An approximation to $\pi(n)$ through the sum of consecutive prime numbers

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Abstract

In this paper it is proved that the sum of consecutive prime numbers under the square root of a given natural number is asymptotically equivalent to the prime counting function $\pi(n)$. Also, it is proved another asymptotic equivalence between the sum of the first $\lfloor\sqrt{n}\rfloor$ prime numbers and the prime counting function $\pi(n)$.

Theorem 1. *Let the prime counting function until a given natural number be*

$$\pi(n) = \#\{p \in P \mid p \leq n\} \tag{1}$$

Let the sum of consecutive prime numbers under the square root of a given natural number be

$$\Upsilon(n) = \sum_{p \leq \sqrt{n}} p \tag{2}$$

It can be stated that

$$\Upsilon(n) \sim \pi(n) \tag{3}$$

Which can be expressed also stating that

$$\lim_{n \to \infty} \frac{\Upsilon(n)}{\pi(n)} = 1 \tag{4}$$

Proof

By partial summation

$$\Upsilon(n) = (\lfloor\sqrt{n}\rfloor\, \pi(\sqrt{n})) - \sum_{m=2}^{\lfloor\sqrt{n}\rfloor - 1} \pi(m) \tag{5}$$

Where $\lfloor\sqrt{n}\rfloor$ denotes the integer part of $\sqrt{n}$.

By the Prime Number Theorem with error term, there exists a constant C such that

$$\left|\pi(x) - \frac{x}{\log x}\right| \leq C\frac{x}{\log^2 x} \qquad \textbf{for } x \geq 2 \tag{6}$$

Therefore, substituting $\pi(\sqrt{n})$ and $\pi(m)$ by the application of the Prime Number Theorem on (6)

$$\Upsilon(n) = (\lfloor\sqrt{n}\rfloor \frac{\sqrt{n}}{\log(\sqrt{n})}) - \sum_{m=2}^{\lfloor\sqrt{n}\rfloor - 1} \frac{m}{\log(m)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{7}$$

Applying Riemman Sums theory to the sum on the right of (5)

$$\sum_{m=2}^{\lfloor\sqrt{n}\rfloor - 1} \frac{m}{\log(m)} = \int_2^{\lfloor\sqrt{n}\rfloor} \frac{x}{\log(x)} dx + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{8}$$

Solving the integral by partial integration, we have that

$$\int_2^{\lfloor\sqrt{n}\rfloor} \frac{x}{\log(x)} dx = \left[\frac{x^2}{2\log(x)}\right]_2^{\lfloor\sqrt{n}\rfloor} + \int_2^{\lfloor\sqrt{n}\rfloor} \frac{x}{2\log^2(x)} =$$

$$= \frac{n}{2\ln(\lfloor\sqrt{n}\rfloor)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right)$$

It is easy to see that

$$\frac{n}{2\log(\lfloor\sqrt{n}\rfloor)} \sim \frac{n}{\log n} \tag{10}$$

Thus

$$\sum_{m=2}^{\lfloor\sqrt{n}\rfloor-1} \frac{m}{\log(m)} \sim \frac{n}{\log(n)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{11}$$

Regarding the left product on (5) it can be seen that

$$\lfloor\sqrt{n}\rfloor \frac{\sqrt{n}}{\log(\sqrt{n})} \sim \frac{n}{\log(\sqrt{n})} = \frac{n}{\frac{1}{2}\log(n)} = \frac{2n}{\log(n)} \tag{12}$$

Substituting (11) and (12) on (5), we have that

$$\Upsilon(n) \sim \frac{2n}{\log(n)} - \frac{n}{\log(n)} + O\left(\frac{n}{\log^2(\sqrt{n})}\right) \tag{13}$$

As

$$\frac{2n}{\log(n)} - \frac{n}{\log(n)} = \frac{n}{\log(n)} \tag{14}$$

Thus

$$\Upsilon(n) \sim \frac{n}{\log(n)} \tag{15}$$

And subsequently, as by the Prime Number Theorem,

$$\pi(n) \sim \frac{n}{\log(n)} \tag{16}$$

It can be stated that

$$\Upsilon(n) \sim \pi(n) \tag{17}$$

Which can be expressed also stating that

$$\lim_{n\to\infty} \frac{\Upsilon(n)}{\pi(n)} = 1 \tag{18}$$

Theorem 2. *Let the prime counting function until a given natural number be*

$$\pi(n) = \# \{p \in P \mid p \leq n\} \tag{19}$$

Let the sum of the first $\lfloor \sqrt{n} \rfloor$ *consecutive prime numbers be*

$$\Psi(n) = \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} p_k \tag{20}$$

It can be stated that

$$\Psi(n) \left(\frac{1}{\log(\sqrt{n})} \right)^2 \sim \pi(n) \tag{21}$$

Which can be expressed also stating that

$$\lim_{n \to \infty} \frac{\Psi(n)}{\pi(n) \log^2(\sqrt{n})} = 1 \tag{22}$$

Proof

From the application of the Prime Number Theorem and the Abel's summation we have

$$\Psi(n) \sim \sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k \, \log(k) \tag{23}$$

$$\sum_{k=1}^{\lfloor \sqrt{n} \rfloor} k \, \log(k) = \frac{\lfloor \sqrt{n} \rfloor (\lfloor \sqrt{n} \rfloor + 1)}{2} \log\left(\lfloor \sqrt{n} \rfloor\right) - \frac{1}{2} \int_1^{\lfloor \sqrt{n} \rfloor} (t+1) \, dt = \tag{24}$$

$$= \frac{\lfloor \sqrt{n} \rfloor^2}{2} \log\left(\lfloor \sqrt{n} \rfloor\right) + \frac{\lfloor \sqrt{n} \rfloor}{2} \log\left(\lfloor \sqrt{n} \rfloor\right) - \frac{\lfloor \sqrt{n} \rfloor^2}{4} - \frac{\lfloor \sqrt{n} \rfloor}{2} + \frac{3}{4} \tag{25}$$

Thus

$$\Psi(n) \sim \frac{\lfloor \sqrt{n} \rfloor^2}{2} \log\left(\lfloor \sqrt{n} \rfloor\right) \sim \frac{n}{2} \log\left(\sqrt{n}\right) \tag{26}$$

As

$$\frac{n}{2} \log\left(\sqrt{n}\right) = \frac{n}{\log(n)} \log^2\left(\sqrt{n}\right) \tag{27}$$

Therefore

$$\Psi(n) \sim \frac{n}{\log(n)} \log^2\left(\sqrt{n}\right) \tag{28}$$

And subsequently, as by the Prime Number Theorem,

$$\pi(n) \sim \frac{n}{\log(n)} \tag{29}$$

It can be stated that

$$\Psi(n) \left(\frac{1}{\log\left(\sqrt{n}\right)}\right)^2 \sim \pi(n) \tag{30}$$

Which can be expressed also stating that

$$\lim_{n\to\infty} \frac{\Psi(n)}{\pi(n)\log^2\left(\sqrt{n}\right)} = 1 \tag{31}$$

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