A Conjecture On The Nature Of Time

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Abstract

In our previous publications we argue that finite mathematics is fundamental, classical mathematics (involving such notions as infinitely small/large, continuity etc.) is a degenerate special case of finite one, and ultimate quantum theory will be based on finite mathematics. We consider a finite quantum theory (FQT) based on a finite field or ring with a large characteristic p and show that standard continuous quantum theory is a special case of FQT in the formal limit $p \to \infty$. Space and time are purely classical notions and are not present in FQT at all. In the present paper we discuss how classical equations of motions arise as a consequence of the fact that p changes, i.e. p is the evolution parameter.

Keywords: finite mathematics, quantum theory, equations of motion

1 The problem of space-time in quantum theory

Although quantum theory exists for more than 90 years, the problem of its foundation is still widely debated. Although it is now obvious that physical intuition based on classical physics usually does not work for explaining quantum phenomena, quantum theory inherited several important notions from classical one.

For example, a rather strange feature of fundamental quantum theories (QED, QCD and electroweak theory) is that their derivation is based on local space-time Lagrangians but the final formulation involves only the S-matrix in momentum representation and space-time is not present in this formulation at all. This is in the spirit of the Heisenberg S-matrix program where description of quantum states at each moment of time t is treated as unphysical and only the description of evolution from the infinite past when $t \to -\infty$ to the distant future when $t \to +\infty$ has a physical meaning.

In many cases fundamental quantum theories give impressive agreements with experiment but nevertheless the problem of substantiation of those theories remains open. The main inconsistency of the theories is that they contain divergent expressions for the S-matrix elements. The main reason is that the Lagrangian densities contain products of local operator fields at the same space-time points. As explained even in textbooks on local quantum field theories (see e.g. Ref. [1]), interacting local quantum fields can be treated only as operator distributions, and a

known fact from the theory of distributions is that their product at the same points is not a correct mathematical operation.

A usual justification of the presence of such products is that they are needed to preserve locality. However, this argument is not consistent for the following reason. Although the construction of the local quantized field $\psi(x)$ (where x is a point in Minkowski space) is based on a single-particle field, the quantized field is an operator in the Fock space for a system with an infinite number of particles, and the argument x does not refer to any particle. It is only an integration parameter for the full Lagrangian. Probably the "strongest" justification is that the philosophy of the absolute majority of physicists is that agreement with experiment is much more important than mathematical rigor.

Let us note that even in classical mechanics particle coordinates and time can be treated in different ways. A standard treatment of this theory is that its goal is to solve equations of motion and get classical trajectories where coordinates and momenta are functions of time t. In Hamiltonian mechanics the action can be written as $S = S_0 - \int H dt$ where H is the Hamiltonian, S_0 does not depend on t and is called the abbreviated action. Suppose now that one wishes to consider a problem which is usually treated as less general: to find not the dependence of the coordinates and momenta on t but only possible forms of trajectories in the phase space without mentioning time at all. If the energy is a conserved physical quantity then, as described in textbooks, this problem can be solved by using the Maupertuis principle involving only S_0.

However, the latter problem *is not* less general than the former one. For illustration we first consider the one-body case. Suppose that by using the Maupertuis principle one has solved the problem with some initial values of coordinates and momenta. Let s be a parameter characterizing the particle trajectory, i.e. the particle radius-vector $\mathbf{r}$, the momentum $\mathbf{p}$ and the energy E are functions of s. The particle velocity $\mathbf{v}$ in units $c = 1$ *is defined* as $\mathbf{v}(s) = \mathbf{p}(s)/E(s)$. At this stage the problem does not contain t yet. One can *define* t by the condition that $dt = |d\mathbf{r}|/|\mathbf{v}|$ and hence the value of t at any point of the trajectory can be obtained by integration. Hence the general problem of classical mechanics can be initially formulated without mentioning t while if for some reasons one prefers to work with t then its value can flow only in the positive direction since $dt > 0$.

Another point of view is that, at least on classical level, time is a primary quantity while the coordinates $\mathbf{r}$ of each free particle should be *defined* in terms of momentum and time as

$$d\mathbf{r} = \mathbf{v}dt = \frac{\mathbf{p}}{E}dt \tag{1}$$

where $E = (m^2 + \mathbf{p}^2)^{1/2}$ and m is the particle mass. Such a definition of coordinates is similar to that in General Relativity (GR) where distances are defined in terms of time needed for light to travel from one point to another.

On quantum level the treatment of particle coordinates and time becomes much more complicated. The postulate of quantum theory is that for any physical quantity there should exist a corresponding selfadjoined operator. As noted by Pauli

(see p. 63 of Ref. [2]), at early stages of quantum theory some authors treated time t as an operator commuting with H as $[H, t] = i\hbar$, i.e. H and t are canonically conjugated. However, there are several reasons why such a treatment is not correct. For example (see e.g. Ref. [3]), the conjugated operators should necessarily have the same spectrum, time has the continuous spectrum in the range $(-\infty, +\infty)$ while the Hamiltonian is usually bounded below and a part of its spectrum may be discrete.

It is usually assumed that in quantum theory the quantity t can be only a classical parameter describing evolution of a quantum system by the time dependent Schrödinger equation. The usual justification of this equation is that in the formal limit $\hbar \to 0$ it becomes the Hamilton-Jacobi equation. Moreover, the justification of standard choice for different operators (e.g. coordinate, momentum, angular momentum operators and others) is that such a choice has a correct classical limit. However, the correct classical limit does not guarantee the correct behavior on quantum level. For example, if A and B are two operators such that B becomes zero in classical limit then the operators A and $A + B$ have the same classical limit but on quantum level they may have considerably different properties.

A problem arises why the principle of quantum theory that every physical quantity is defined by an operator does not apply to time. In the literature the problem of time is also often formulated such that "the time of GR and of ordinary Quantum Theory are mutually incompatible notions" (see e.g. Ref. [4]). As noted by several authors, (see e.g. Refs. [5, 6, 7]), t cannot be treated as a fundamental physical quantity. The reason is that all fundamental physical laws do not require time and the quantity t is obsolete on fundamental level.

In quantum theory a problem arises "how to forget time" (by analogy with the Maupertuis principle), construct a theory (in particular quantum gravity) which does not involve time at all and in what approximations classical time can be reconstructed. This is a very complicated problem which has been discussed in detail in Refs. [5, 6].

One can also consider a situation when a quantum system under consideration is a small subsystem of a big system where the big subsystem - the environment, is strongly classical. Then one can define t for the environment as described above. The author of Ref. [7] considers a scenario when the system as a whole is described by the stationary Schrödinger equation $H\Psi = E\Psi$ but the small quantum subsystem is described by the time dependent Schrödinger equation where t is defined for the environment as $t = \partial S_0/\partial E$. In this scenario it is clear why a quantum system is described by the Schrödinger equation depending on the classical parameter t which is not an operator: because t is the physical quantity characterizing not the quantum system but the environment. This scenario seems also natural because it is in the spirit of the Copenhagen interpretation of quantum theory: the evolution of a quantum system can be characterized only in terms of measurements which in the Copenhagen interpretation are treated as interactions with classical objects. However, this scenario encounters several problems. For example, the environment can be a classical object only in some approximation and, as noted in Ref. [7], the above scenario does not solve the problem of quantum jumps.

The authors of Ref. [3] state that the Pauli objection can be circumvented if one uses an external system to track time, so that "time arises as correlations between the system and the clock". In this case, the time operator can be defined. It is not conjugate to the system Hamiltonian, but its eigenvalues still satisfy the Schrödinger equation for arbitrary Hamiltonians. Such an approach is to some extent in the spirit of Ref. [7]. The authors of Ref. [3] refer to the extensive literature where the time operator has been discussed. In any case, the problem to deal or not with the time operator depends on the physical situation and there is no universal choice of the time operator which follows from first principles of quantum theory.

In contrast to time, it is usually believed that in quantum theory the position operator has a clear physical meaning. For example, in nonrelativistic quantum mechanics the position and momentum operators are related to each other by the Fourier transform. As a consequence, we have the famous Heisenberg uncertainty relations or vice versa, from these relations it follows that the operators are related to each other by the Fourier transform. Many authors (including Heisenberg, Dirac and others) gave different arguments in favor of such relations. A great success of the early quantum theory was that the nonrelativistic Schrödinger equation gives a good description of the hydrogen energy levels and the Dirac equation gives a good description of the fine structure of those levels in the approximation $(v/c)^2$.

However, from the point of view of the present knowledge, the Schrödinger and Dirac equations should be treated as follows. As follows from Feynman diagrams for the one-photon exchange, in the approximation up to $(v/c)^2$ the electron in the hydrogen atom can be described in the potential formalism where the potential acts on the wave function (WF) in momentum space. So for calculating energy levels one should solve the eigenvalue problem for the Hamiltonian with this potential. This is an integral equation which can be solved by different methods. One of the convenient methods is to apply the Fourier transform and get standard Schrödinger or Dirac equation in coordinate representation with the Coulomb potential. Hence the fact that the results for energy levels are in good agreement with experiment shows that QED defines the potential correctly and *standard coordinate Schrödinger and Dirac equations are only convenient mathematical ways of solving the eigenvalue problem in the approximation up to* $(v/c)^2$. For this problem the physical meaning of the position operator is not important at all. One can consider other transformations of the original integral equation and define other position operators. The fact that for non-standard choices one might obtain something different from the Coulomb potential is not important on quantum level.

The Schrödinger and Dirac equations work with a high accuracy because the fine structure constant α is small and, as a consequence, the effects beyond the single-particle approximation (e.g. the Lamb shift) are small. However, consider a hypothetical situation where a Universe is such that the value of α is of the order of unity or greater. Although it is not known (even if α is small) whether the perturbation series of QED converges or not, the logical structure of QED remains the same. At the same time, the single-particle approximation is not valid anymore and the Schrödinger and Dirac equations do not define the hydrogen energy levels even

approximately. In other words, in this situation the application of those equations for calculating the hydrogen energy level does not have a physical meaning.

The fact that in our world the Schrödinger and Dirac equations describe the hydrogen energy level with a high accuracy, is usually treated as a strong argument that the coordinate and momentum representations should be related to each other by the Fourier transform. However, as follows from the above considerations, this fact takes place only because we are lucky that the value of α in our Universe is small. Therefore this argument is not physical and cannot be used.

As shown by Newton and Wigner [8], relativistic position operator differs from the nonrelativistic one but the basic feature that the momentum and position operators are related to each other by the Fourier transform remains in the Newton-Wigner construction as well. This postulate is a good illustration of the fact mentioned at the beginning of this section that quantum theory inherited many its features from the classical one. The relation between the coordinates and momenta is analogous to the one between the coordinates and wave vectors in classical electrodynamics. A known effect here is the wave packet spreading (WPS). In classical electrodynamics the wave packet consists of many particles but in quantum theory the effect takes place even for a single-particle WF.

At the very beginning of quantum theory several physicists (e.g. de Broglie) argued that the WPS effect should not take place in quantum theory and the single particle should not be described by the time dependent Schrödinger equation. On the other hand, as shown by Darwin [9], for macroscopic particles the WPS effect is negligible. It is also believed that in experiments with atoms and elementary particles the time is so small that the WPS effect does not manifest itself. Probably for those reasons the majority of physicists do not treat the WPS effect as a drawback of the theory.

However, photons from distant stars can travel to Earth even for billions of years and for them the WPS effect cannot be neglected. As shown in Ref. [10], the WPS effect for such photons results in a fundamental quantal paradox that predictions of the theory contradict our experience on how we observe stars. The paradox can be resolved if the position operator is essentially different from the standard one and the coordinate and momentum representations are not related by the Fourier transform.

One can discuss different choices of the position operator but in any case the choice is not dictated by first principles of quantum theory. History of physics tells us that in any theory it is desirable to have the least possible amount of notions. Quantum theory is believed to be more general than classical one and so at some conditions it should reproduce all the results of classical theory including classical equations of motion. However, it does not mean that quantum theory should explicitly involve particle coordinates and time.

The main results of the paper are described in Sec. 8. Here it is shown that there exist scenarios when classical equations of motion can be obtained from quantum theory without using any classical notions such as coordinates, time, position operator, standard semiclassical approximation etc. The goal of the preceding sections is to prepare the reader for understanding these results. Here the consideration is

based on the results obtained in our previous publications, mainly in Refs. [11, 12, 13]. Those results have been obtained with extensive calculations but in this paper we explain the meaning of the results and argue that they are very natural.

2 Why quantum theory should be based on finite mathematics

A belief of the overwhelming majority of scientists is that classical mathematics (involving the notions of infinitely small/large, continuity etc.) is fundamental while finite mathematics is something inferior what is used only in special applications. In our publications (see e.g. Refs. [11, 13, 14]) we argue that the situation is the opposite: finite mathematics is fundamental, classical mathematics is a degenerate special case of finite one and ultimate quantum theory will be based on finite mathematics. In this section we give basic arguments in favor of this statement.

Historically the notions of infinitely small/large, continuity etc. have arisen from a belief based on everyday experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts. Classical physics is based on classical mathematics developed mainly when people did not know about existence of elementary particles. However, from the point of view of the present knowledge those notions look problematic.

For example, a glass of water contains approximately 10^{25} molecules. We can divide this water by ten, million, etc. but when we reach the level of atoms and elementary particles the division operation loses its meaning and we cannot obtain arbitrarily small parts. So, *any description of macroscopic phenomena using continuity and differentiability can be only approximate*. In nature there are no continuous curves and surfaces. For example, if we draw a line on a sheet paper and look at this line by a microscope then we will see that the line is strongly discontinuous because it consists of atoms.

The official birth of quantum theory is 1925, and even the word "quantum" reflects a belief that nature is discrete. The founders of this theory were highly educated physicists but they used only classical mathematics and even now mathematical education at physics departments does not involve discrete and finite mathematics. In view of the above remarks it is reasonable to think that in quantum theory classical mathematics might be used for solving special problems but ultimate quantum theory should not be based on classical mathematics.

Classical mathematics is not in the spirit of the philosophy of quantum theory and the Viennese school of logical positivism that *"A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false"*. For example, it cannot be determined whether the statement that $a + b = b + a$ for all natural numbers a and b is true or false. Let us also pose a problem whether 10+20 equals 30. Then we should describe an experiment which will solve this problem. Any computer can operate only with a finite number of bits

and can perform calculations only modulo some number p. Say $p = 40$, then the experiment will confirm that 10+20=30 while if $p = 25$ then we will get that 10+20=5. So the statements that 10+20=30 and even that $2 \cdot 2 = 4$ are ambiguous because they do not contain *explicit* information on how they should be verified. On the other hand, the statements

$$10 + 20 = 30\,(mod\,40),\ 10 + 20 = 5\,(mod\,25),\ 2 \cdot 2 = 4\,(mod\,5),\ 2 \cdot 2 = 2\,(mod\,2)$$

are well defined because they do contain such an information. So only operations modulo some number are well defined. This example shows that classical mathematical is based on the implicit assumption that in principle one can have any desired amount of resources and, in particular, one can work with computers having as many bits as desired.

Classical mathematics starts from natural numbers and the famous Kronecker's expression is: *"God made the natural numbers, all else is the work of man"*. However here only addition and multiplication are always possible. In order to make addition invertible we introduce negative integers and their only goal is to get the ring of integers Z. However, if instead of all natural numbers we consider only a set R_p of p numbers 0, 1, 2, ... $p-1$ where addition and multiplication are defined as usual but modulo p then we get a ring without adding new elements.

For example, if p is odd then one can consider R_p as a set of elements $\{0, \pm i\}$ $(i = 1, ...(p-1)/2)$ and such elements are called minimal residues. For elements $a \in R_p$ such that $|a| \ll p$ the addition, subtraction and multiplication are the same as in Z, i.e. such elements do not feel the existence of p. When p increases, the bigger and bigger part of R_p becomes the same as Z. Hence Z can be treated as a degenerate case of R_p in the formal limit $p \to \infty$ because in this limit operations modulo p disappear. *Therefore, at the level of rings standard mathematics is a degenerate case of finite one when formally* $p \to \infty$.

The transition from R_p to Z is similar to the procedure, which in group theory is called contraction. This notion is used when the Lie algebra of a group with a lower symmetry can be treated as a formal limit of the Lie algebra of a group with a higher symmetry when some parameter goes to zero or infinity. Known examples are the contraction from the de Sitter to the Poincare group and from the Poincare to the Galilei group.

The above construction has a well-known historical analogy. For many years people believed that the Earth was flat and infinite, and only after a long period of time they realized that it was finite and curved. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature. Analogously one might think that the set of numbers describing physics in our Universe has a "curvature" defined by a very large number p but we do not notice it when we deal only with numbers much less than p.

One might argue that introducing a new fundamental number p is not justified. However, history of physics tells us that more general theories arise when a parameter, which in the old theory was treated as infinitely small or infinitely

large, becomes finite. For example, nonrelativistic physics is the degenerate case of relativistic one in the formal limit $c \to \infty$ and classical physics is the degenerate case of quantum one in the formal limit $\hbar \to 0$. Therefore, it is natural to think that in quantum physics the number p should be not infinitely large but finite.

It is known that if p is prime then R_p becomes the field F_p where it is also possible to divide by numbers not equal to zero. In general, division in F_p is considerably different from the division in the field of rational numbers. For example, $1/2$ equals $(p-1)/2$, i.e. a large number if p is large. However, this does not mean that quantum theory cannot be based on a finite field. This is clear from the fact that spaces in quantum theory are projective (see Refs. [11, 13, 14]).

On the other hand, the above remarks indicate that division is not a fundamental operation. Since history of physics tells us that it is desirable to introduce the least necessary number of notions, one might think that the ultimate quantum theory will be based on a finite ring, not field. This problem is discussed in Refs. [13, 14].

From mathematical point of view, standard quantum theory can be treated as a theory of representations of special real Lie algebras in complex Hilbert spaces. In Refs. [11, 13] and other publications we have proposed an approach called FQT (Finite Quantum Theory) when Lie algebras and representation spaces are over a finite field or ring with characteristic p. It has been shown that in the formal limit $p \to \infty$ FQT recovers predictions of standard continuous theory. Therefore classical mathematics describes many experiments with a high accuracy as a consequence of the fact that the number p is very large. However, since classical mathematics has foundational problems by its own nature (as follows, for example, from Gödel's incompleteness theorems), ultimate quantum theory cannot be based on classical mathematics.

In physics p is the standard notation for the momentum but in number theory it is the standard notation for the characteristic of a ring or field. In what follows it will be obvious in what context the notation p is used.

3 Symmetry on quantum level

In relativistic quantum theory the usual approach to symmetry on quantum level follows. Since the Poincare group is the group of motions of Minkowski space, quantum states should be described by representations of this group. This implies that the representation generators commute according to the commutation relations of the Poincare group Lie algebra:

$$\begin{aligned} &[P^\mu, P^\nu] = 0 \quad [P^\mu, M^{\nu\rho}] = -i(\eta^{\mu\rho}P^\nu - \eta^{\mu\nu}P^\rho) \\ &[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma}) \end{aligned} \tag{2}$$

where $\mu, \nu = 0, 1, 2, 3$, P^μ are the operators of the four-momentum and $M^{\mu\nu}$ are the operators of Lorentz angular momenta. This approach is in the spirit of Klein's Erlangen program in mathematics. However, as we argue in Refs. [13, 15] and Sec.

1, quantum theory should not be based on classical space-time background and the approach should be the opposite. Each system is described by a set of independent operators. By definition, the rules how they commute with each other define the symmetry algebra. In particular, *by definition*, Poincare symmetry on quantum level means that the operators commute according to Eq. (2). This definition does not involve Minkowski space at all.

Such a definition of symmetry on quantum level has been proposed by Leonid Avksent'evich Kondratyuk during our collaboration. I believe that this replacement of the standard paradigm is fundamental for understanding quantum theory, and I did not succeed in finding a similar idea in the literature. This idea is to some extent in the spirit of Ref. [16]. Here Dirac proposed different forms of relativistic dynamics which are defined by choosing which operators in Eq. (2) are free and which of them are interaction dependent.

Analogously, the definition of the de Sitter (dS) symmetry on quantum level should not involve the fact that the dS group is the group of motions of the dS space. Instead, *the definition* is that the operators M^{ab} $(a, b = 0, 1, 2, 3, 4,\ M^{ab} = -M^{ba})$ describing the system under consideration satisfy the commutation relations *of the dS Lie algebra* so(1,4), *i.e.*,

$$[M^{ab}, M^{cd}] = -i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{3}$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$. The *definition* of the AdS symmetry on quantum level is given by the same equations but $\eta^{44} = 1$.

With such a definition of symmetry on quantum level, dS and AdS symmetries look more natural than Poincare symmetry. In the dS and AdS cases all the ten representation operators of the symmetry algebra are angular momenta while in the Poincare case only six of them are angular momenta and the remaining four operators represent standard energy and momentum. If we *define* the operators P^μ as $P^\mu = M^{4\mu}/R$ where R is a parameter with the dimension *length* then in the formal limit when $R \to \infty$, $M^{4\mu} \to \infty$ but the quantities P^μ are finite, Eqs. (3) become Eqs. (2). This procedure is called contraction and in the given case it is the same for the dS or AdS symmetry.

In the literature, Poincare, dS and AdS symmetries are usually associated not with the corresponding algebras but (in the spirit of the Erlangen program) with the background space invariant under the action of the corresponding group. Those spaces are characterized by the curvature called the cosmological constant Λ (CC) such that $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$ respectively. The expressions for Λ in terms of R are $\Lambda = 0$, $\Lambda = 3/R^2$ and $\Lambda = -3/R^2$, respectively.

It is obvious that FQT can involve only numbers and cannot contain any dimensionful quantities. Equations (3) contain no parameters and it is often said that those expressions are written in units $\hbar = c = 1$. This phrase might create a wrong impression that expressions with $\hbar$ and c are primary while Eqs. (3) are secondary, but the situation is the opposite. Let us make a few remarks on this question.

Consider a measurement of a component of angular momentum. The result depends on the system of units. As shown in quantum theory, in units $\hbar/2 = 1$ the result is given by an integer $0, \pm 1, \pm 2, ...$ We can reverse the order of units and say that in units where the angular momentum is an integer l, its value in $kg \cdot m^2/sec$ is $(1.05457162 \cdot 10^{-34} \cdot l/2) kg \cdot m^2/sec$. Which of those two values has more physical significance? In units where the angular momentum components are integers, the commutation relations between the components are

$$[M_x, M_y] = 2iM_z \quad [M_z, M_x] = 2iM_y \quad [M_y, M_z] = 2iM_x \tag{4}$$

and they do not depend on any parameters. Then the meaning of l is clear: it shows how big the angular momentum is in comparison with the minimum nonzero value 1. At the same time, the measurement of the angular momentum in units $kg \cdot m^2/sec$ reflects only a historic fact that at macroscopic conditions on the Earth in the period between the 18th and 21st centuries people measured the angular momentum in such units.

For quantum theory itself the quantity $\hbar$ is not needed. Classical theory is a good approximation for quantum one when all angular momenta in question are very large. From the formal point of view $\hbar$ is needed only as a formal intermediate step for getting classical theory from quantum one: we first write quantum theory with $\hbar$ and then take the limit $\hbar \to 0$.

Analogous remarks can be given on the quantity c (see e.g. Ref. [13]). Nonrelativistic theory is a good approximation for relativistic one when all velocities in question are much less than unity. Relativistic theory by itself does not need c. It is needed only as a formal intermediate step for getting nonrelativistic theory from relativistic one: we first write relativistic theory with c and then take the limit $c \to \infty$.

In the literature the $c\hbar G$ cube of physical theories is sometimes discussed with the meaning that any relativistic theory should contain c, any quantum theory should contain $\hbar$ and any gravitational theory should contain the gravitational constant G. The problem of G will be discussed below but as far as c and $\hbar$ are concerned, the situation is the opposite: relativistic theory *should not* contain c and quantum theory *should not* contain $\hbar$. The impression that nonrelativistic classical theory without gravitation does not contain parameters is not correct because it contains three parameters: (kg, m, s).

Let us now define the notion of elementary particle. Although theory of elementary particles exists for a rather long period of time, there is no commonly accepted definition of elementary particle in this theory. In the spirit of the above definition of symmetry on quantum level and Wigner's approach to Poincare symmetry [17], a general definition, not depending on the choice of the classical background and on whether we consider a local or nonlocal theory, is that a particle is elementary if the set of its WFs is the space of an irreducible representation (IR) of the symmetry algebra in the given theory.

The explicit construction of IRs of the dS and AdS algebras describing elementary particles (see e.g. Refs. [11, 13]) shows that it is possible to find a basis

where the spectrum of all the representation operators is discrete. Therefore such IRs can be used in both, standard theory and FQT. At the same time, for IRs describing elementary particles in Poincare invariant theory the spectrum of some operators is necessarily continuous. Therefore such IRs cannot be used in FQT.

By definition, the tensor product of IRs corresponding to N particles describes a system where those particles are free. The representation operators for the free N-particle systems are sums of the corresponding single-particle operators. In the present paper we consider only systems of free particles, i.e. there is no interaction between the particles. A problem arises whether the cosmological repulsion and gravity can take place in systems of free particles.

In standard nonrelativistic approximation, gravity is characterized by the term $-Gm_1m_2/r$ in the mean value of the two-particle mass operator. Here m_1 and m_2 are the particle masses and r is the distance between the particles. Since the kinetic energy is always positive, the free nonrelativistic mass operator is positive definite and therefore there is no way to obtain gravity in the framework of the free theory. Analogously, in Poincare invariant theory the spectrum of the free two-particle mass operator belongs to the interval $[m_1 + m_2, \infty)$ while the existence of gravity necessarily requires that the spectrum should contain values less than m_1+m_2.

In theories where the symmetry algebra is the AdS algebra so(2,3), the structure of IRs is well-known (see e.g. Refs. [11, 13]). In particular, for positive energy IRs the AdS Hamiltonian has the spectrum in the interval $[m, \infty)$ and m has the meaning of the AdS mass. Therefore the situation is pretty much analogous to that in Poincare invariant theories. In particular, the free two-particle mass operator again has the spectrum in the interval $[m_1 + m_2, \infty)$ and therefore there is no way to reproduce gravitational effects in the free AdS theory.

In contrast to the situation in Poincare and AdS theories, the free two-particle mass operator in dS theory is not bounded below by the value of $m_1 + m_2$. The results of Ref. [12, 13] show that this property by no means implies that the theory is unphysical. In addition, the existing experimental data (see e.g. Ref. [18]) practically exclude the possibility that $\Lambda \leq 0$. As shown in Ref. [12] (see also the next section) the cosmological repulsion naturally arises in free systems described in the framework of the dS theory. Therefore if one has a choice between Poincare, AdS and dS symmetries then the only chance to describe the cosmological repulsion and gravity in a free theory is to choose dS symmetry.

4 A system of two particles in standard quantum dS theory

As shown in Ref. [12], by using the results of the book [19] on IRs of the dS group one can explicitly construct IRs of the dS algebra describing elementary particles. In this paper we are interested not in elementary particles but in macroscopic bodies. If we consider systems of particles such that the distances between them are much greater than their sizes then the internal structure of the particles is not important and it

suffices to describe each particle only by the variables characterizing the motion of each particle as a whole. The WFs describing such a motion are the same as for elementary particles i.e. we can use IRs of the dS algebra. Since spin is a pure quantum notion which disappears in classical limit, we will consider only spinless IRs and will not consider massless and tachyon representations.

In Poincare theory any massive IR can be implemented in the Hilbert space of functions $\chi(\mathbf{v})$ on the Lorenz 4-velocity hyperboloid with the points $v = (v_0, \mathbf{v})$, $v_0 = (1 + \mathbf{v}^2)^{1/2}$ such that $\int |\chi(\mathbf{v})|^2 d\rho(\mathbf{v}) < \infty$ and $d\rho(\mathbf{v}) = d^3\mathbf{v}/v_0$ is the Lorenz invariant volume element. For positive and negative energy IRs the value of energy is $E = \pm mv_0$ respectively where m is the particle mass *defined as the positive square root* $(E^2 - \mathbf{P}^2)^{1/2}$. Therefore for massive IRs, $m > 0$ by definition.

It is usually assumed that the energy for real particles should be positive. However, the choice of the energy sign is only the matter of convention but not the matter of principle. It is only important that the energy sign for all the particles in question is the same because otherwise the conservation of energy will not take place. In the literature the positive energy IRs are usually associated with particles and the negative energy IRs — with the corresponding antiparticles. Then after the second quantization the energies of both, particles and antiparticle become positive.

In contrast to Poincare theory, IRs in dS theory can be implemented only on two Lorenz hyperboloids, i.e. the Hilbert space for such IRs consist of sets of two functions $(\chi_1(\mathbf{v}), \chi_2(\mathbf{v}))$ such that

$$\int (|\chi_1(\mathbf{v})|^2 + |\chi_2(\mathbf{v})|^2) d\rho(\mathbf{v}) < \infty$$

In Poincare limit one dS IR splits into two IRs of the Poincare algebra with positive and negative energies. In Ref. [12] we argue that this implies that one IR of the dS algebra describes a particle and its antiparticle simultaneously. Since in the present paper we do not deal with antiparticles, we give only expressions for the action of the operators on the upper hyperboloid [12]:

$$\mathbf{M} = l(\mathbf{v}), \quad \mathbf{N} = -iv_0 \frac{\partial}{\partial \mathbf{v}}, \quad \mathbf{B} = m_{dS}\mathbf{v} + i[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}]$$
$$\mathcal{E} = m_{dS}v_0 + iv_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}) \tag{5}$$

where $\mathbf{M} = \{M^{23}, M^{31}, M^{12}\}$, $\mathbf{N} = \{M^{01}, M^{02}, M^{03}\}$, $\mathbf{B} = \{M^{41}, M^{42}, M^{43}\}$, $\mathbf{l}(\mathbf{v}) = -i\mathbf{v} \times \partial/\partial\mathbf{v}$, $\mathcal{E} = M^{40}$ and m_{dS} is a positive quantity.

This implementation of the IR is convenient for the transition to Poincare limit. Indeed, the operators of the Lorenz algebra in Eq. (5) are the same as in the IR of the Poincare algebra. Suppose that the limit of m_{dS}/R when $R \to \infty$ is finite and denote this limit as m. Then in the limit $R \to \infty$ we get standard expressions for the operators of the IR of the Poincare algebra where m is the standard mass, $E = \mathcal{E}/R = mv_0$ and $\mathbf{P} = \mathbf{B}/R = m\mathbf{v}$. For this reason m_{dS} has the meaning of the dS mass. In contrast to m, m_{dS} is dimensionless. Since Poincare symmetry is a special case of dS one, m_{dS} is more fundamental than m. Since Poincare symmetry

works with a high accuracy, the value of R is supposed to be very large. Then even dS masses of elementary particles are very large.

For example, according to Ref. [18], $R \approx 10^{26}m$. The conclusion of this work on R is based not on the consideration of the dS algebra but from the fit to the Friedman-Robertson-Walker model. This value of R is in the spirit of modern cosmology that the Universe has approximately the same size. However, the model depends on parameters and therefore the validity of the conclusion cannot be accepted for granted. In particular, the value of R may be much greater than $10^{26}m$. However, even for this value of R the dS masses of the electron, the Earth and the Sun are of the order of 10^{39}, 10^{93} and 10^{99}, respectively. The fact that even the dS mass of the electron is very large poses a question whether the electron is a true elementary particle.

Consider the non-relativistic approximation when $|\mathbf{v}| \ll 1$. If we wish to work with units where the dimension of velocity is $meter/sec$, we should replace $\mathbf{v}$ by $\mathbf{v}/c$. If $\mathbf{p} = m\mathbf{v}$ then it is clear from the expression for $\mathbf{B}$ in Eq. (5) that $\mathbf{p}$ becomes the real momentum $\mathbf{P}$ only in the limit $R \to \infty$. At this stage we do not have any coordinate space yet. However, if we assume that semiclassical approximation is valid, then, by analogy with standard quantum mechanics, we can *define* the position operator $\mathbf{r}$ as $i\partial/\partial\mathbf{p}$.

In classical approximation we can treat $\mathbf{p}$ and $\mathbf{r}$ as usual vectors. Then as follows from Eq. (5)

$$\mathbf{P} = \mathbf{p} + mc\mathbf{r}/R, \quad H = \mathbf{p}^2/2m + c\mathbf{pr}/R, \quad \mathbf{N} = -m\mathbf{r} \tag{6}$$

where $H = E - mc^2$ is the classical nonrelativistic Hamiltonian. As follows from these expressions,

$$H(\mathbf{P}, \mathbf{r}) = \frac{\mathbf{P}^2}{2m} - \frac{mc^2\mathbf{r}^2}{2R^2} \tag{7}$$

The last term in Eq. (7) is the dS correction to the non-relativistic Hamiltonian. It is interesting to note that the non-relativistic Hamiltonian depends on c although it is usually believed that c can be present only in relativistic theory. This illustrates the fact mentioned in Sec. 3 that the transition to nonrelativistic theory understood as $|\mathbf{v}| \ll 1$ is more physical than that understood as $c \to \infty$. The presence of c in Eq. (7) is a consequence of the fact that this expression is written in standard units. In nonrelativistic theory c is usually treated as a very large quantity. Nevertheless, the last term in Eq. (7) is not large since we assume that R is very large.

As follows from Eq. (7) and the Hamilton equations, in dS theory a free particle moves with the acceleration given by

$$\mathbf{a} = \mathbf{r}c^2/R^2 \tag{8}$$

where $\mathbf{a}$ and $\mathbf{r}$ are the acceleration and the radius vector of the particle, respectively. Since R is very large, the acceleration is not negligible only at cosmological distances when $|\mathbf{r}|$ is of the order of R. The result (8) can be obtained not only from Hamilton

equations but by different ways. For example, assuming that the Hamiltonian is a conserved physical quantity, this result can be obtained from the Maupertuis principle or from Eq. (1) as noted in Sec. 1.

Let us now consider whether the result (8) is compatible with GR. The dS space is a four-dimensional manifold in the five-dimensional space defined by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = R^2 \tag{9}$$

In the formal limit $R \to \infty$ the action of the dS group in a vicinity of the point $(0, 0, 0, 0, x_4 = R)$ becomes the action of the Poincare group on Minkowski space. The dS space can be parameterized without using the quantity R at all if instead of x_a $(a = 0, 1, 2, 3, 4)$ we define dimensionless variables $\xi_a = x_a/R$. It is also clear that the elements of the SO(1,4) group do not depend on R since they are products of conventional and hyperbolic rotations. So the dimensionful value of R appears only if one wishes to measure coordinates on the dS space in terms of coordinates of the flat five-dimensional space where the dS space is embedded in. This requirement does not have a fundamental physical meaning. Therefore the value of R defines only a scale factor for measuring coordinates in the dS space.

With the parameterization of dS space as in Eq. (9) the metric tensor on this space is

$$g_{\mu\nu} = \eta_{\mu\nu} - x_\mu x_\nu/(R^2 + x_\rho x^\rho) \tag{10}$$

where $\mu, \nu, \rho = 0, 1, 2, 3$, $\eta_{\mu\nu}$ is the Minkowski metric tensor, and a summation over repeated indices is assumed. It is easy to calculate the Christoffel symbols in the approximation where all the components of the vector x are much less than R: $\Gamma_{\mu,\nu\rho} = -x_\mu \eta_{\nu\rho}/R^2$. Then a direct calculation shows that in the nonrelativistic approximation the equation of motion for a single particle is the same as in Eq. (8).

Another way to show that Eq. (8) is compatible with GR follows. The known result of GR is that if the metric is stationary and differs slightly from the Minkowskian one then in the nonrelativistic approximation the curved space-time can be effectively described by a gravitational potential $\varphi(\mathbf{r}) = (g_{00}(\mathbf{r}) - 1)/2c^2$. We now express x_0 in Eq. (9) in terms of a new variable t as $x_0 = t + t^3/6R^2 - t\mathbf{x}^2/2R^2$. Then the expression for the interval becomes

$$ds^2 = dt^2(1 - \mathbf{r}^2/R^2) - d\mathbf{r}^2 - (\mathbf{r}d\mathbf{r}/R)^2 \tag{11}$$

Therefore, the metric becomes stationary and $\varphi(\mathbf{r}) = -\mathbf{r}^2/2R^2$ in agreement with Eq. (8).

Consider now a system of two free particles in dS space. Let $(\mathbf{r}_i, \mathbf{a}_i)$ $(i = 1, 2)$ be their radius vectors and accelerations, respectively. Then Eq. (8) is valid for each particle if $(\mathbf{r}, \mathbf{a})$ is replaced by $(\mathbf{r}_i, \mathbf{a}_i)$, respectively. Now if we define the relative radius vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the relative acceleration $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ then they will satisfy the same Eq. (8) which shows that the dS antigravity is repulsive. It terms of Λ it reads $\mathbf{a} = \Lambda \mathbf{r} c^2/3$ and therefore in the AdS case we have attraction rather than repulsion.

Let us now consider a system of two free particles in the framework of the representation of the dS algebra. The particles are described by the variables $\mathbf{P}_j$ and $\mathbf{r}_j$ $(j = 1, 2)$. Define the standard nonrelativistic variables

$$\mathbf{P}_{12} = \mathbf{P}_1 + \mathbf{P}_2, \quad \mathbf{q} = (m_2\mathbf{P}_1 - m_1\mathbf{P}_2)/(m_1 + m_2)$$
$$\mathbf{R}_{12} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{12}$$

Then, as follows from Eq. (6), in the nonrelativistic approximation the two-particle quantities $\mathbf{P}$, $\mathbf{E}$ and $\mathbf{N}$ are given by

$$\mathbf{P} = \mathbf{P}_{12}, \quad E = M + \frac{\mathbf{P}_{12}^2}{2M} - \frac{Mc^2\mathbf{R}_{12}^2}{2R^2}, \quad \mathbf{N} = -M\mathbf{R}_{12} \tag{13}$$

where

$$M = M(\mathbf{q}, \mathbf{r}) = m_1 + m_2 + H_{nr}(\mathbf{r}, \mathbf{q}), \quad H_{nr}(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - \frac{m_{12}c^2\mathbf{r}^2}{2R^2} \tag{14}$$

and m_{12} is the reduced two-particle mass. Here the operator M acts in the space of functions $\chi(\mathbf{q})$ such that $\int |\chi(\mathbf{q})|^2 d^3\mathbf{q} < \infty$ and $\mathbf{r}$ acts in this space as $\mathbf{r} = i\partial/\partial\mathbf{q}$.

It now follows from Eq. (5) that M has the meaning of the two-body mass. This can also be shown ([12, 13]) from the fact that M is the Casimir operator, i.e. it commutes with all representation operators. As follows from Eq. (3), in the dS case the Casimir operator of the second order is

$$I_2 = -\frac{1}{2}\sum_{ab} M_{ab}M^{ab} = \mathcal{E}^2 + \mathbf{N}^2 - \mathbf{B}^2 - \mathbf{J}^2 \tag{15}$$

According to the known Schur lemma in representation theory, all elements in the space of IR are eigenvectors of the Casimir operators with the same eigenvalue. A direct calculation shows that for the operators (5) the numerical value of I_2 is $m_{dS}^2 + 9/4$. One can also show [12] that for IRs with spin $I_2 = m_{dS}^2 - \mathbf{s}^2 + 9/4$ where $\mathbf{s}$ is the spin operator. Then the explicit calculation [12] shows that for the two-body system $I_2 = M^2 - \mathbf{S}^2 + 9/4$ where $\mathbf{S}$ is the spin operator for the two-body system, i.e. the angular momentum in the rest frame. Therefore $M(\mathbf{q}, \mathbf{r})$ is the internal two-body Hamiltonian. Then, by analogy with the derivation of Eq. (8), in can be shown in different ways that in semiclassical approximation the relative acceleration is given by the same expression (8) but now $\mathbf{a}$ is the relative acceleration and $\mathbf{r}$ is the relative radius vector.

The fact that two free particles have a relative acceleration is known for cosmologists who consider dS symmetry on classical level. This effect is called the dS antigravity. The term antigravity in this context means that the particles repulse rather than attract each other. In the case of the dS antigravity the relative acceleration of two free particles is proportional (not inversely proportional!) to the distance between them. As shown above, this classical result is a direct consequence of GR.

The experimental results obtained in 1998 (see e.g. Ref. [18]) is that R is of the order of $10^{26}m$, i.e. Λ is very small but, as stated in Ref. [18]), the accuracy

of the experiment is of the order of 5% and therefore the cases $\Lambda \leq 0$ are practically excluded. This created the following problem.

In textbooks written before 1998 (when the cosmological acceleration was discovered) it is often claimed that Λ is not needed since its presence contradicts the philosophy of GR: matter creates curvature of space-time, so in the absence of matter space-time should be flat (i.e. Minkowski) while empty dS space is not flat. This philosophy has historical roots in view of the well-known fact that first Einstein introduced Λ into his equations and then said that it was the greatest blunder of his life. The problem whether the empty space-time may have a nonzero curvature was also discussed in the dispute between Einstein and de Sitter.

However, such a philosophy has no physical meaning since the curvature is only a mathematical way to describe the motion of real bodies and therefore the curvature does not have a physical meaning for the empty space-time. However, in view of the above statement, in the literature, the discovery of the fact that $\Lambda \neq 0$ has ignited many discussions. The most popular approach is as follows. One can move the term with Λ in the Einstein equations from the left-hand side to the right-hand one. Then the term with Λ is treated as the stress-energy tensor of a hidden matter which is called dark energy. With such an approach one implicitly returns to Einstein's point of view that a curved space-time cannot be empty. In other words, this is an assumption that the Poincare symmetry is fundamental while the dS one is emergent. With the observed value of Λ this dark energy contains approximately 75% of the energy of the Universe. In this approach G is treated as a fundamental constant and one might try to express Λ in terms of G. The existing quantum theory of gravity cannot perform this calculation unambiguously since the theory contains strong divergences. With a reasonable cutoff parameter, the result for Λ is such that in units where $\hbar = c = 1$, $G\Lambda$ is of the order of unity. This result is expected from dimensionful considerations since in these units, the dimension of G is $length^2$ while the dimension of Λ is $1/length^2$. However, this value of Λ is greater than the observed one by 122 orders of magnitude. In supergravity the disagreement can be reduced but even in best scenarios it exceeds 40 orders of magnitude. This problem is called the CC problem or dark energy problem.

Several authors criticized this approach from the following considerations. GR without the contribution of Λ has been confirmed with a good accuracy in experiments in Solar System. If Λ is as small as it has been observed then it can have a significant effect only at cosmological distances while for experiments in Solar System the role of such a small value is negligible. The authors of Ref. [20] titled "Why All These Prejudices Against a Constant?", note that since the solution of the Einstein equations depends on two arbitrary constants G and Λ it is not clear why we should think that only a special case $\Lambda = 0$ is allowed.

In our approach the result for the cosmological acceleration has been obtained without using dS space, its metric, connection etc. The fact that $\Lambda \neq 0$ is a consequence of dS symmetry on quantum level: since dS symmetry is more general than Poincare one then on classical level Λ *must* be nonzero. This has nothing to do with gravity, existence or nonexistence of dark energy and with the problem whether

or not empty space-time should be necessarily flat. The parameter R is fully analogous to c and $\hbar$. This parameter *should not* be used in pure dS theory and its only purpose is to get a less general theory (Poincare one) as a formal limit $R \to \infty$ of dS theory. The question of why R is as it is, is not a matter of fundamental physics since the answer is: because we want to measure distances in meters. In particular, there is no guaranty that the CC is really a constant, i.e. does not change with time.

At the same time, our derivation depends on two assumptions. The result $\mathbf{r} = i\partial/\partial\mathbf{q}$ follows from the assumption that the momentum and position operators are related to each other by the Fourier transform. As noted in Sec. 1, such a choice of the position operator is problematic in view of the WPS effect. It has been noted that for macroscopic bodies this effect is negligible. However, from the first principles of quantum theory it is not clear whether there exists a universal choice of the position operator and whether this operator is needed at all. Another problem is that since the meaning of time on quantum level is not clear, the physical meaning of the prescription on how classical equations of motion arise from quantum theory is not clear as well.

5 Construction of IRs in discrete basis

As noted in Sec. 3, for IRs of the so(1,4) algebra it is possible to find a basis such that all representation operators have only discrete spectrum. This is important for understanding the relation between standard theory and FQT.

First of all, to make relations between standard theory and FQT more straightforward, we will modify the commutation relations (3) by writing them in the form

$$[M^{ab}, M^{cd}] = -2i(\eta^{ac}M^{bd} + \eta^{bd}M^{ac} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{16}$$

One might say that these relations are written in units $\hbar/2 = c = 1$. However, as noted in Sect. 3, fundamental quantum theory should not involve quantities $\hbar$ and c at all, and Eq. (16) indeed does not contain those quantities. The reason for writing the commutation relations in the form (16) rather than (3) is that in this case the minimum nonzero value of the angular momentum is 1 instead of 1/2. Therefore the spin of fermions is odd and the spin of bosons is even. This will be convenient in FQT where 1/2 is a very large number (see Sec. 2). As noted in Sec. 4, for our goals it suffices to consider massive spinless IRs.

By analogy with the method of little group for constructing standard IRs, we first define the rest states and then the other states can be obtained from them by the action of representation operators. As noted in the preceding section, for the operators (5) the numerical value of I_2 is $m_{dS}^2 + 9/4$. However, if the commutation relations are defined by Eq. (16) then the relation betweem I_2 and m_{dS} is $I_2 = m_{dS}^2 + 9$.

In spinless case the space of rest states is one-dimensional and its basis consists of only one vector which we denote as e_0. Since $\mathbf{B}$ is the dS analog of $\mathbf{P}$ (see Sec. 4) and in the spinless case the angular momentum of the rest state is zero, we

define e_0 as the vector satisfying the conditions

$$\mathbf{B}e_0 = \mathbf{J}e_0 = 0, \quad I_2 e_0 = (w+9)e_0 \tag{17}$$

i.e. w has the meaning of m_{dS}^2.

We define $e_1 = 2\mathcal{E}e_0$ and

$$e_{n+1} = 2\mathcal{E}e_n - [w + (2n+1)^2]e_{n-1} \tag{18}$$

These definitions make it possible to find e_n for any $n = 0, 1, 2....$

We use the notation $J_x = J^1$, $J_y = J^2$, $J_z = J^3$ and analogously for the operators $\mathbf{N}$ and $\mathbf{B}$. Instead of the (xy) components of the vectors it is convenient to use the $\pm$ components such that $J_x = J_+ + J_-$, $J_y = -i(J_+ - J_-)$ and analogously for the operators $\mathbf{N}$ and $\mathbf{B}$. We now define the elements e_{nkl} as

$$e_{nkl} = \frac{(2k+1)!!}{k!l!}(J_-)^l (B_+)^k e_n \tag{19}$$

It can be shown [13] that e_{nkl} is the eigenvector of the operator $\mathbf{B}^2$ with the eigenvalue $4n(n+2) - 4k(k+1)$, the eigenvector of the operator $\mathbf{J}^2$ with the eigenvalue $4k(k+1)$ and the eigenvector of the operator J_z with the eigenvalue $2(k-l)$. Therefore different vectors e_{nkl} are mutually orthogonal. It can be shown [13] that the scalar product compatible with the Hermiticity of the operators $(\mathcal{E}, \mathbf{B}, \mathbf{N}, \mathbf{J})$ can be defined such that

$$(e_{nkl}, e_{nkl}) = (2k+1)! C_{2k}^l C_n^k C_{n+k+1}^k \prod_{j=1}^{n} [w + (2j+1)^2] \tag{20}$$

where $C_n^k = n!/[(n-k)!k!]$ is the binomial coefficient. At this point we do not normalize basis vectors to one since, as will be discussed below, the normalization (20) has its own advantages. At a fixed value of n, k takes the values $k = 0, 1, ...n$, l takes the values $l = 0, 1, ...2k$ and if l and k are not in this range then $e_{nkl} = 0$.

Instead of l we define a new quantum number $\mu = k - l$ which can take values $-k, -k+1, ...k$. Each element of the representation space can be written as $x = \sum_{nk\mu} c(n,k,\mu) e_{nk\mu}$ where the set of the coefficients $c(n,k,\mu)$ can be called the WF in the $(nk\mu)$ representation. Assuming that we work with a finite field, a direct calculation (see Ref. [13]) shows that

$$\begin{aligned} \mathcal{E}c(n,k,\mu) &= \frac{n-k}{2n} c(n-1,k,\mu) + \frac{n+2+k}{2(n+2)} [w + (2n+3)^2] c(n+1,k,\mu) \\ J_z c(n,k,\mu) &= 2\mu c(n,k,\mu) \end{aligned} \tag{21}$$

and for the expressions for other representation operators see Ref. [13]. It is seen from the second expression that the meaning of the quantum number μ is such that $c(n,k,\mu)$ is the eigenfunction of the operator J_z with the eigenvalue 2μ, i.e. μ is the standard magnetic quantum number.

We use $\tilde{c}(n,k,\mu)$ to denote the WF in the basis where the basis elements are normalized to one. Then a direct calculation [13] shows that the action of the representation operators is given by

$$\mathcal{E}\tilde{c}(n,k,\mu) = \frac{1}{2}[\frac{(n-k)(n+k+1)}{n(n+1)}(w+(2n+1)^2)]^{1/2}\tilde{c}(n-1,k,\mu) +$$
$$\frac{1}{2}[\frac{(n+1-k)(n+k+2)}{(n+1)(n+2)}(w+(2n+3)^2)]^{1/2}\tilde{c}(n+1,k,\mu)]$$
$$N_z\tilde{c}(n,k,\mu) = -\frac{i}{2}[\frac{(k-\mu)(k+\mu)}{(2k-1)(2k+1)(n+1)}]^{1/2}$$
$$\{[\frac{(n+k)(n+k+1)}{n}(w+(2n+1)^2)]^{1/2}\tilde{c}(n-1,k-1,\mu) -$$
$$[\frac{(n+2-k)(n+1-k)}{n+2}(w+(2n+3)^2)]^{1/2}\tilde{c}(n+1,k-1,\mu)\} -$$
$$\frac{i}{2}[\frac{(k+1-\mu)(k+1+\mu)}{(2k+1)(2k+3)(n+1)}]^{1/2}$$
$$\{[\frac{(n-k)(n-k-1)}{n}(w+(2n+1)^2)]^{1/2}\tilde{c}(n-1,k+1,\mu) -$$
$$[\frac{(n+k+2)(n+k+3)}{n+2}(w+(2n+3)^2)]^{1/2}\tilde{c}(n+1,k+1,\mu)]\}$$
$$B_z\tilde{c}(n,k,\mu) = -2[\frac{(k-\mu)(k+\mu)(n+1-k)(n+1+k)}{(2k-1)(2k+1)}]^{1/2}\tilde{c}(n,k-1,\mu)$$
$$-2[\frac{(k+1-\mu)(k+1+\mu)(n-k)(n+k+2)}{(2k+1)(2k+3)}]^{1/2}\tilde{c}(n,k+1,\mu) \quad (22)$$

and for the expressions for other representation operators see Ref. [13].

As noted in Secs. 4, the operator $\mathbf{B}$ is the dS analog of the usual momentum $\mathbf{P}$ such that in Poincare limit $\mathbf{B} = 2R\mathbf{P}$ (the factor 2 arises because we now use the commutation relations (16) instead of (3)). At the same time, the operator $\mathbf{J}$ has the same meaning as in Poincare invariant theory.

Consider now the semiclassical approximation in the normalized basis. In view of the usual understanding of the structure of semiclassical WFs (see e.g. Ref. [13]) a necessary condition for the semiclassical approximation is that the quantum numbers $(nk\mu)$ are much greater than 1. We assume that a state is semiclassical if its WF has the form

$$\tilde{c}(n,k,\mu) = a(n,k,\mu)exp[i(-n\varphi + k\alpha - \mu\beta)] \quad (23)$$

where $a(n,k,\mu)$ is an amplitude, which is not small only in some vicinities of $n = n_0$, $k = k_0$ and $\mu = \mu_0$. We also assume that when the quantum numbers $(nk\mu)$ change by one, the main contribution comes from the rapidly oscillating exponent. Then, as follows from the first expression in Eq. (22), the action of the dS energy operator can be written as

$$\mathcal{E}\tilde{c}(n,k,\mu) \approx \frac{1}{n_0}[(n_0-k_0)(n_0+k_0)(w+4n_0^2)]^{1/2}cos\varphi\tilde{c}(n,k,\mu) \quad (24)$$

Therefore the semiclassical WF is approximately the eigenfunction of the dS energy operator with the eigenvalue

$$\frac{1}{n_0}[(n_0 - k_0)(n_0 + k_0)(w + 4n_0^2)]^{1/2} cos\varphi.$$

When $n_0 \gg k_0$ and $\varphi \ll 1$ the eigenvalue equals $(w + 4n_0^2)^{1/2}$. Since $w = m_{dS}^2$, this result shows that n is the dS analog of the magnitude of the momentum, i.e. in Poincare approximation $n \approx R|\mathbf{P}|$.

6 System of two particles in discrete basis

Consider now a system of two free particles in dS theory. Then the two-particle operator M_{ab} is a sum of the corresponding single-particle operators and the two-body Casimir operator can be defined by Eq. (15) with the two-particle operators M_{ab}. By analogy with the single-particle case, one can define the two-body operator W which is an analog of the quantity w:

$$I_2 = W - \mathbf{S}^2 + 9 \tag{25}$$

where $\mathbf{S}$ is the two-body spin operator.

By analogy with standard theory, it is convenient to consider the two-body mass operator if individual particle dS momenta n_1 and n_2 are expressed in terms of the total and relative dS momenta N and n. In the c.m. frame we can assume that $\mathbf{B}_1$ is directed along the positive direction of the z axis and then $\mathbf{B}_2$ is directed along the negative direction of the z axis. Therefore the quantum number N characterizing the total dS momentum can be defined as $N = n_1 - n_2$. In nonrelativistic theory the relative momentum is defined as $\mathbf{q} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2)$ and in relativistic theory as $\mathbf{q} = (E_2\mathbf{p}_1 - E_1\mathbf{p}_2)/(E_1 + E_2)$. Therefore, taking into account the fact that in the c.m. frame the particle momenta are directed in opposite directions, one might define n as $n = (m_2n_1 + m_1n_2)/(m_1 + m_2)$ or $n = (E_2n_1 + E_1n_2)/(E_1 + E_2)$. These definitions involve Poincare masses and energies. Another possibility is $n = (n_1 + n_2)/2$. In all these cases we have that $n \to (n+1)$ when $n_1 \to (n_1+1)$, $n_2 \to (n_2+1)$ and $n \to (n-1)$ when $n_1 \to (n_1 - 1)$, $n_2 \to (n_2 - 1)$. In what follows, only this feature is important.

Consider the space of functions $\tilde{c}(n)$ such that

$$\sum_{n=0}^{\infty} |\tilde{c}(n)|^2 < \infty$$

Let $\mathcal{B}$ be the operator which acts in this space as

$$\mathcal{B}\tilde{c}(n) = \frac{1}{2}[\tilde{c}(n+1) + \tilde{c}(n-1)] \tag{26}$$

and $G = 1 - \mathcal{B}$. As shown in Ref. [13], in the approximation when $n_j \gg k_j$ $(j = 1, 2)$

$$W = W_0 - 2(w_1 + 4n_1^2)^{1/2}(w_2 + 4n_2^2)^{1/2}G \tag{27}$$

where

$$W_0 = w_1 + w_2 + 2(w_1 + 4n_1^2)^{1/2}(w_2 + 4n_2^2)^{1/2} - 2\mathbf{B}_1\mathbf{B}_2 \tag{28}$$

This operator can be represented as $W_0 = 4R^2M_0^2$ where $M_0^2 = (p_1+p_2)^2$, p_j $(j = 1, 2)$ is the standard Poincare four-momentum of particle j and therefore M_0^2 is the free mass operator squared in Poincare invariant theory.

Since classical mechanics works with a very high accuracy at macroscopic level, one might think that the validity of semiclassical approximation at this level is beyond any doubts. However, to the best of our knowledge, this question has not been investigated quantitatively. In quantum theory a physical quantity is treated as semiclassical if its uncertainty is much less than its mean value. Consider WFs describing the motion of macroscopic bodies as a whole (say the WFs of the Sun, the Earth, the Moon etc.). It is obvious that uncertainties of coordinates in these WFs are much less than the corresponding macroscopic dimensions. What are those uncertainties for the Sun, the Earth, the Moon, etc.? What are the uncertainties of their momenta?

If A is a physical quantity then we use ΔA to denote the uncertainty of this quantity in some state. In standard quantum mechanics, the validity of semiclassical approximation is defined by the product $\Delta r\Delta p$ while each uncertainty by itself can be rather large. It is known that if, for example, the coordinate and momentum WFs are Gaussian then $\Delta r\Delta p$ is of the order of unity. On the other hand, as noted in Sec. 1, the validity of the standard position operator is problematic. Do we know what scenario for the distribution of momenta and coordinates takes place for macroscopic bodies?

In view of the correspondence between standard theory and FQT we will consider only WFs with a finite support. One might think that a necessary condition for the validity of semiclassical approximation is that the exponent in the semiclassical WF makes many oscillations in the region where the WF is not small. We will consider WFs $\psi(n)$ containing $exp(-i\varphi n)$ such that $\psi(n)$ can be different from zero only if $n \in [n_{min}, n_{max}]$. Then, if $\delta = n_{max} - n_{min}$, the exponent makes $|\varphi|\delta/2\pi$ oscillations on $[n_{min}, n_{max}]$ and φ should satisfy the condition $|\varphi| \gg 1/\delta$. The problem arises whether this condition is sufficient.

As already noted, a quantity can be semiclassical only if its means value is much greater than its uncertainty. In particular, a quantity cannot be semiclassical if its mean value is zero or very small. In Poincare theory the exponent is written as $exp(-ipr)$. Since n is the Poincare analog of Rp, one might think that φ is the Poincare analog of r/R. Since Poincare limit is treated as $R \to \infty$, in Poincare limit φ is not zero only for cosmological distances. This poses the problem whether φ can be semiclassical for non-cosmological distances. The dS analog of $\Delta r\Delta p$ is $\Delta\varphi\Delta n = \Delta\varphi\delta$ and the problem arises whether there exist states where this product is of the order of unity.

In Ref. [13] we discussed in detail the choice of the two-body relative distance operator. Since the functions $\tilde{c}(n)$ are discrete and have a finite support, we now do not have an option to choose the momentum and coordinate WFs Gaussian.

As shown in Ref. [13], if the coordinate r is treated as φR and $r \ll R$ then even for favorable scenarios $\Delta\varphi$ is of the order of $1/\delta^{1/2}$. Therefore $\Delta\varphi\delta$ is a very large value of the order of $\delta^{1/2}$ and this is unacceptable. We argue that the coordinate is semiclassical if $exp(-i\varphi n)$ is replaced by $exp(-i\theta n)$ where $\theta = const/(\delta\varphi)^{1/2}$.

The mean value of the operator W can be written as $\overline{W} = 4R^2M_0^2 + \overline{\Delta W}$ where the last term is the dS correction to the result in Poincare theory.

If the exponent in the internal WF is $exp(-i\varphi n)$, φ is understood as r/R and $\varphi \ll 1$ then as follows from Eq. (27) [13]

$$\overline{\Delta W} = -4R^2[(m_1^2 + \mathbf{p}_1^2)(m_2^2 + \mathbf{p}_2^2)]^{1/2}\varphi^2 \tag{29}$$

As noted above, this can be justified if r is cosmological but still much less than the parameter R. As follows from Eq. (29), in the nonrelativistic approximation we get the same result as in Eq. (14).

If the exponent in the internal WF is $exp(-i\theta n)$, $\theta \ll 1$ then as follows from Eq. (27) [13]

$$\overline{\Delta W} = -const^2[(w_1 + 4n_1^2)(w_2 + 4n_2^2)]^{1/2}\frac{\delta_1 + \delta_2}{\delta_1\delta_2|\varphi|} \tag{30}$$

and the result for the classical nonrelativistic Hamiltonian is

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - \frac{m_1m_2Rconst^2}{2(m_1 + m_2)r}(\frac{1}{\delta_1} + \frac{1}{\delta_2}) \tag{31}$$

where δ_j $(j = 1, 2)$ is the width of the n-distribution for particle j. We see that the correction disappears if the width of the dS momentum distribution for each body becomes very large. In standard theory (over complex numbers) the only limitation is that the width of the dS momentum distribution should be much less than the mean value of this momentum. This is not a serious restriction and the width can be arbitrarily large. In the next section we argue that in FQT it is natural that the width of the momentum distribution for a macroscopic body is inversely proportional to its mass. Then we recover the Newton gravitational law. Namely, if

$$\delta_j = \frac{R}{m_jG'} \quad (j = 1, 2), \quad const^2G' = 2G \tag{32}$$

then

$$H(\mathbf{r}, \mathbf{q}) = \frac{\mathbf{q}^2}{2m_{12}} - G\frac{m_1m_2}{r} \tag{33}$$

In Ref. [13] we also discussed relativistic corrections to the Newton law. We conclude that in our approach gravity is not an interaction but simply the dS correction to standard free nonrelativistic Hamiltonian.

As noted above, classical equations of motions can be obtained from the Hamiltonian in different ways. If $m_2 \gg m_1$ then the Newton law for particle 1 can

be obtained from the single-particle operators discussed in the preceding section. In this case $\delta_1 \gg \delta_2$ and, as follows from Eqs. (31) and (32),

$$H(\mathbf{r},\mathbf{q}) = \frac{\mathbf{q}^2}{2m_1} - G\frac{m_1 m_2}{r} \tag{34}$$

if

$$\delta_2 = \frac{Rconst^2}{2m_2 G} \tag{35}$$

Therefore for particle 1 the presence of the heavy body is manifested such that the single-particle width δ_1 should be replaced by the width of the n-distribution which equals δ_2.

7 Semiclassical states in FQT

For any new theory there should exist a correspondence principle that at some conditions this theory and standard well tested one should give close predictions. As noted in Sec. 2, we treat standard quantum theory as a special case of FQT in the formal limit $p \to \infty$. A detailed discussion of FQT has been given in Refs. [11, 13, 14]. Here we describe only basic facts needed for further presentation.

In conventional quantum theory the state of a system is described by a vector $\tilde{x}$ from a separable Hilbert space H. We use a "tilde" to denote elements of Hilbert spaces and complex numbers while elements of linear spaces over a finite ring or field and elements of the corresponding ring or field will be denoted without a "tilde".

Let $(\tilde{e}_1, \tilde{e}_2, ...)$ be a basis in H. This means that $\tilde{x}$ can be represented as

$$\tilde{x} = \tilde{c}_1\tilde{e}_1 + \tilde{c}_2\tilde{e}_2 + ... \tag{36}$$

where $(\tilde{c}_1, \tilde{c}_2, ...)$ are complex numbers. It is assumed that there exists a complete set of commuting selfadjoint operators $(\tilde{A}_1, \tilde{A}_2, ...)$ in H such that each $\tilde{e}_i$ is the eigenvector of all these operators: $\tilde{A}_j\tilde{e}_i = \tilde{\lambda}_{ji}\tilde{e}_i$. Then the elements $(\tilde{e}_1, \tilde{e}_2, ...)$ are mutually orthogonal: $(\tilde{e}_i, \tilde{e}_j) = 0$ if $i \neq j$ where (...,...) is the scalar product in H. In that case the coefficients can be calculated as

$$\tilde{c}_i = \frac{(\tilde{e}_i, \tilde{x})}{(\tilde{e}_i, \tilde{e}_i)} \tag{37}$$

Their meaning is that $|\tilde{c}_i|^2(\tilde{e}_i, \tilde{e}_i)/(\tilde{x}, \tilde{x})$ represents the probability to find $\tilde{x}$ in the state $\tilde{e}_i$. In particular, when $\tilde{x}$ and the basis elements are normalized to one, the probability equals $|\tilde{c}_i|^2$.

Let us note that the Hilbert space contains a big redundancy of elements, and we do not need to know all of them. Indeed, with any desired accuracy we can approximate each $\tilde{x} \in H$ by a finite linear combination

$$\tilde{x} = \tilde{c}_1\tilde{e}_1 + \tilde{c}_2\tilde{e}_2 + ...\tilde{c}_n\tilde{e}_n \tag{38}$$

where $(\tilde{c}_1, \tilde{c}_2, ...\tilde{c}_n)$ are rational complex numbers. This is a consequence of the known fact that the set of elements given by Eq. (38) is dense in H. In turn, this set is redundant too. Indeed, we can use the fact that Hilbert spaces in quantum theory are projective: ψ and $\tilde{c}\psi$ $(\tilde{c} \neq 0)$ represent the same physical state. Then we can multiply both parts of Eq. (38) by the common denominator of the numbers $(\tilde{c}_1, \tilde{c}_2, ...\tilde{c}_n)$. As a result, we can always assume that in Eq. (38) $\tilde{c}_j = \tilde{a}_j + i\tilde{b}_j$ where $\tilde{a}_j$ and $\tilde{b}_j$ are integers. The meaning of the fact that Hilbert spaces in quantum theory are projective is very clear because not the probability itself but the relative probabilities of different measurement outcomes have a physical meaning.

In FQT we can consider complex analogs of finite rings or fields. For example, we can consider the rings $R_{p^2} = R_p + iR_p$ or fields $F_{p^2} = F_p + iF_p$. The latter definition is valid if p is prime and $p = 3\ (mod\ 4)$ but quadratic extensions of F_p can be also used if $p = 1\ (mod\ 4)$ [11, 13].

Since complex conjugation is the automorphism of R_{p^2} (and F_{p^2}) then, by analogy with conventional quantum theory, in FQT it is possible to formally consider situations when linear spaces over R_{p^2} (or F_{p^2}) used for describing physical states, are supplied by a scalar product and it is also possible to consider analogs of Hermitian operators.

Summarizing this discussion, we conclude that if p is large then there exists a correspondence between the description of physical states on the language of Hilbert spaces and self-adjoint operators in them on one hand, and on the language of linear spaces over R_{p^2} and Hermitian operators in them on the other. However, in FQT probabilistic interpretation can be only approximate: it is valid only for states the norm of which is much less than p.

As noted in Sec. 5, a single-particle WF can be written as

$$x = \sum_{nk\mu} c(n, k, \mu) e_{nk\mu}$$

For the validity of semiclassical approximation the condition

$$\sum_{nk\mu} ||e(n, k, \mu)||^2 |c(n, k, \mu)|^2 \ll p \tag{39}$$

should be satisfied. A detailed analysis in Ref. [13] shows that if $n \gg k$ then this condition can be satisfied if

$$\delta lnw \ll lnp \tag{40}$$

Therefore not only the number p should be very large, but even lnp should be very large. Note that in finite mathematics there is no logarithm but in number theory it is rather often used for estimations. For example, the famous prime number theorem describing the asymptotic distribution of primes involves logarithm.

For elementary particles the condition (40) is always valid without any doubts. Consider now what happens in the case of a macroscopic body which consists of many elementary particles. In quantum theory, state vectors of a system of N bodies belong to the Hilbert space which is the tensor product of single-body Hilbert

spaces. This means that state vectors of the N-body systems are all possible linear combinations of functions

$$\psi(n_1, k_1, l_1, ...n_N, k_N, l_N) = \psi_1(n_1, k_1, l_1) \cdots \psi_N(n_N, k_N, l_N) \tag{41}$$

By definition, the bodies do not interact if all representation operators of the symmetry algebra for the N-body systems are sums of the corresponding single-body operators. For example, the energy operator $\mathcal{E}$ for the N-body system is a sum $\mathcal{E}_1 + \mathcal{E}_2 + ... + \mathcal{E}_N$ where the operator $\mathcal{E}_i$ $(i = 1, 2, ...N)$ acts nontrivially over its "own" variables (n_i, k_i, l_i) while over other variables it acts as the identity operator.

If we have a system of noninteracting bodies in standard quantum theory, each $\psi_i(n_i, k_i, l_i)$ in Eq. (41) is fully independent of states of other bodies. However, in FQT the situation is different. Here, as shown in the preceding section, a necessary condition for the WF to have a probabilistic interpretation is given by Eq. (40). As noted above, for elementary particles this is not a serious restriction. However, if a system consists of N components, a necessary condition that the WF of the system has a probabilistic interpretation is

$$\sum_{i=1}^{N} \delta_i lnw_i \ll lnp \tag{42}$$

where $\delta_i = \Delta n_i$ and $w_i = 4R^2 m_i^2$ where m_i is the mass of the subsystem i. This condition shows that in FQT the greater the number of components is, the stronger is the restriction on the width of the dS momentum distribution for each component. This is a crucial difference between standard theory and FQT. A naive explanation is that if p is finite, the same set of numbers which was used for describing one body is now shared between N bodies. In other words, if in standard theory each body in the free N-body system does not feel the presence of other bodies, in FQT this is not the case. This might be treated as an effective interaction in the free N-body system.

The existing quantum theory does not make it possible to reliably calculate the width of the total dS momentum distribution for a macroscopic body and at best only a qualitative estimation of this quantity can be given. The above discussion shows that the greater the mass of the macroscopic body is, the stronger is the restriction on the dS momentum distribution for each subsystem of this body. Suppose that a body with the mass M can be treated as a composite system consisting of similar subsystems with the mass m. Then the number of subsystems is $N = M/m$ and, as follows from Eq. (42), the width δ of their dS momentum distributions should satisfy the condition $N\delta lnw \ll lnp$ where $w = 4R^2 m^2$. Since the greater the value of δ is, the more accurate is the semiclassical approximation, a reasonable scenario is that each subsystem tends to have the maximum possible δ but the above restriction allows to have only such value of δ that it is of the order of magnitude not exceeding $lnp/(Nlnw)$.

The next question is how to estimate the width of the total dS momentum distribution for a macroscopic body. For solving this problem one has to change variables from individual dS momenta of subsystems to total and relative dS momenta.

Now the total dS momentum and relative dS momenta will have their own momentum distributions which are subject to a restriction similar to that given by Eq. (42). If we assume that all the variables share this restriction equally then the width of the total momentum distribution also will be a quantity not exceeding $lnp/(Nlnw)$. Suppose that $m = N_1 m_0$ where m_0 is the nucleon mass. The value of N_1 should be such that our subsystem still can be described by semiclassical approximation. Then the estimation of δ is

$$\delta = N_1 m_0 lnp/[2Mln(2RN_1 m_0)] \tag{43}$$

Suppose that N_1 can be taken to be the same for all macroscopic bodies. For example, it is reasonable to expect that when N_1 is of the order of 10^3, the subsystems still can be described by semiclassical approximation but probably this is the case even for smaller values of N_1.

In summary, although calculation of the width of the total dS momentum distribution for a macroscopic body is a very difficult problem, FQT gives a reasonable qualitative explanation why this quantity is inversely proportional to the mass of the body. With the estimation (43), the result given by Eq. (31) can be written in the form (33) where

$$G = \frac{2const\, Rln(2RN_1 m_0)}{N_1 m_0 lnp} \tag{44}$$

In Sec. 3 we argued that in theories based on dS invariance there should be no dimenful quantities. In particular, neither the gravitational nor cosmological constant can be fundamental. In units $\hbar/2 = c = 1$, the dimension of G is $length^2$ and its numerical value is l_P^2 where l_P is the Planck length ($l_P \approx 10^{-35} m$). Equation (44) is an additional indication that this is the case since G depends on R (or the cosmological constant) and there is no reason to think that it does not change with time. Since $G_{dS} = G\Lambda$ is dimensionless in units $\hbar/2 = c = 1$, this quantity should be treated as the gravitational constant in dS theory. If $\mu = 2Rm_0$ is the dS nucleon mass then Eq. (44) can be written as

$$G_{dS} = \frac{12const\, ln(N_1 \mu)}{N_1 \mu lnp} \tag{45}$$

As noted in Sect. 4, standard cosmological constant problem arises when one tries to explain the value of Λ from quantum theory of gravity assuming that this theory is QFT, G is fundamental and dS symmetry is a manifestation of dark energy (or other fields) on flat Minkowski background. Such a theory contains strong divergences and the result depends on the value of the cutoff momentum. With a reasonable assumption about this value, the quantity Λ is of the order of $1/G$ and this is reasonable since G is the only parameter in this theory. Then Λ is by more than 120 orders of magnitude greater than its experimental value. However, in our approach we have an additional fundamental parameter p. Equation (45) shows that $G\Lambda$ is not of the order of unity but is very small since not only p but even lnp is very large. For a rough estimation, we assume that the values of $const$ and N_1 in this expression are of the order of unity. Then if, for example, R is of the order of $10^{26} m$, we have that μ

is of the order of 10^{42} and lnp is of the order of 10^{80}. Therefore p is a huge number of the order of $exp(10^{80})$. As noted in Sec. 4, the value of R may be even much greater than $10^{26}m$ and in that case the value of p will be even much greater than $exp(10^{80})$.

Concluding this section we would like to make remarks about the hierarchy of physical theories. As discussed in Secs. 3 and 4, transition from a more general theory to a less general one can be accomplished such that the more general theory can be written with some finite parameter and the less general theory is obtained as a formal limit when this parameter goes to zero or infinity. From this point of view, FQT is the most general theory since all other theories can be obtained from FQT by this procedure. Since FQT is based on finite mathematics it should depend on a finite parameter p which is roughly the greatest possible number in the theory and no physical quantity can exceed this number. As noted above, in our approach gravity is a consequence of the fact that p is finite. It is also obvious that FQT should not depend on any dimenful parameters. When we take a formal limit $p \to \infty$ we obtain standard dS or AdS theories. They still do not depend on dimensionful quantities. However, when we introduce the quantity R and take the limit $R \to \infty$ we obtain quantum Poincare theory in which the dimensionful parameters can have only the dimension of *length* or its powers. When we take the limit $\hbar \to 0$ or $c \to \infty$ we obtain less general theories with greater number of dimensions. The less general theory is classical nonrelativistic theory which depend on dimensions (kg, m, s).

8 Classical equations of motions in FQT

8.1 Preliminary remarks

In standard quantum theory a necessary condition for a WF to be semiclassical is that it contains a rapidly oscillating exponent. However, in finite mathematics there is no exponent. In view of the discussion in Sec. 1, a problem arises whether classical equations of motion can be derived from quantum theory without using classical notions of space-time or even without using classical notions at all. In this section we propose a possible approach for tackling this very difficult problem. This area of quantum theory is *terra incognita* and probably the problem can be solved only with fundamentally new ideas. History of physics tells us that in any theory it is desirable to use the least necessary number of notions. For this reason we first discuss which notions of the present quantum theory might be obsolete in the future theory.

We first discuss whether it is necessary that quantum theory should involve complex numbers. The present quantum theory is based on complex numbers for several reasons. First, the theory involves momenta and coordinates which are related to each other by the Fourier transform. As noted in Sec. 1, this property is inherited from classical electrodynamics and in quantum theory it is problematic. Another reason is that quantum theory involves selfadjoined operators in Hilbert spaces and, according to the spectral theorem, the spectral decomposition for them is always valid only in complex Hilbert spaces.

However, if we accept arguments given in Sec. 2 then fundamental quantum theory can be based only on finite mathematics. The field of complex numbers C is algebraically closed, i.e. any equation of the nth power in C has exactly n solutions. On the other hand, no finite ring or field is algebraically closed; in particular here an equation of the nth power may have no solution at all. However, in single-particle IRs on finite rings or fields discussed in Ref. [11, 13] the spectrum of all necessary physical operators in question is defined explicitly by construction and therefore the fact that finite rings and fields are not algebraically closed is not important in this case.

If $\mathbf{A}$ is the operator of a vector quantity then in quantum theory one can discuss the operators (A_x, A_y, A_z) representing projections of the vector quantity on coordinate axes. However, the notion of coordinate axes is pure classical and it does not seem natural that this notion is present in quantum theory. In classical approximation $\mathbf{A}$ becomes a vector and in this approximation quantum theory should determine the projection of the vector on coordinate axes but the notion of coordinate axis should be used only on classical level.

As an example, consider the operators of the so(3) (or su(2)) algebra. They satisfy the commutation relations (4) which necessarily involve i because the operators are selfadjoined. However, from the point of view of theory of Lie algebras, the most natural basis of operators in the representation space is not (M_x, M_y, M_z) but the Cartan-Weyl basis (M_+, M_-, M_0) where M_0 is the representation operator of the basis element of the Cartan subalgebra and $M_\pm$ are the representation operators of the root elements in the algebra. The commutation relations between these operators are:

$$[M_0, M_-] = -2M_-, \quad [M_0, M_+] = 2M_+, \quad [M_+, M_-] = M_0 \tag{46}$$

and the Casimir operator is

$$K = M_0^2 - 2M_0 + 4M_+M_- = M_0^2 + 2M_0 + 4M_-M_+ \tag{47}$$

These relations do not involve i and if the basis of the representation space consists of eigenvectors of the operator M_0 then the matrix elements of all representation operators are real.

If we now *define* $M_z = M_0$, $M_x = M_+ + M_-$, $M_y = -i(M_+ - M_-)$ then the relations (46) become (4) and $K = \mathbf{M}^2$. We expect that in classical approximation the operators (M_x, M_y, M_z) become real values but it is clear that if the representation is considered in the space over real numbers then it is possible to obtain real values only for $(M_x, M_z, \mathbf{M}^2)$ but not for M_y. However, the real value for the magnitude of M_y can be found since $M_y^2 = \mathbf{M}^2 - M_x^2 - M_z^2$ and so only the sign of M_y is not defined. It seems unnatural that for defining only the sign of M_y we must extend the representation space to the space over complex numbers. A problem arises whether the real WF has a property which defines the direction of M_y in classical limit. Below we discuss a similar problem and show that such a property does exist.

For the dS algebra, the commutation relations in the Cartan-Weyl basis and the relations between the Cartan-Weyl operators and the operators M_{ab} are given in Ref. [13].

8.2 One-dimensional model

Consider a system of two particles with the masses m_1 and m_2 such that $m_2 \gg m_1$. Then, as noted at the end of Sec. 6, particle 1 can be considered in the framework of single-particle problem but the width of the n_1 distribution should be replaced by the width of the n distribution which equals $\delta = \delta_2$. For simplicity we will consider the case when on classical level the particle is moving along the z-axes. The corresponding semiclassical WF is the eigenstate of the operator J_z with the eigenvalue $\mu = 0$ and such that the parameter α in Eq. (23) is zero or π. Our goal is to obtain classical results without using standard semiclassical approximation, position operators and time but proceeding only from quantum states. However, the semiclassical results give a hint that if $k \ll n$ then a simple case which we can consider is the one-dimensional model where the WF $c(n)$ depends only on n and, as follows from the first expression in Eq. (21)

$$\mathcal{E}c(n) = \frac{1}{2}c(n-1) + \frac{1}{2}[w + (2n+3)^2]c(n+1) \tag{48}$$

Although we work in FQT, it will be helpful to compare the results with those obtained in standard theory because our physical intuition is based on that theory. Here, as follows from Eq. (22), the dS energy operator acts on the normalized WF as

$$\mathcal{E}\tilde{c}(n) = \frac{1}{2}[(w + (2n+1)^2)]^{1/2}\tilde{c}(n-1) + \frac{1}{2}[w + (2n+3)^2)]^{1/2}\tilde{c}(n+1) \tag{49}$$

For the correspondence with standard theory, in FQT it is desirable to work with least possible numbers in order to avoid comparisons modulo p whenever possible. We now use n_1 and n_2 to define the minimum and maximum values of n in the support of $c(n)$. Then by using the fact that the space of states is projective, as follows from Eq. (20), the normalization of the elements e_n can be chosen as

$$(e_n, e_n) = \prod_{j=n_1+1}^{n} [w + (2j+1)^2] \quad (n \in [n_1, n_2]) \tag{50}$$

Then up to a normalization factor the relation between the WFs in FQT and in standard theory can be written in the form

$$\tilde{c}(n_2 - l) = c(n_2 - l)W^{-l/2}\{\prod_{m=0}^{l-1} \frac{1}{W}[w + (2n_2 - 2m + 1)^2]\}^{-1/2} \tag{51}$$

where $l = n_2 - n$ and $W = w + (2n_2 + 1)^2$.

Since $c(n)$ has a finite support it cannot be the eigenstate of the operator $\mathcal{E}$. For example, $c(n_2 + 1) = 0$ but, as follows from Eq. (48), $\mathcal{E}c(n_2 + 1) = c(n_2)/2 \neq 0$. We will see below that the uncertainty of $\mathcal{E}$ is minimal when $\mathcal{E}c(n) = \lambda c(n)$ for $n \in [n_1, n_2]$. Since the norm of e_n is maximal when $n = n_2$, we want to work with least possible numbers, the states are projective, the minimum possible value of $c(n_2)$

in FQT is $c(n_2) = \pm 1$ then we choose $c(n_2) = 1$. Then, as follows from Eq. (48), for $n \in [n_1, n_2]$ all the values $c(n)$ can be found consecutively:

$$c(n-1) = 2\lambda c(n) - [w + (2n+3)^2]c(n+1) \tag{52}$$

In particular, $c(n_2 - 1) = 2\lambda$, $c(n_2 - 2) = 4\lambda^2 - W$ etc. However, it is problematic to find an explicit expression for $c(n)$ if n ia arbitrary.

In the nonrelativistic case $w \gg n_2^2$ and for semiclassical WFs $\delta = (n_2 - n_1) \ll n_2$. So one might think that a good approximation is to neglect the variations of $[w+(2n+1)^2]$ at $n \in [n_1, n_2]$ and consider the following approximation of Eq. (52):

$$c(n-1) = 2\lambda c(n) - Wc(n+1) \tag{53}$$

Then it is easy to prove by induction that

$$c(n_2 - l) = \sum_{m=0} \frac{(-1)^m (l-m)!}{m!(l-2m)!} (2\lambda)^{l-2m} W^m \tag{54}$$

where the upper limit is defined by the condition that $1/(l-2m)! = 0$ if $l < 2m$. As follows from Eq. (51), in this approximation

$$\tilde{c}(n_2 - l) = C(l) = C(l, x) = \sum_{m=0} \frac{(-1)^m (l-m)!}{m!(l-2m)!} (2x)^{l-2m} \tag{55}$$

where $x = \lambda / W^{1/2}$. This is the Gegenbauer polynomial which in the literature is denoted as $C_l^1(x)$, and it is known that if $x = cos\theta$ then $C(l) = sin((l+1)\theta)/sin\theta$.

Suppose that $sin((\delta+2)\theta) = 0$. Then $(\delta+2)\theta = k\pi$ where k is an integer, $sin((\delta+1)\theta) = (-1)^{k+1} sin\theta$ and

$$Norm^2 = \sum_{l=0}^{\delta} C(l)^2 = \frac{1}{sin^2\theta} \sum_{l=0}^{\delta} sin^2((l+1)\theta) = \frac{\delta+2}{2sin^2\theta} \tag{56}$$

In this case $\mathcal{E}\tilde{c}(n) = \lambda \tilde{c}(n)$ for all $n \in [n_1, n_2]$, λ is exactly the mean value of the operator $\mathcal{E}$:

$$\bar{\mathcal{E}} = \frac{1}{Norm^2} (\tilde{c}, \mathcal{E}\tilde{c}) = \frac{1}{Norm^2} \sum_{n=n_1}^{n_2} \tilde{c}(n) \mathcal{E} \tilde{c}(n) = \lambda, \tag{57}$$

and the uncertainty of $\mathcal{E}$ is

$$\Delta\mathcal{E} = \frac{1}{Norm} (\tilde{c}, (\mathcal{E} - \bar{\mathcal{E}})^2 \tilde{c})^{1/2} = \frac{1}{Norm} ||(\mathcal{E} - \bar{\mathcal{E}})\tilde{c}|| = (\frac{W}{\delta+2})^{1/2} |sin\theta| \tag{58}$$

Note that $(\mathcal{E} - \bar{\mathcal{E}})\tilde{c}(n)$ is not zero only if $n = n_1 - 1$ or $n = n_2 + 1$.

As already noted, the choice of the energy sign is only the matter of convention but not the matter of principle. Consider, for example, the case of nonrelativistic classical mechanics. Here the Hamiltonian is $H = \mathbf{p}^2/(2m) + U(\mathbf{r})$ where $U(\mathbf{r})$ is the potential energy. The equations of motions can be obtained from the minimum action

principle. However, if we define $H = -\mathbf{p}^2/(2m) - U(\mathbf{r})$ then the same equations can be obtained from the maximum action principle. Also, if we require that the energy is a conserved physical quantity and coordinates and momenta are related to each other by Eq. (1) then the sign of H is not important at all. Let us note that in finite mathematics the notions of $<$, $>$, minimum and maximum can be only approximate in some situations. They become well defined only in the degenerate case when we take the limit $p \to \infty$.

As follows from Eq. (21), if $k \ll n$ then the energy of the dS particle which is far from other particles is $\mathcal{E} \approx \pm W^{1/2}$ and, as follows from Eqs. (7) and (34), for nonrelativistic particles the interaction gives a small correction to $|\mathcal{E}|$. Therefore λ can be positive or negative but $|\lambda|/W^{1/2}$ is close to 1 but is less than 1. Therefore one can choose θ such that $cos\theta = \lambda/W^{1/2}$ and θ is close to zero or π. Therefore, as follows from Eq. (58), $\Delta\mathcal{E}/|\bar{\mathcal{E}}| \approx |sin\theta|/\delta^{1/2}$ is very small because δ is very large and $|sin\theta|$ is small. Indeed, a simple estimation shows that if the kinetic and potential energies are of the same order then θ is of the order of v/c and for the cosmological repulsion θ is of the order of r/R. As a consequence, the particle state is strongly semiclassical.

As follows from Eq. (51), in FQT the WF also makes many oscillations on $[n_1, n_2]$. In finite mathematics there is no sine function and Eq. (43) is not the exact prescription for δ but an estimation. In view of the above discussion it is reasonable to think that within this estimation the semiclassical function tends to choose such a value of δ that the formal result (55) for $c(n_1 - 1)$ tends to be as small as possible.

We denote $y = 2n_2 + 1$. Since $w + (2n+3)^2 = W - 4(l-1)y + 4(l-1)^2$ and $4(l-1)^2 \ll 4(l-1)y \ll W$, one might think that the first-order correction in y will give a small correction to the solution of Eq. (48) given by Eq. (55). We now denote $c^{(0)}(n)$ the zero-order solution given by Eq. (55) and $c^{(1)}(n)$ the first-order in y. Then, as follows from Eq. (48)

$$c^{(1)}(n_2 - l) = 2y(l-2)\sum_{m=0} \frac{(-1)^m(l-m-1)!}{m!(l-2m-2)!}(2\lambda)^{l-2m-2}W^m = \\ -2y(l-2)\frac{d}{dW}\{\sum_{m=0} \frac{(-1)^m(l-m)!}{m!(l-2m)!}(2\lambda)^{l-2m}W^m\} \tag{59}$$

We are interested in cases where l is order of δ, i.e. is very large. Then, as follows from Eqs. (55), (59) and (51), that if the terms linear in y are taken into account then in first order in y the result in standard theory is

$$C(l) = (1 - \frac{l^2 y}{W})\frac{sin((l+1)\theta)}{sin\theta} - \frac{2yl}{W^{l/2}}\frac{d}{dW}\{\frac{sin((l+1)\theta)}{sin\theta}W^{l/2}\} = \\ \frac{sin((l+1)\theta)}{sin\theta} - \frac{yl\lambda}{W(W-\lambda^2)^{1/2}}[l\frac{cos((l+1)\theta)}{sin\theta} - \frac{sin((l+1)\theta)}{sin^2\theta}cos\theta] \tag{60}$$

If $lsin\theta \gg 1$ then the last term can be neglected and with the same accuracy

$$C(l) = \frac{1}{sin\theta}sin[(l+1)\theta - \frac{l^2 y cos\theta}{W sin\theta}] \tag{61}$$

Note that the term with y depends on the sign of λ because $cos\theta = \lambda/W^{1/2}$.

The above calculation is based on the assumption that the terms linear in y give small corrections to the main term, in particular we assumed that $yl^2/W \ll 1$. The value of W is the Poincare analog of the energy squared: $W = 4R^2(m^2 + \mathbf{p}^2)$, n_2 is the Poincare analog of $R|\mathbf{p}|$ and, as follows from Eq. (35), δ is of the order of R/r_g where r_g is the gravitational (Schwarzschild) radius of the heavy body. Then if l is of the order of δ and R is of the order of $10^{26}m$ then indeed $yl^2/W \ll 1$. However, as noted above, the value of R may be much greater than $10^{26}m$, Poincare limit is defined as $R \to \infty$ and in the formal limit $R \to \infty$, $y\delta^2/W \to \infty$. So there is no guaranty that the second term of the argument of sine in Eq. (61) is small. In addition, as noted above, if the kinetic energy is of the same order that the potential one then θ is of the order of v/c but for the cosmological repulsion θ is of the order of r/R. Therefore only the exact solution of Eq. (48) can solve the problem.

Nevertheless, regardless of the relation between the first and second terms in the argument of sine, the function given by Eq. (61) is an approximate solution of Eq. (49) if $\mathcal{E}\tilde{c}(n) = \lambda\tilde{c}(n)$ for $n \in [n_1, n_2]$. Indeed, we can neglect the difference between $[w + (2n+1)^2]^{1/2}$ and $[w + (2n+3)^2]^{1/2}$ because it is very small and, since $y/(Wsin\theta) \ll 1$ we have

$$C(l+1) = sin[(l+1)\theta - \frac{l^2 ycos\theta}{Wsin\theta} + \theta - \frac{2lycos\theta}{Wsin\theta}]$$
$$C(l-1) = sin[(l+1)\theta - \frac{l^2 ycos\theta}{Wsin\theta} - \theta + \frac{2lycos\theta}{Wsin\theta}] \quad (62)$$

and therefore

$$C(l+1) + C(l-1) = 2sin[(l+1)\theta - \frac{l^2 ycos\theta}{Wsin\theta}]cos(\theta - \frac{2lycos\theta}{Wsin\theta}) \quad (63)$$

Then Eq. (49) will be satified if

$$[w + 4(n+1)^2]^{1/2} cos(\theta - \frac{2lycos\theta}{Wsin\theta}) = \lambda \quad (64)$$

Since $yl \ll W$ then

$$[w + 4(n+1)^2]^{1/2} \approx W^{1/2}(1 - 2ly/W),\ cos(\theta - \frac{2lycos\theta}{Wsin\theta}) \approx cos\theta(1 + 2ly/W) \quad (65)$$

and Eq. (64) is indeed satisfied in first order in yl/W.

However, the fact that the function given by Eq. (61) is an approximate solution does not mean that this solution is unique. The argument of the sine can contain additional terms which are not small but the function still will satisfy Eq. (49). Meanwhile, the solution should be unique because it is fully defined by the conditions $c(n_2) = 1,\ c(n_2 - 1) = 2\lambda$. We conclude that even in the one-dimensional model discussed above the problem of finding the exact solution of Eq. (48) is very difficult.

8.3 Classical equations of motion

As already noted, Eq. (43) gives the estimation of the width of the relative dS momentum if the mass of particle 2 is much greater than the mass of particle 1. It also follows from Eq. (44) that not only p is a very large number but even lnp is very large. Suppose now that p changes. We do not say that p changes with time because time is a classical notion while we are considering a pure quantum problem. Below we propose a scenario that classical time arises as a consequence of the fact that p changes. As noted in Ref. [13], there are reasons to think that at early stages of the Universe p was much less than now i.e. p is increasing.

If p changes by Δp then Δp cannot be infinitely small because, roughly speaking, p is an integer. Moreover, a possible scenario is that at every step p is multiplied by a number k and if $k \gg 1$ then $\Delta p \gg p$. However, in that case lnp changes by $\Delta lnp = lnk$. This quantity also cannot be infinitely small but it is possible that $\Delta lnp/lnp$ is a very small real number. As follows from Eq. (43), $\Delta\delta/\delta = \Delta lnp/lnp$. Therefore $\Delta\delta/\delta$ does not depend on the heavy mass and depends only on the change of p. Since time is a dimensionful parameter, we *define* time such that its variation is given by $\Delta t = R\Delta lnp/lnp$. In that case Δt also cannot be infinitely small but can be very small in comparison with macroscopic times.

In view of Eq. (44) and the definition of time the following problem arises. If p changes then does it mean that G changes? In our approach the number p is fundamental while G is not. In view of the remarks in Secs. 3 and 7, a problem also arises whether dimensionful quantities can be fundamental. In particular, as noted in Sec. 7, the quantity G_{dS} given by Eq. (45) is more fundamental than G because it is dimensionless. Equation (44) shows that G depends not only on p but also on R. This parameter has the dimension of meter because people want to deal with Poincare momenta and not with dimensionless dS angular momenta. So it is not even clear whether R expressed in meters changes or not. In any case, among the constants which are treated as fundamental, G is measured with the least accuracy and its value is known only for approximately 300 years. If $\Delta lnp \ll lnp$ then it is quite possible that the change of G could not be noticed for such a short period of time. In view of these remarks we assume that relative variations of such quantities as R and δ are much smaller than relative variations of standard momenta and coordinates characterizing the particle under consideration. In what follows we use p to denote the magnitude of standard momentum.

The problem arises how n_2 changes with the change of δ. Understanding this problem is very difficult because, as discussed in the preceding subsection, even the problem of finding exact solutions of Eq. (48) is very difficult. For this reason we can only make assumptions about the dependence of the variation of n_2 on the variation of δ. We assume that the ratio $c(n_2-1)/c(n_2)$ does not change with the change of δ and then, since the ratio is 2λ, λ is the conserved quantity. For simplicity, in what follows we will write n instead of n_2 and consider only nonrelativistic approximation.

Consider first a possibility that

$$\Delta n = \pm(W - \lambda^2)^{1/2}\frac{\Delta\delta}{2\delta} \tag{66}$$

where the sign depends on whether λ is positive or negative, respectively. In nonrelativistic approximation $cos\theta \approx \pm 1$ depending on the sign of λ and Eq. (61) contains $ycos\theta$. Therefore it is natural to think that the sign of Δn depends on the sign of λ. We treat Eq. (66) as an approximate consequence of FQT formulated in terms of real numbers and so we can use classical mathematics for treating this expression with a good approximation.

Consider a situation in standard theory when a particle is moving along the z-axis and is attracted or repulsed by a body in the origin. Then the sign of the change of the distance to the origin depends on whether the particle momentum and radius-vector are parallel or antiparallel. In standard dS theory the values of the momentum and radius-vector are defined by the operators B_z and N_z (see Eq. (22)). However, the expressions for N_z contains i and this is not in the spirit of the discussion in subsection 8.1. As noted in this discussion, in standard theory complex numbers are necessary only for defining the direction of M_y. In standard theory it is assumed that the energy is always positive while, as noted in Subsec. 8.2, the energy sign maybe positive and negative as well. Now we see that in the above scenario the sign of λ defines whether the variation of n is positive or negative.

If θ is *defined* such that $cos\theta = \lambda/W^{1/2}$ and $sin\theta$ is positive then θ is closed to 0 or π. In the last case we replace $\pi - \theta$ by θ. Then $\theta \approx sin\theta = (1 - \lambda^2/W)^{1/2}$ and

$$\lambda \approx \pm W^{1/2}(1 - \theta^2/2) \approx 2R(m + p^2/2m - m\theta^2/2), \quad p\Delta p = m^2\theta\Delta\theta \tag{67}$$

The last relation follows from the fact that λ is a conserved quantity. Finally, if we *define* $r = R\theta$ and note that $n = Rp$ then, as follows from the definition of time and Eqs. (66) and (67)

$$\Delta p = \pm\frac{mr}{R^2}\Delta t, \quad \Delta r = \pm\frac{p}{m}\Delta t \tag{68}$$

In view of the remarks on Eq. (1), the second expression shows that the quantity r defined above indeed has the meaning of the coordinate. Since the quantities p and r are positive by construction, it is clear that in our one-dimensional model the sign is $\pm$ when the momentum and radius-vector are collinear and anticollinear, respectively.

In the approximation when Δt in Eq. (68) can be treated as infinitely small, we get $\dot{p} = \pm mr/R^2$, $\dot{r} = \pm p/m$, i.e. exactly the Hamilton equations obtained from the Hamiltonian $H = p^2/(2m) - mr^2/(2R^2)$. It follows from these relations that $\ddot{r} = r/R^2$ in agreement with Eq. (8) (taking into account that we work in units where $c = 1$). Therefore we have repulsion as it should be in accordance with the consideration in Sec. 4. Here it has been noted that the result for dS antigravity is compatible with the prescription of standard quantum theory that the coordinate and momentum representations should be related to each other by the Fourier transform.

Consider now a possibility that

$$\Delta n = \pm\frac{(W - \lambda^2)^2}{4const^2W^{3/2}}\Delta\delta \tag{69}$$

where *const* is the same as in Eq. (32). We can define θ, assume that $\theta \ll 1$ and use Eq. (67) as above. Then $\Delta n = \pm W^{1/2}\theta^4\Delta\delta/(4const^2)$. However, if we define r as above then this quantity will not satisfy the second condition in Eq. (68), i.e. it will not have the meaning of coordinate. Therefore in the given case the momenta and coordinates cannot be related by the Fourier transform. In accordance with Sec. 6, we now *define* $\theta = const/(\delta\varphi)^{1/2}$ where $\varphi = r/R$. Then as follows from the definition of time and Eqs. (32) and (67)

$$\Delta p = \pm\frac{MmG}{r^2}\Delta t, \quad \Delta r = \mp\frac{p}{m}\Delta t \tag{70}$$

where M is the mass of the heavy particle 2. As follows from the second expression, the quantity r has now the meaning of the coordinate in view of the remarks on Eq. (1). We conclude that the sign in Eq. (69) should be opposite to that in Eq. (66): it is $\pm$ when the momentum and radius-vector are anticollinear and collinear, respectively. In the approximation when Δt is infinitely small we get $\dot p = \pm MmG/r^2$, $\dot r = \mp p/m$ and $\ddot r = -MG/r^2$. The last relation shows that in this case we have attraction as it should be for gravity.

We have considered two cases when Δn is given by Eqs. (66) and (69), respectively. The first case reproduces standard dS antigravity and the second case — standard gravity. The comparison of those expressions shows that the first case takes place when $\delta\theta^3 \ll 1$ and the second case — in the opposite situation when $\delta\theta^3 \gg 1$. As follows from Eq. (32), δ is of the order R/r_g where r_g is the gravitational radius of the heavy particle 2. As shown above, $\theta = r/R$ in the first case and $\theta = const(R/\delta r)^{1/2} \approx (r_g/r)^{1/2}$ in the second one. Therefore the above conditions are indeed satisfied if R is very large.

9 Conclusion

In Secs. 1 and 2 we argue that, although quantum theory exists for more than 90 years and it is now clear that classical physical intuition typically does not work here, quantum theory inherited many its notions from classical theory. Quantum theory is treated as more general as classical one and at some conditions quantum theory should reproduce all results of classical theory, including classical equations of motions. However, in quantum theory the notion of space-time is unnatural and should not be present at all. In addition, quantum theory is based on the results of classical mathematics developed mainly when people did not know about the existence of atoms and elementary particles. From the point of view of quantum theory, the notions of infinitely small/large, continuity etc. also are unnatural and the theory should be based on finite mathematics.

A natural generalization of standard quantum theory is such that quantum states are elements of a finite ring or field with the characteristic p such that no physical quantity exceeds p. This number is a fundamental parameter defining physical laws. However, this does not mean that this number is always the same in the

history of Universe. We do not say that the number is the same at all times because time is a pure classical notion and should not be present in quantum theory. We argue that *the existence of classical time is a consequence of the fact that p changes* and in Sec. 8 we *define* time such that its variation Δt is related to the variation of p as

$$\Delta t = \frac{R}{c}\frac{\Delta lnp}{lnp} \tag{71}$$

where R is the de Sitter (dS) radius. Then as shown in Subsec. 8.3, there exist scenarios when classical equations of motions can be obtained from pure quantum notions without using space, time and standard semiclassical approximation.

In this scenario the goal of quantum theory is to determine how mean values of dS angular momenta change when the widths of their distribution change. As shown in Subsec. 8.2, even in the one-dimensional model discussed in this subsection the problem of finding exact solutions is very difficult. However, in Subsec. 8.3 we consider two possibilities when classical equations of motion in standard dS antigravity and standard gravity can be obtained from pure quantum theory without involving any classical notions and standard semiclassical approximation.

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