093908382459529999}{2^{31}\cdot 5^{30}}\right) \tag{37}$$

Example 11: $p=2, q=5$

$$\pi = 8\sin^{-1}\left(\frac{2}{5}\right) - 2\sin^{-1}\left(\frac{47}{5^{4}}\right) \tag{38}$$

$$\pi = 7\sin^{-1}\left(\frac{2}{5}\right) + \sin^{-1}\left(\frac{20158}{5^{7}}\right) \tag{39}$$

$$\pi = 8\sin^{-1}\left(\frac{2}{5}\right) - \frac{2}{3}\sin^{-1}\left(\frac{54662833}{5^{12}}\right) \tag{40}$$

$$\pi = \frac{15}{2}\sin^{-1}\left(\frac{2}{5}\right) + \frac{1}{2}\sin^{-1}\left(\frac{3363290162}{5^{15}}\right) \tag{41}$$

$$\pi = \frac{23}{3}\sin^{-1}\left(\frac{2}{5}\right) - \frac{1}{3}\sin^{-1}\left(\frac{478014292027682}{5^{23}}\right) \tag{42}$$

Example 12: $p=3, q=7$

$$\pi = 7\sin^{-1}\left(\frac{3}{7}\right) + \sin^{-1}\left(\frac{33933}{7^7}\right) \tag{43}$$

$$\pi = \frac{36}{5}\sin^{-1}\left(\frac{3}{7}\right) - \frac{2}{5}\sin^{-1}\left(\frac{192381759695969}{7^{18}}\right) \tag{44}$$

Example 13: $p = 3, q = 8$

$$\pi = 8\sin^{-1}\left(\frac{3}{8}\right) + 2\sin^{-1}\left(\frac{17}{2^9}\right) \tag{45}$$

$$\pi = 8\sin^{-1}\left(\frac{3}{8}\right) + \frac{2}{3}\sin^{-1}\left(\frac{3337423}{2^{25}}\right) \tag{46}$$

Example 14: $p = 3, q = 10$

$$\pi = 12\sin^{-1}\left(\frac{3}{10}\right) - 2\sin^{-1}\left(\frac{3977}{5^6}\right) \tag{47}$$

$$\pi = 11\sin^{-1}\left(\frac{3}{10}\right) - \sin^{-1}\left(\frac{20359947}{2\cdot 5^{11}}\right) \tag{48}$$

Example 15: $p = 4, q = 11$

$$\pi = 8\sin^{-1}\left(\frac{4}{11}\right) + 2\sin^{-1}\left(\frac{1201}{11^4}\right) \tag{49}$$

Example 16: $p = 7, q = 10$

$$\pi = 4\sin^{-1}\left(\frac{7}{10}\right) + 2\sin^{-1}\left(\frac{1}{50}\right) \tag{50}$$

$$\pi = 4\sin^{-1}\left(\frac{7}{10}\right) + \frac{2}{3}\sin^{-1}\left(\frac{937}{5^6}\right) \tag{51}$$

Example 17: $p = 353, q = 500$

$$\pi = 4\sin^{-1}\left(\frac{353}{500}\right) + 2\sin^{-1}\left(\frac{391}{2^3\cdot 5^6}\right) \tag{52}$$

Example 18: $p = 12, q = 17$

$$\pi = 4\sin^{-1}\left(\frac{12}{17}\right) + 2\sin^{-1}\left(\frac{1}{17^2}\right) \tag{53}$$

Example 19: $p = 29, q = 41$

$$\pi = 4\sin^{-1}\left(\frac{29}{41}\right) - 2\sin^{-1}\left(\frac{1}{41^2}\right) \tag{54}$$

Example 20: $p=70, q=99$

$$\pi = 4\sin^{-1}\left(\frac{70}{99}\right) + 2\sin^{-1}\left(\frac{1}{3^4 \cdot 11^2}\right) \tag{55}$$

Example 21: $p=169, q=239$

$$\pi = 4\sin^{-1}\left(\frac{169}{239}\right) - 2\sin^{-1}\left(\frac{1}{239^2}\right) \tag{56}$$

Example 22: $p=5, q=13$

$$\pi = 8\sin^{-1}\left(\frac{5}{13}\right) - 2\sin^{-1}\left(\frac{239}{13^4}\right) \tag{57}$$

Example 23: $p=13, q=34$

$$\pi = 8\sin^{-1}\left(\frac{13}{34}\right) + 2\sin^{-1}\left(\frac{239}{2 \cdot 17^4}\right) \tag{58}$$

Example 24: $p=31, q=81$

$$\pi = 8\sin^{-1}\left(\frac{31}{81}\right) - 2\sin^{-1}\left(\frac{6079}{3^{16}}\right) \tag{59}$$

Example 25: $p=7, q=27$

$$\pi = 12\sin^{-1}\left(\frac{7}{27}\right) - 2\sin^{-1}\left(\frac{1059449}{3^{18}}\right) \tag{60}$$

Example 26: $p=15, q=58$

$$\pi = 12\sin^{-1}\left(\frac{15}{58}\right) + 2\sin^{-1}\left(\frac{732871}{29^6}\right) \tag{61}$$

Example 27: $p=22, q=85$

$$\pi = 12\sin^{-1}\left(\frac{22}{85}\right) - 2\sin^{-1}\left(\frac{10505503}{5^6 \cdot 17^6}\right) \tag{62}$$

Example 28: $p=7, q=36$

$$\pi = 16\sin^{-1}\left(\frac{7}{36}\right) + 2\sin^{-1}\left(\frac{116103649}{2^9 \cdot 3^{16}}\right) \tag{63}$$

Example 29: $p=8, q=41$

$$\pi = 16\sin^{-1}\left(\frac{8}{41}\right) - 2\sin^{-1}\left(\frac{2060044223}{41^8}\right) \tag{64}$$

Example 30: $p=5, q=32$

$$\pi = 20\sin^{-1}\left(\frac{5}{32}\right) + 2\sin^{-1}\left(\frac{4106929927}{2^{41}}\right) \tag{65}$$

Example 31: $p=105, q=307$

$$\pi = 9\sin^{-1}\left(\frac{105}{307}\right) + \sin^{-1}\left(\frac{139054095093375945}{307^{9}}\right) \tag{66}$$

Example 32: $p=1, q=\sqrt{10}$

$$\pi = 9\sin^{-1}\left(\frac{1}{\sqrt{10}}\right) + \sin^{-1}\left(\frac{481}{625\sqrt{10}}\right) \tag{67}$$

4. Series Representations

Inverse sine function:

$$\sin^{-1} x = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} x^{2n+1} = x F\left(\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{3}{2}\right\}, x^2\right) \quad , |x| \le 1 \tag{68}$$

Remark: $F\left(\{a,b\},\{c\},x\right)$ is the hypergeometric function.

Using (68) we obtain:

Formula (14):

$$\pi = 4\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2n+1} \left(\frac{1}{6}\right)^{2n+1} \left(4 + \left(\frac{17}{27}\right)^{2n+1}\right) \tag{69}$$

Formula (23):

$$\pi = 4\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2n+1} \left(\frac{1}{8}\right)^{2n+1} \left(6 + \left(\frac{7}{32}\right)^{2n+1}\right) \tag{70}$$

Formula (26):

$$\pi = 4\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{2n+1} \left(\frac{1}{10}\right)^{2n+1} \left(8 - \left(\frac{15647}{5^{7}}\right)^{2n+1}\right) \tag{71}$$

Formula (28):

$$\pi = 4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{1}{12}\right)^{2n+1}\left(10-\left(\frac{12223}{3^9}\right)^{2n+1}\right) \tag{72}$$

Formula (31):

$$\pi = 4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{1}{16}\right)^{2n+1}\left(12+\left(\frac{2241857}{2^{22}}\right)^{2n+1}\right) \tag{73}$$

Formula (38):

$$\pi = 4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{1}{5}\right)^{2n+1}\left(4-\left(\frac{47}{250}\right)^{2n+1}\right) \tag{74}$$

Formula (45):

$$\pi = 4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{3}{16}\right)^{2n+1}\left(4+\left(\frac{17}{192}\right)^{2n+1}\right) \tag{75}$$

Formula (65):

$$\pi = 4\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{5}{64}\right)^{2n+1}\left(10+\left(\frac{41069229927}{2^{36}\cdot 5}\right)^{2n+1}\right) \tag{76}$$

Formula (67):

$$\pi = \frac{1}{\sqrt{10}}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{2n+1}\left(\frac{1}{40}\right)^{n}\left(1+\left(\frac{481}{625}\right)^{2n+1}\right) \tag{77}$$

References

1. Abramowitz, M. and Stegun, I.A.: Handbook of Mathematical Functions. New York: Dover , 1965.
2. Gradshteyn, I.S. and Ryzhik, I.M.: Table of Integrals, Series, and Products. Seventh Edition. Edited by Alan Jeffrey and Daniel Zwillinger, Academic Press, 2007.