Intuitive Geometric Significance of Pauli Matrices and Others in a Plane

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June 2017

Abstract

The geometric significance of complex numbers is well known, such as the meaning of imaginary unit i is to rotate a vector with 90°, etc. In this article, we will try to find some intuitive geometric significances of Pauli matrices, split-complex numbers, SU_2, SO_3, and their relations, and some other operators often used in quantum physics, including a new method to lead to the spinor-space and Dirac equation.

1. Retrospection about a few Matrices

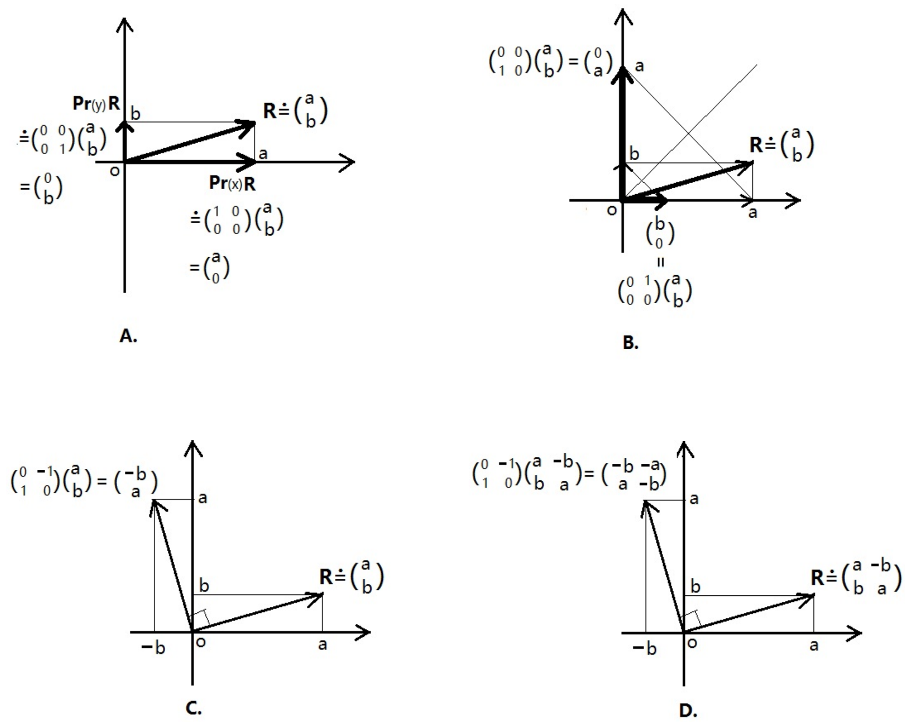

Figure 1

As shown in Figure 1, suppose there is a point in a Euclidean plane, its horizontal and vertical coordinates are a and b. Assume its position-vector is represented by a column-matrix:

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

An operator represented by a matrix acts on the vector, transforms it to a new one, like the three simple examples:

Figure 1.A. Projection-operators and their matrix-representations. They project a vector onto horizontal x axis and vertical y axis separately.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}.$$

Figure 1.B. Exchange-projection-operators. They exchange the vector-projections on x and on y axis for each other.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}.$$

Figure 1.C. 90° counterclockwise rotation operator.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

It is corresponding to the imaginary number unit i in the complex plane.

$$i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad i^2 = -1 \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We can also instead of i with the rotation matrix in Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$, and in the infinitesimal rotation-transformation: $1 + i\epsilon$.

Figure 1.D. We can use a 2×2 matrix to represent the position-vector:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} a \\ b \end{pmatrix}$$

thus we can use a 2×2 matrix with the special form to represent three things: a point in a plane, the position-vector of the point, and an operator which can act on a position-vector. The matrix is corresponding to a complex number in a complex plane:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \sim a + bi.$$

2. Pauli Matrices

Pauli matrices:

$$\sigma_x = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(where $i^2 = -1$), originally act on vectors in 2-dimensional complex space, but we can find that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

can be regarded as matrices (operators) in 2D real space, a real plane, thus we can find their intuitive geometric meanings in an usual plane. (Notice that $i\sigma_y$ is an effective operator as a whole, but there is no direct definition of σ_y in this space in this case.)

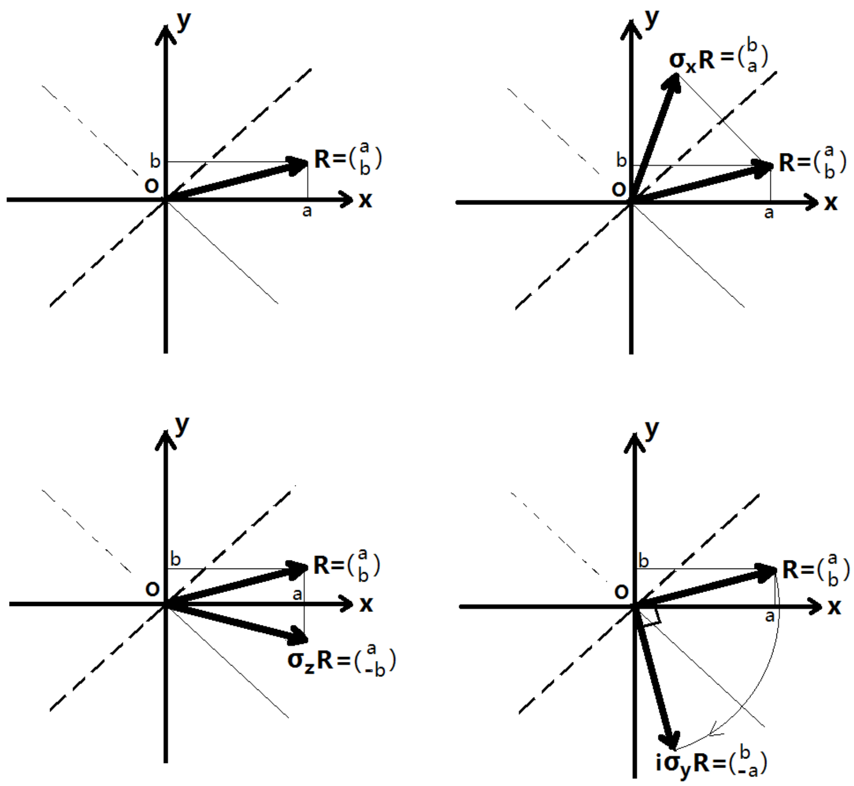

Figure 2

Suppose that the Descartes coordinate of a point is (a, b), where a and b are both real numbers, we now write the representation of its position-vector as:

$$\mathbf{R} \doteq \begin{pmatrix} a \\ b \end{pmatrix}$$

thus

$$\sigma_x \mathbf{R} = \begin{pmatrix} b \\ a \end{pmatrix}, \quad i\sigma_y \mathbf{R} = \begin{pmatrix} b \\ -a \end{pmatrix}, \quad \sigma_z \mathbf{R} = \begin{pmatrix} a \\ -b \end{pmatrix},$$

so, here we can say:

σ_x is a operator which changes a vector to its mirror image about the 45° line (line $y = x$),

$i\sigma_y$ is a operator which rotates a vector with 90° clockwise, and,

σ_z is a operator which changes a vector to its mirror image about the x axis (horizontal axis). As shown in Figure 2.

Because giving a point twice mirror-reflecting about a line can just return to the original one, so we can understand intuitively and immediately:

$\sigma_x \sigma_x \mathbf{R} = \mathbf{R}$, or shortly note: $\sigma_x^2 = 1$, and similarly, $\sigma_z^2 = 1$.

Since operating a vector with twice 90° clockwise rotation just gets its inverse one, so we can write: $(i\sigma_y)^2 = -1$, this indirectly means $\sigma_y^2 = 1$.

Besides $\sigma_x^2 = 1$, $\sigma_z^2 = 1$, and $(i\sigma_y)^2 = -1$, below we can also see these relations by intuitive observation, as shown in Figure 3:

$\sigma_x(i\sigma_y) = -\sigma_z$, $(i\sigma_y)\sigma_z = -\sigma_x$, $\sigma_z\sigma_x = i\sigma_y$, and

$$(i\sigma_y)\sigma_x - \sigma_x(i\sigma_y) = 2\sigma_z$$

$$\sigma_z(i\sigma_y) - (i\sigma_y)\sigma_z = 2\sigma_x$$

$$\sigma_z\sigma_x - \sigma_x\sigma_z = 2(i\sigma_y),$$

that is

$$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x = i[(i\sigma_y)\sigma_x - \sigma_x(i\sigma_y)] = 2i\sigma_z,$$

$$[\sigma_y, \sigma_z] = \sigma_y\sigma_z - \sigma_z\sigma_y = i[\sigma_z(i\sigma_y) - (i\sigma_y)\sigma_z] = 2i\sigma_x,$$

$$[\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y,$$

or

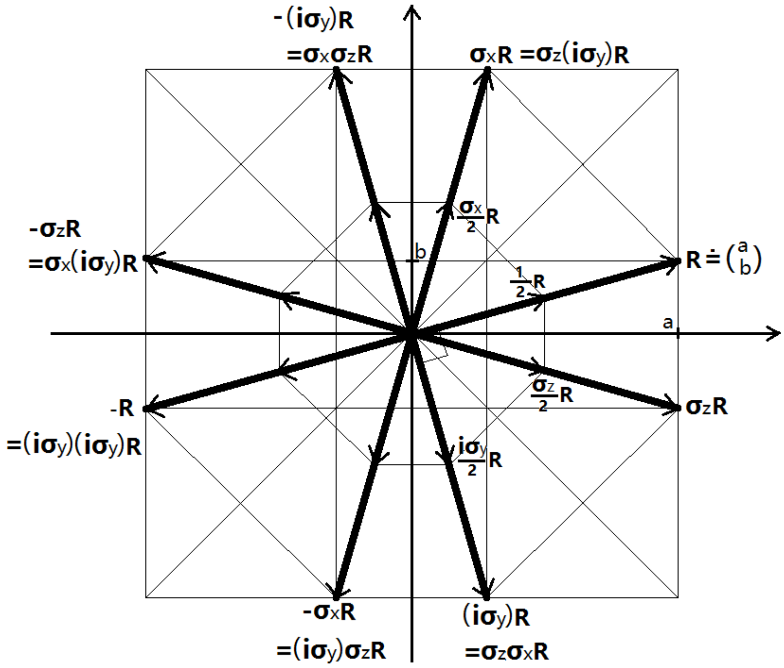

Figure 3

$$[\frac{\sigma_x}{2}, \frac{\sigma_y}{2}] = i\frac{\sigma_z}{2}, \quad [\frac{\sigma_y}{2}, \frac{\sigma_z}{2}] = i\frac{\sigma_x}{2}, \quad [\frac{\sigma_z}{2}, \frac{\sigma_x}{2}] = i\frac{\sigma_y}{2}.$$

We can see the Lie algebra clearly in Figure 2.

3. Eigenvectors and Eigenvalues of σ_z and σ_x

In an intuitive perspective, an operator (or a matrix) usually can change a vector to another, when an operator acts on a vector in a real plane, if it only changes the length but keeps the direction of the vector (just reversing a vector does not been regarded as changing its direction), then the vector is called an eigenvector of the operator. This is just equal to a number multiplying the vector (a negative number multiplying the vector can reverse it, that means the direction unchanged), thus the number is called an eigenvalue of the operator about the eigenvector. Maybe an operator has more than one eigenvectors or eigenvalues, maybe has none of them.

Thus we can intuitively find out:

Each of the vectors on x axis or on y axis is an eigenvector of σ_z, its eigenvalue is 1 or -1; each of the vectors on the 45° line or on the 135° line is an eigenvector of σ_x, its eigenvalue is 1 or -1; and, $i\sigma_y$ has no eigenvector in the plane in this case.

Let us consider $\sigma_z/2$, select two eigenvectors of it:

$$|z+\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |z-\rangle = \begin{pmatrix}0\\1\end{pmatrix},$$

one is on the x axis and another is on the y axis. Correspondingly, there are two eigenvalues, 1/2 and -1/2, of $\sigma_z/2$ about them.

These can be seen in Figure 4.

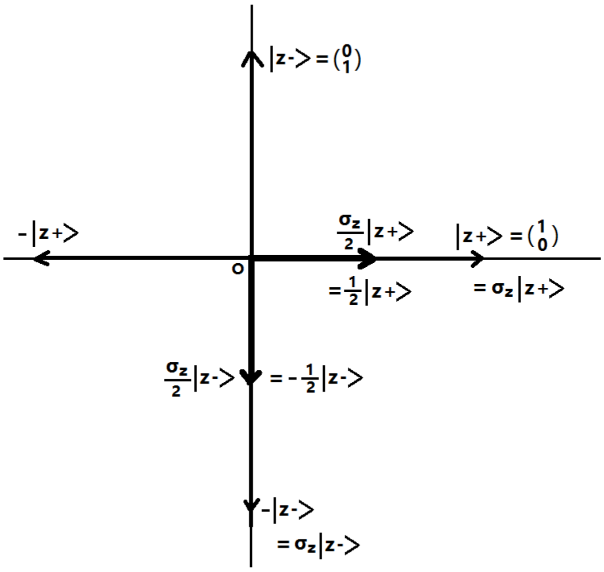

Figure 4

4. Raising and Lowering Operators

Raising operator J_+ and lowering operator J_- are defined as:

$$J_+ = \frac{\sigma_x}{2} + \frac{i\sigma_y}{2} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \quad J_- = \frac{\sigma_x}{2} - \frac{i\sigma_y}{2} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix},$$

In Figure 5, we can see the results of what they act on a vector $\mathbf{R}$. Intuitively, J_+ projects the 45° mirror image of a vector onto y axis, and J_- projects the 45° mirror image of the vector onto x axis:

$$J_+ = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\sigma_x, \quad J_- = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\sigma_x.$$

where

$$\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

are operators of projections onto x and y axes, as in Figure 1.A.

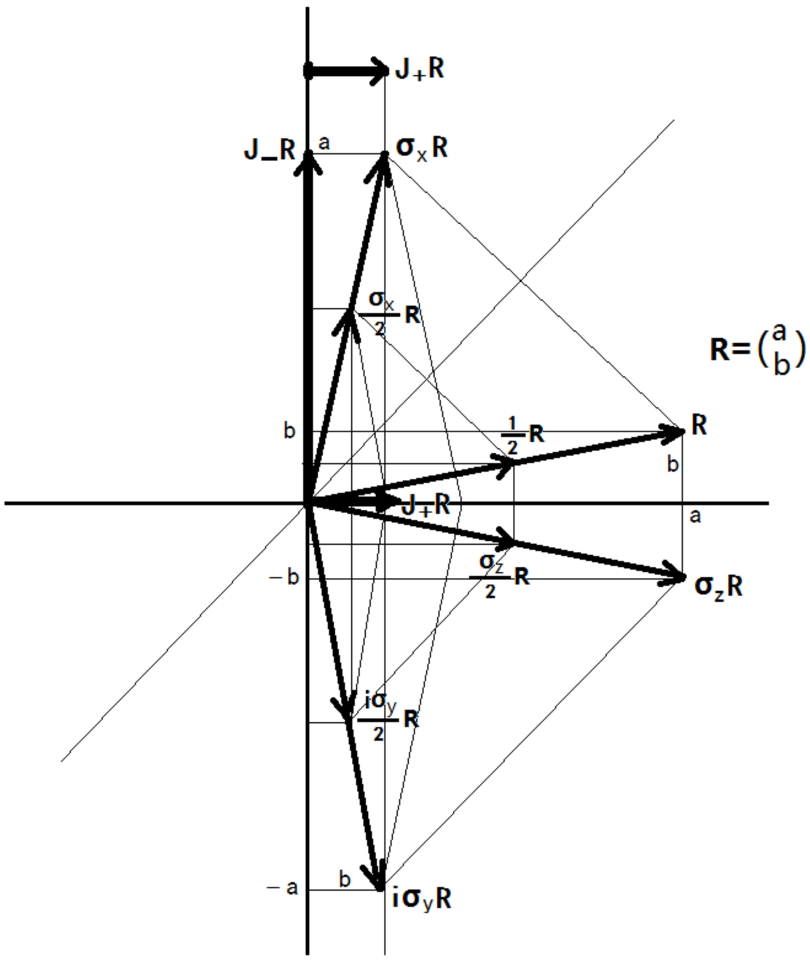

Figure 5

In another words, actually, J_+ first projects a vector onto $|-\rangle$ (or y axis), then rotating the projection clockwise onto $|+\rangle$ (or x axis), and J_- first projects a vector onto $|+\rangle$ (or x axis) then rotating the projection counterclockwise onto $|-\rangle$ (or y axis). They are the operators of **exchange-projection**, as in Figure 1.B.

So we can see immediately, as shown in Figure 6:

$$J_+ \, |z-\rangle = |z+\rangle \,, \quad J_- \, |z+\rangle = |z-\rangle \,,$$

that means, J_+ acting on the eigenvector of $\sigma_z/2$ with -1/2 eigenvalue gets the other one with 1/2 eigenvalue, and J_- acting on the eigenvector of $\sigma_z/2$ with 1/2 eigenvalue gets the other one with -1/2 eigenvalue.

When we consider a space expanded only by $|0\rangle$ and $|1\rangle$ which respectively represent the ground-state and the first exited-state of a scalar field, we can

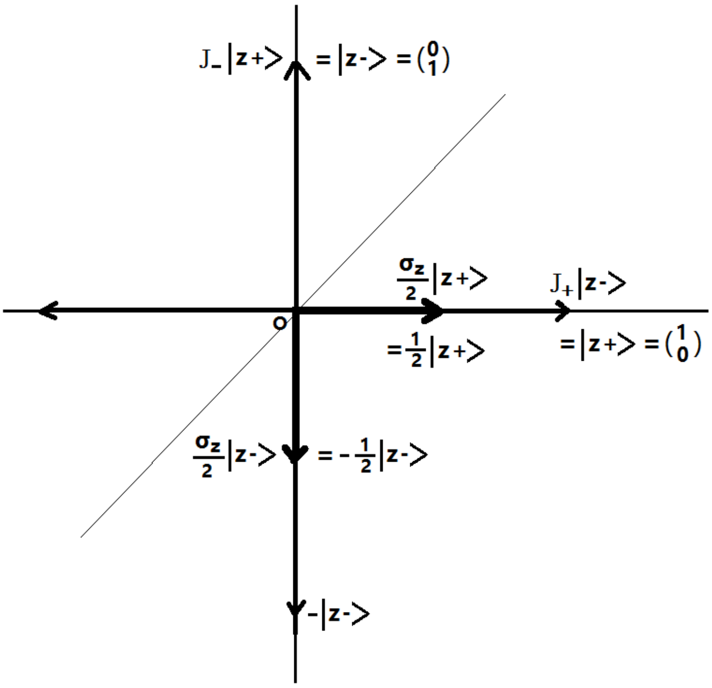

Figure 6

find that its creation operator $\hat{a}^\dagger$ and annihilation operator $\hat{a}$ are very like to the raising operator J_+ and lowering operator J_-. But this sentence is just temporarily correct in a sense, because in fact, there must be infinite numbers of exited-states, not only one. But it can help us to obtain and remember some useful formulas of relations about the relative operators.

As shown in Figure 7, suppose $\hat{a}^\dagger$ is an operator of projecting a vector onto $|0\rangle$ (on x axis) and then rotating the projection counterclockwise onto $|1\rangle$ (on y axis), $\hat{a}$ is an operator of projecting a vector onto $|1\rangle$ (on y axis) then rotating the projection clockwise onto $|0\rangle$ (on x axis), and, $\hat{n}$ is an operator of projecting a vector onto $|1\rangle$ (on y axis). We can see in Figure 6 that:

$$\hat{a}^\dagger \hat{a} R = \hat{n} R, \quad \hat{n}\hat{a}^\dagger R - \hat{a}^\dagger \hat{n} R = \hat{a}^\dagger R, \quad etc.$$

where R is a arbitrary vector, so we can write:

$$\hat{a}^\dagger \hat{a} = \hat{n}, \quad \hat{n}\hat{a}^\dagger - \hat{a}^\dagger \hat{n} = \hat{a}^\dagger, \quad ect.$$

We can also find that an eigenvector $|0\rangle$ of $\hat{n}$ is on x axis, its eigenvalue is 0, and another eigenvector $|1\rangle$ of $\hat{n}$ is on y axis, its eigenvalue is 1:

$$\hat{n}\,|0\rangle = 0\,|0\rangle\,, \quad \hat{n}\,|1\rangle = 1\,|1\rangle\,.$$

5. Symmetric Using with $i\sigma_x$, σ_y and $i\sigma_z$

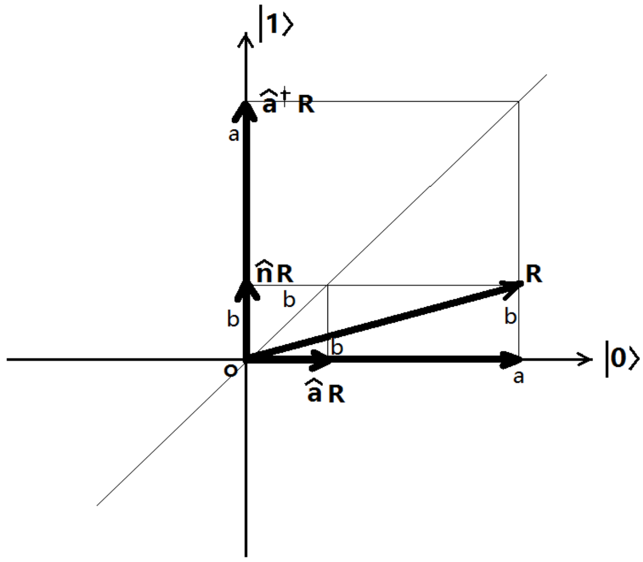

Figure 7

As above, we give (a, b), the coordinates of a point in a real plane, where a and b are real numbers, and give a column matrix to represent its position-vector:

$$\mathbf{R} \doteq \begin{pmatrix} a \\ b \end{pmatrix},$$

and we also effectively give the operating definitions of σ_x, $i\sigma_y$ and σ_z.

Now let we change the form of representations of the vectors, with other forms of space:

1) As shown in Figure 8, we know (a, b) is the coordinates of a point in a real plane, where a and b are real numbers, now we give a new form of representation of its position-vector like this:

$$\mathbf{R} \doteq \begin{pmatrix} a \\ bi \end{pmatrix},$$

where $i^2 = -1$, then we can give the definitions of operators correspondingly:

$$i\sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

When $i\sigma_x$, σ_y or σ_z acts on $\mathbf{R}$:

$$i\sigma_x\mathbf{R} = \begin{pmatrix} -b \\ ai \end{pmatrix}, \quad \sigma_y\mathbf{R} = \begin{pmatrix} b \\ ai \end{pmatrix}, \quad \sigma_z\mathbf{R} = \begin{pmatrix} a \\ -bi \end{pmatrix},$$

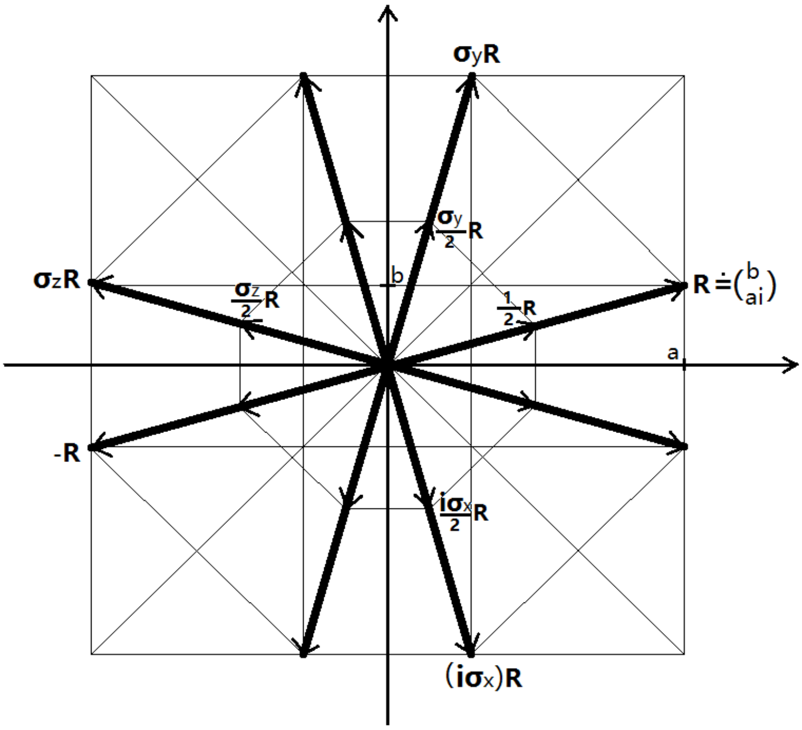

Figure 8

means that the point (a, b) is respectively changed to $(-b, a)$, (b, a) and $(a, -b)$, correspondingly, its position-vector is respectively changed with $90°$ counterclockwise rotation, to the mirror image about $45°$ line and the mirror image about x axis.

We can find that the algebraic relationships between σ_x, σ_y and σ_z are the same as in the section 1.

2) As shown in Figure 9, (a, b) still is the coordinates of a point in a real plane, where a and b are real numbers, once again we give a new form of representation of its position-vector as below:

$$\mathbf{R} \doteq \begin{pmatrix} a+bi \\ a-bi \end{pmatrix},$$

where $i^2 = -1$, since

$$\sigma_x \mathbf{R} = \begin{pmatrix} a+(-b)i \\ a-(-b)i \end{pmatrix}, \quad \sigma_y \mathbf{R} = \begin{pmatrix} (-b)+(-a)i \\ (-b)-(-a)i \end{pmatrix}, \quad i\sigma_z \mathbf{R} = \begin{pmatrix} (-b)+ai \\ (-b)-ai \end{pmatrix},$$

in this case, we can find that σ_x, σ_y or $i\sigma_z$ changes a position-vector to its mirror image about x axis, about $135°$ line or with a $90°$ counterclockwise rotation.

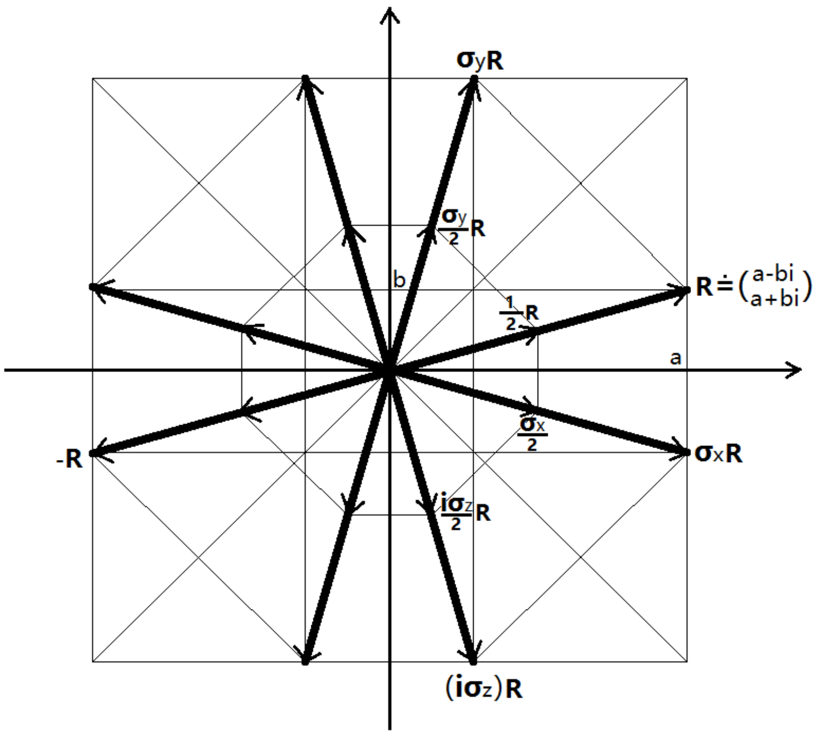

Figure 9

We can also find that the algebraic relationships between σ_x, σ_y and σ_z are the same as in the section 1.

6. Pauli Matrices, Split-Complex Number and Lorentz Transformation

It is well known that a complex number can be represented with a 2×2 real matrix. Generally, some square matrices not only can be an operator acting on position-vectors, but also can be the representation of the position-vector itself.

We now suppose there are three 1+1 dimensional planes separately, they are t-x plane, t-y plane and t-z plane. There are three points separately in each of the planes, their coordinates separately are (t, x), (t, y) and (t, z), where x, y, z and t are all real numbers. As shown in Figure 10.

1) In t-x plane, suppose the form of the representation of the position-vector of a point (t, x) is:

$$\mathbf{R}(t, x) \doteq t + x\sigma_x = \begin{pmatrix} t & x \\ x & t \end{pmatrix}$$

thus

$$\sigma_x \mathbf{R}(t, x) = \sigma_x(t + x\sigma_x) = x + t\sigma_x = \begin{pmatrix} x & t \\ t & x \end{pmatrix}$$

So we know that in this system of the representation in this plane, σ_x changes a vector to its mirror image about 45° line.

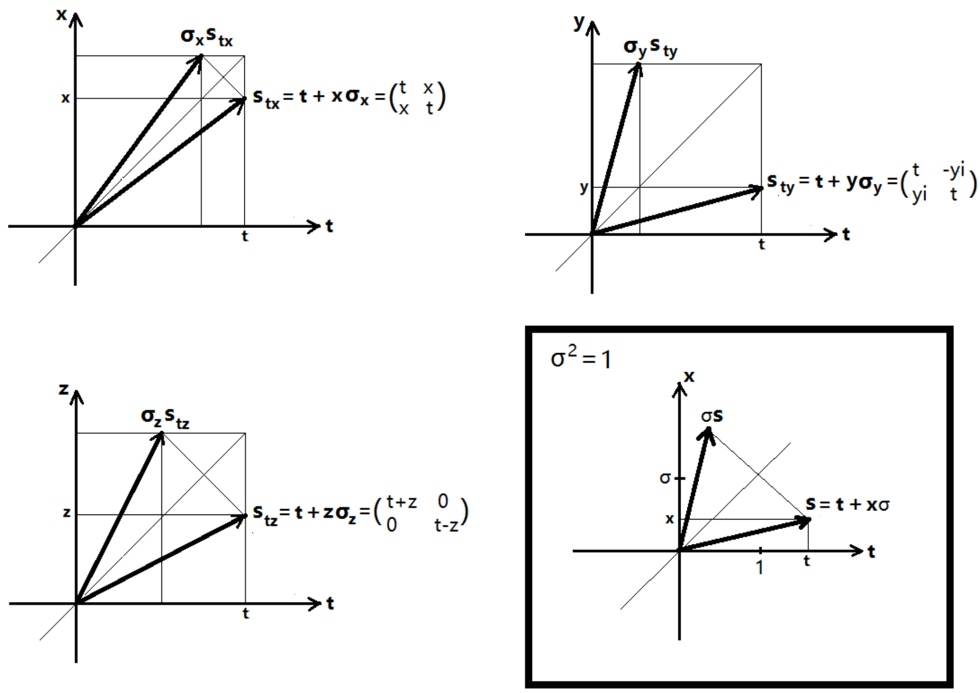

Figure 10

2) In t-y plane, suppose the form of the representation of the position-vector of a point (t, y) is:

$$\mathbf{R}(t, y) \doteq t + y\sigma_y = \begin{pmatrix} t & -yi \\ yi & t \end{pmatrix}$$

thus

$$\sigma_y \mathbf{R}(t, y) = \sigma_y(t + y\sigma_y) = y + t\sigma_y = \begin{pmatrix} y & -ti \\ ti & y \end{pmatrix}$$

So we know that in this system of the representation in this plane, σ_y also changes a vector to its mirror image about 45° line.

3) In t-z plane, suppose the form of the representation of position-vector of a point (t, z) is:

$$\mathbf{R}(t, z) \doteq t + z\sigma_z = \begin{pmatrix} t + z & 0 \\ 0 & t - z \end{pmatrix}$$

thus

$$\sigma_z \mathbf{R}(t, z) = \sigma_z(t + z\sigma_z) = z + t\sigma_z = \begin{pmatrix} z + t & 0 \\ 0 & z - t \end{pmatrix}$$

So we know that σ_z also changes a vector to its mirror image about 45° line, in this plane and in this system of representation.

We can give an united description: giving a point in a plane, if its coordinates is (t, x), where t and x are real number, then its position vector $\mathbf{S}$ can be represented by:

$$\mathbf{S} = t + x\sigma, \quad \text{where} \quad \sigma^2 = 1.$$

This actually is the **split-complex number**. Where σ is traditionally noted by j, now we know its geometric meaning: as a operator, σ changes a vector to its mirror image about $45°$ line, as a vector, σ represents the position-vector corresponding to the point with Descartes coordinates $(0, 1)$.

Because $\sigma^2 = 1$, by using the Taylor expansion, we have:

$$e^{\sigma\theta} = \cosh\theta + \sigma \sinh\theta$$

where θ is a real parameter.

So the position vector of each point (t, x) in t-x plane can be denoted:

$$\mathbf{S} = t + x\sigma$$

when $t > x$, it can be noted as

$$\mathbf{S} = t + x\sigma = \tau e^{\sigma\theta} = \tau(\cosh\theta + \sigma \sinh\theta),$$

here, $t = \tau\cosh\theta$, $x = \tau\sinh\theta$, τ is a real number.

when $t < x$, it can be noted as

$$\mathbf{S} = t + x\sigma = s\sigma e^{\sigma\theta} = s(\sinh\theta + \sigma \cosh\theta),$$

here, $t = s\sinh\theta$, $x = s\cosh\theta$, s is a real number.

When $\tau = const.$, $s = const.$, and only θ changes continuously, we get a hyperbolic curve. If we give a new definition of "distance", or "square of distance": $s^2 = t^2 - x^2$, we can say that the distances from O to every of the points on a hyperbolic curve are equal. They are in a **hyperbolic plane**.

In (1+1) space-time of special theory of relativity, suppose each of all different inertial coordinate systems moves at different relative speeds, and an event $\mathbf{O}$ is that each origins of all the coordinate-systems coincide at the time of 0 in every system, let us consider another event $\mathbf{S}$. The sets of the records of the space-time coordinates of the same one event $\mathbf{S}$ in all systems (at different speeds), can form a hyperbolic curve (we let the speed of light $c = 1$):

1) If $t > x$, $\mathbf{S}$ is a time-like event, as shown in Figure 11 and 12, we have:

$$\mathbf{S} = \tau e^{\sigma\theta} = \tau(\cosh\theta + \sigma \sinh\theta),$$

because $\cosh^2\theta - \sinh^2\theta = 1$, so

$$t^2 - x^2 = \tau^2, \quad (\tau \text{ is the proper time}).$$

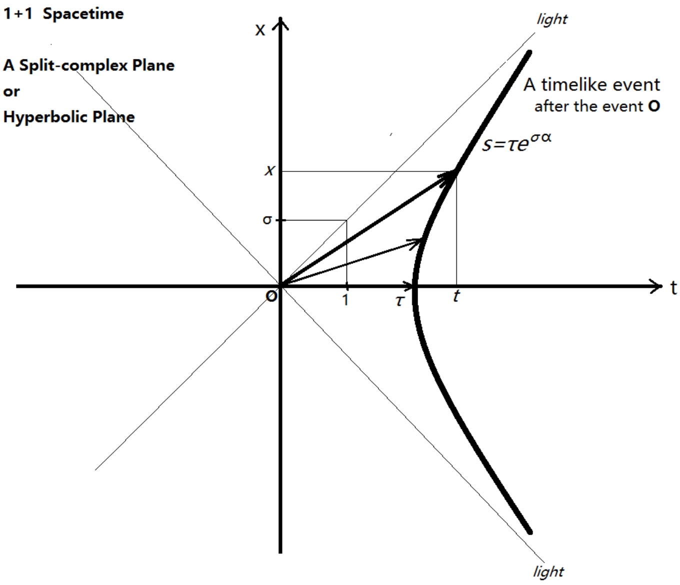

Figure 11

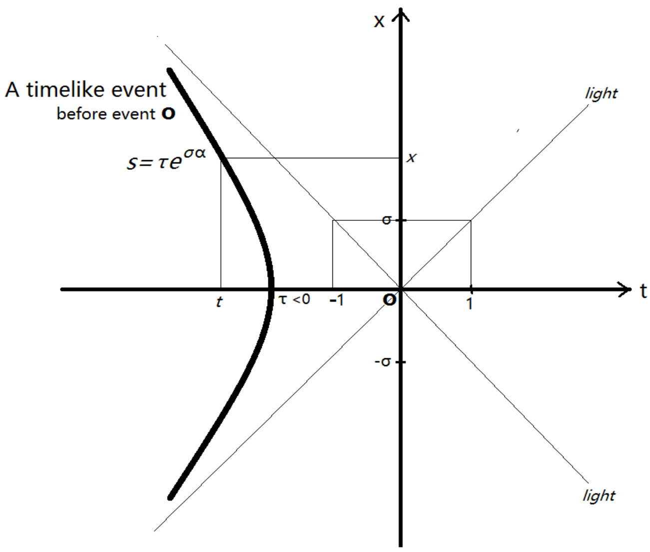

Figure 12

2) If $x > t$, $\mathbf{S}$ is a space-like event, as shown in Figure 13 and 14, we note:

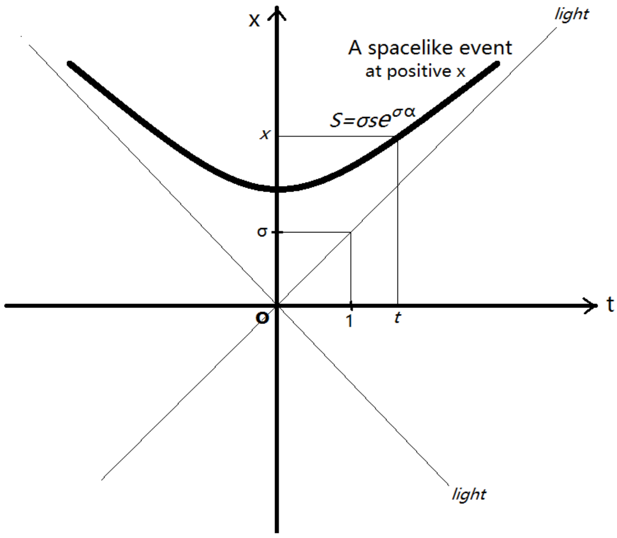

Figure 13

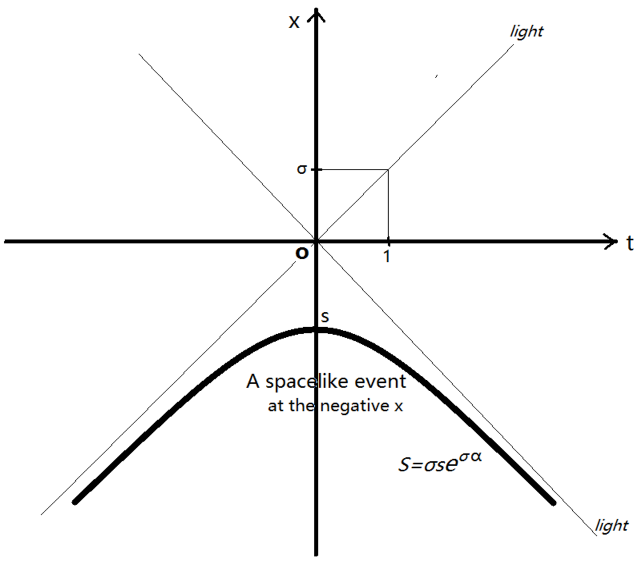

Figure 14

$$\mathbf{S} = s\sigma e^{\sigma\theta} = s(\sinh\theta + \sigma\cosh\theta).$$

$$x^2 - t^2 = s^2, \qquad (s \text{ is the proper length}).$$

Notice that in this way 1) or 2), all points in one hyperbolic curve just represent **one event**, different points in the same hyperbolic curve are only different records of space-time-coordinates from different reference-systems for **one event**. By the way, we can clearly see the relativity of simultaneity in Figure 13 or 14, just see the even O and another event represented by a point in the curve, have or do not have the same coordinate $t = 0$.

As shown in Figure 15, the relation between the two points in one hyperbolic curve line, is the coordinates transformation, and just is the **Special Lorentz Transformation**:

$$S' = e^{\sigma\phi} S, \quad \text{where} \tanh\phi = v/c,$$

(we let the speed of light c=1 in this article.)

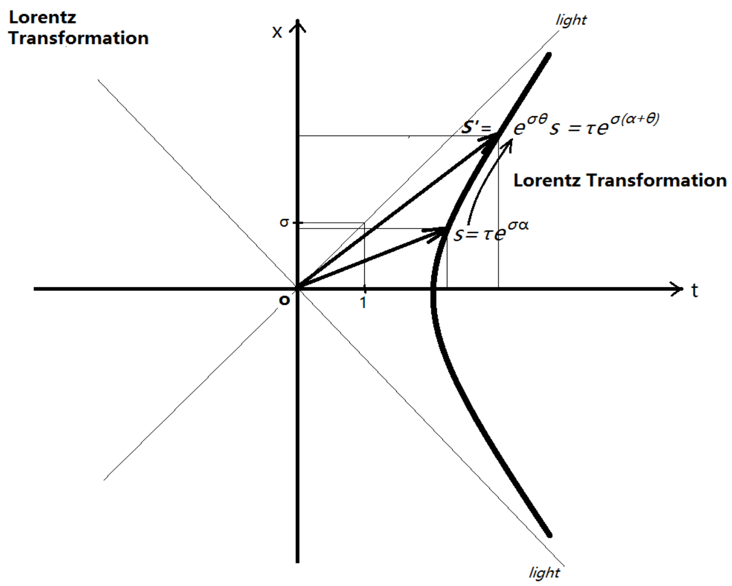

Figure 15

Now from

$$\tanh(\alpha - \beta) = \frac{\tanh\alpha - \tanh\beta}{1 - \tanh\alpha \tanh\beta},$$

we can get the formula of relative speeds transformation:

$$v_{12} = \frac{v_1 - v_2}{1 - v_1 v_2}$$

7. Spinor and Dirac Equation

We can also represent the energy-momentum space with the hyperbolic plane, different points Z in a hyperbolic curve are different records of energy-momentum of a body with m mass ($m = const.$) from different systems of reference at different speeds in 1+1 space-time. As shown in Figure 16,

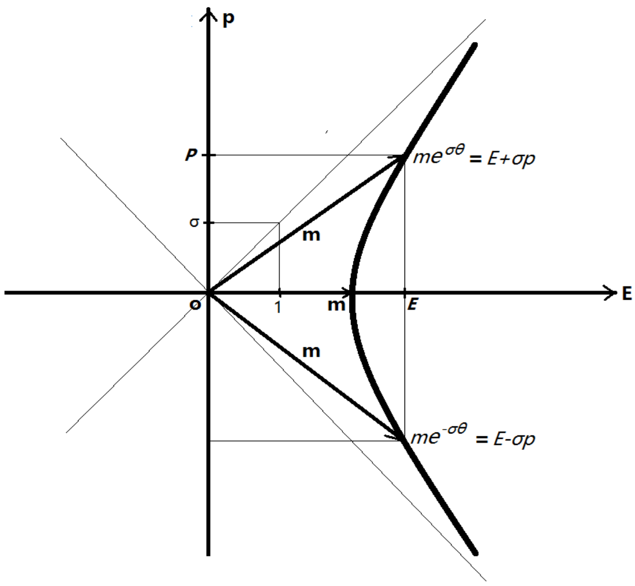

Figure 16

$$Z = E + \sigma p = me^{\sigma\theta} = m(\cosh\theta + \sigma\sinh\theta)$$

so

$$E^2 - p^2 = m^2 = const.$$

Notice the operator of transformation from stasis to movement:

$$e^{\sigma\theta} = \frac{E + \sigma p}{m},$$

its conjugate:

$$e^{-\sigma\theta} = \frac{E - \sigma p}{m}$$

represents the operator of transformation from stasis to the inverse movement.

Further more, we can represent a point in 3+1 D Minkowski space by:

$$S = t + x\sigma_x + y\sigma_y + z\sigma_z$$

A point in the energy-momentum space can be represented by:

$$Z = E + \sigma_x p_x + \sigma_y p_y + \sigma_z p_z = E + \sigma^1 p_1 + \sigma^2 p_2 + \sigma^3 p_3 = E + \sigma^i p_i,$$

$$Z = me^{\sigma^i \theta_i} = E + \sigma^i p_i,$$

and its conjugate, representing the m moving inversely:

$$Z^* = me^{-\sigma^i \theta_i} = E - \sigma^i p_i$$

Now suppose $e^{\sigma^i \theta_i}$ and its conjugate $e^{-\sigma^i \theta_i}$ act in another hyperbolic space called the spinor space, a spinor has two components φ and ψ, as shown in Figure 17, and they satisfy:

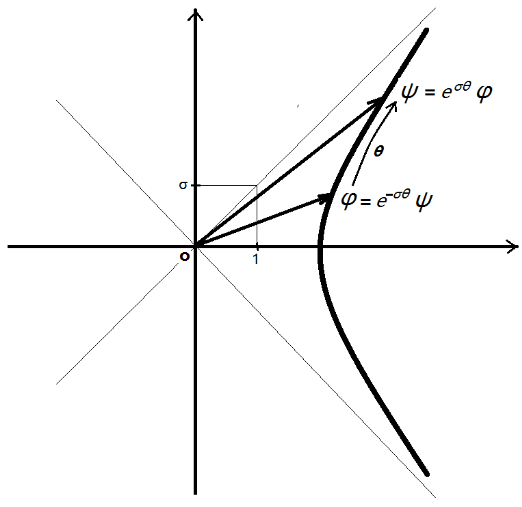

Figure 17

$$\begin{cases} e^{\sigma^i \theta_i} \varphi = \psi \\ e^{-\sigma^i \theta_i} \psi = \varphi \end{cases} \quad \text{or} \quad \begin{cases} \frac{E+\sigma^i p_i}{m} \varphi = \psi \\ \frac{E-\sigma^i p_i}{m} \psi = \varphi \end{cases}$$

write in form of matrix:

$$\begin{pmatrix} -m & (E + \sigma^i p_i) \\ (E - \sigma^i p_i) & -m \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = 0$$

or

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} E - \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} p_i - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} m \right] \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = 0$$

by quantum mechanics, $E \to i\frac{\partial}{\partial x^0} = i\partial_0$, and $p_i \to -i\frac{\partial}{\partial x^i} = -i\partial_i$, so we can obtain the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\phi = 0$$

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad and \quad \phi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$$

So two components of Dirac spinors or $SU2 \otimes SU2$ are related by unit transformations in a split-complex plane (a hyperbolic plane), just like in the plane of energy-momentum or 1+1 space-time.

8. $i\sigma$ and rotations

Now let we consider $i\sigma_x, i\sigma_y$, and $i\sigma_z$ again.

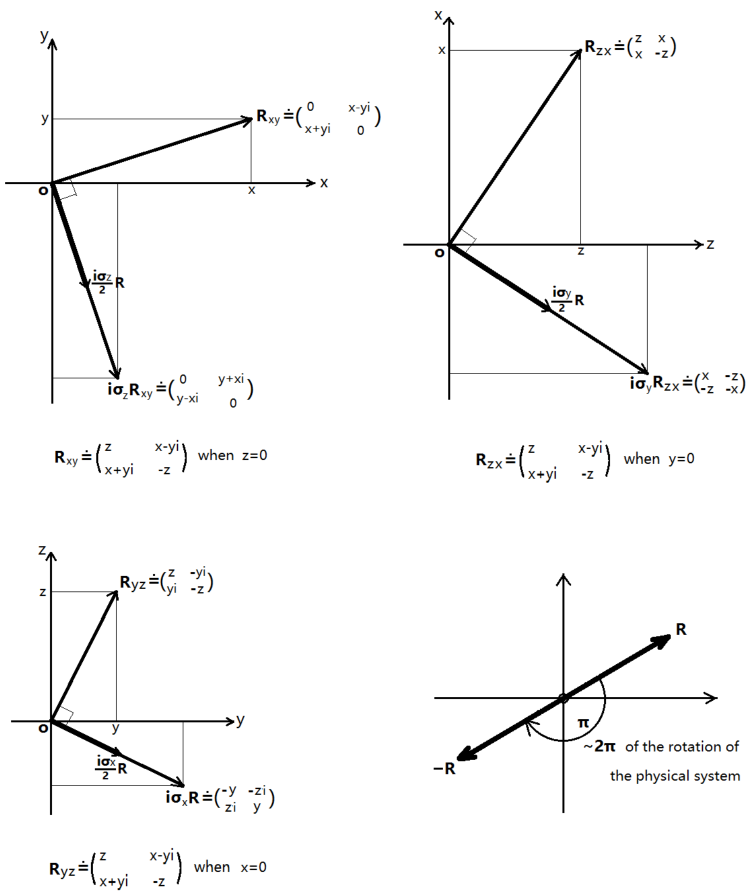

Figure 18

In the corresponding spaces of representation as shown in Figure 18, σ_x, σ_y and σ_z can rotate the vector with 90° clockwise separately in three different planes. Just like the function of the unit complex numbers in a complex plane, $e^{i\sigma_\alpha/2}$, $e^{i\sigma_\beta/2}$ and $e^{i\sigma_\gamma/2}$ also can represent rotations with $\alpha/2$, $\beta/2$ and $\gamma/2$

angles in its space.

Since R and $e^{i\sigma\pi}R = -R$ in these spaces can be corresponding to the relation of 2π rotation of the physical system (that is identical system), so we use like *angle*/2 phase form in these spaces (spinor spaces), the *angle* is just corresponding to the angle of the rotation of the physical system.

Let

$$\hat{z} = -\frac{i\sigma_z}{2}, \quad \hat{y} = -\frac{i\sigma_y}{2}, \quad \hat{x} = -\frac{i\sigma_x}{2},$$

then we have

$$[\hat{x}, \hat{y}] = \hat{z}, \quad [\hat{y}, \hat{z}] = \hat{x}, \quad [\hat{z}, \hat{x}] = \hat{y}.$$

9. U_1 and SO_2

U_1 is a rotation group, a Lie group, but now we call its elements U_1 too. U_1 acts on a vector in a complex plane (that is 1D complex space), changes the vector but maintains its length. The vector in a complex plane can be represented by a complex number $z = re^{i\phi}$, so U_1 changes the phase of the complex number but keeps its modulus length unchanged. Obviously, a representation of U_1 is $e^{-i\theta}$. Selecting $-\theta$ means the vector rotating clockwise (negative angle), corresponding to the coordinate system rotating anti-clockwise (positive angle) relatively.

Let we transform the coordinate system to 2D real plane:

$$i \sim \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad z = re^{i\phi} = x + yi \sim \begin{pmatrix} x \\ y \end{pmatrix}$$

thus

$$e^{-i\theta} = \cos\theta - i\sin\theta \sim \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad e^{-i\theta}z \sim \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

notice:

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1,$$

All of these mean:

$$U_1 \cong SO_2$$

When SO_2 acts on a position-vector of the point (x, y) in a 2D real plane, it only changes the direction of the vector, keeping its length unchanged. That is $x'^2 + y'^2 = x^2 + y^2$, and the determinant of the representation-matrix of SO_2 is 1.

In Figure 19, we let $r_1 = x, \quad r_2 = y$ both lie down in x axis, using an ordered pair of vectors in only one real axis instead two axis, to represent a vector in a 2D real plane, can also maintain their algebraic relations.

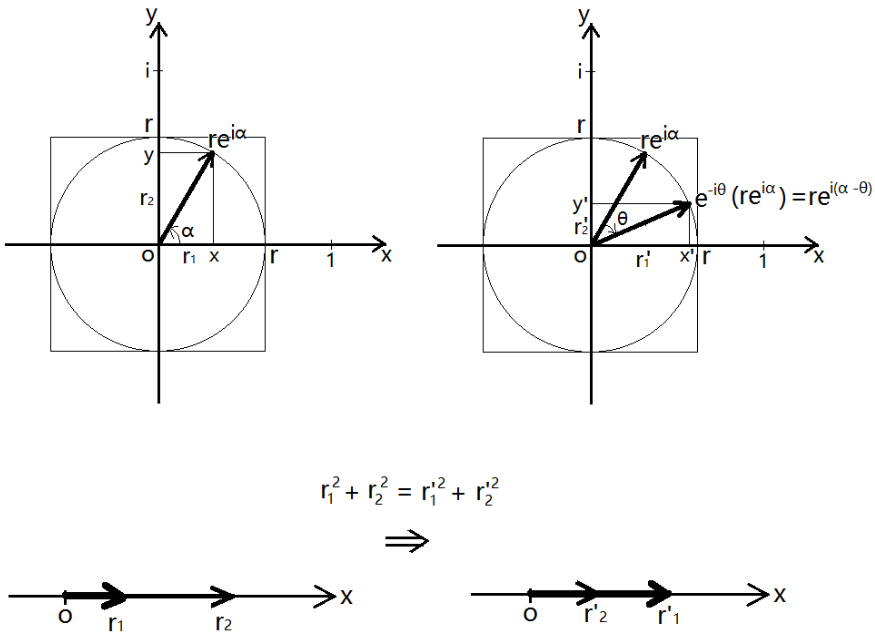

Figure 19

10. Intuitive Meanings of SU_2 in a Plane

SU_2 originally acts on the vectors in 2D complex space. But we can use it to act on an ordered pair of complex numbers in one complex plane.

U_2 acts on an ordered pair of complex numbers $(r_1e^{i\theta_1/2}, r_2e^{i\theta_2/2})$, changes it to $(r'_1e^{i\theta'_1/2}, r'_2e^{i\theta'_2/2})$, but must satisfy ${r'_1}^2 + {r'_2}^2 = {r_1}^2 + {r_2}^2$, where r and $\theta/2$ are real numbers. Besides this condition, SU_2 must satisfy: the determinant of its representation-matrix is 1. We will explain again why we use 1/2 angles later.

At first, we have

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1e^{i\theta_1/2} \\ r_2e^{i\theta_2/2} \end{pmatrix}$$

this is an arbitrary ordered pair of complex numbers. How can we make some operations, to realize that:

$$\begin{pmatrix} r_1e^{i\theta_1/2} \\ r_2e^{i\theta_2/2} \end{pmatrix} \longrightarrow \begin{pmatrix} r \\ r \end{pmatrix}, \quad \text{where} \quad r_1^2 + r_2^2 = 2r^2,$$

the right column is an ordered pair of real numbers which is a special ordered pair of complex numbers both in the real axis, it can be called the goal-vector, or the basic form. On the contrary, we can begin from the basic form vector, change it to an arbitrary vector as the left side, maintaining the sum of the square of the two modulus-lengths.

As shown in Figure 20, our plan includes three steps: **1) only change the phases of the two complex numbers reversely, make them both onto their angular bisector simultaneously, maintaining their modulus-lengths, 2) then only change their modulus-lengths both to r, maintaining their common phase, 3) finally only change their new common phase to 0° simultaneously.**

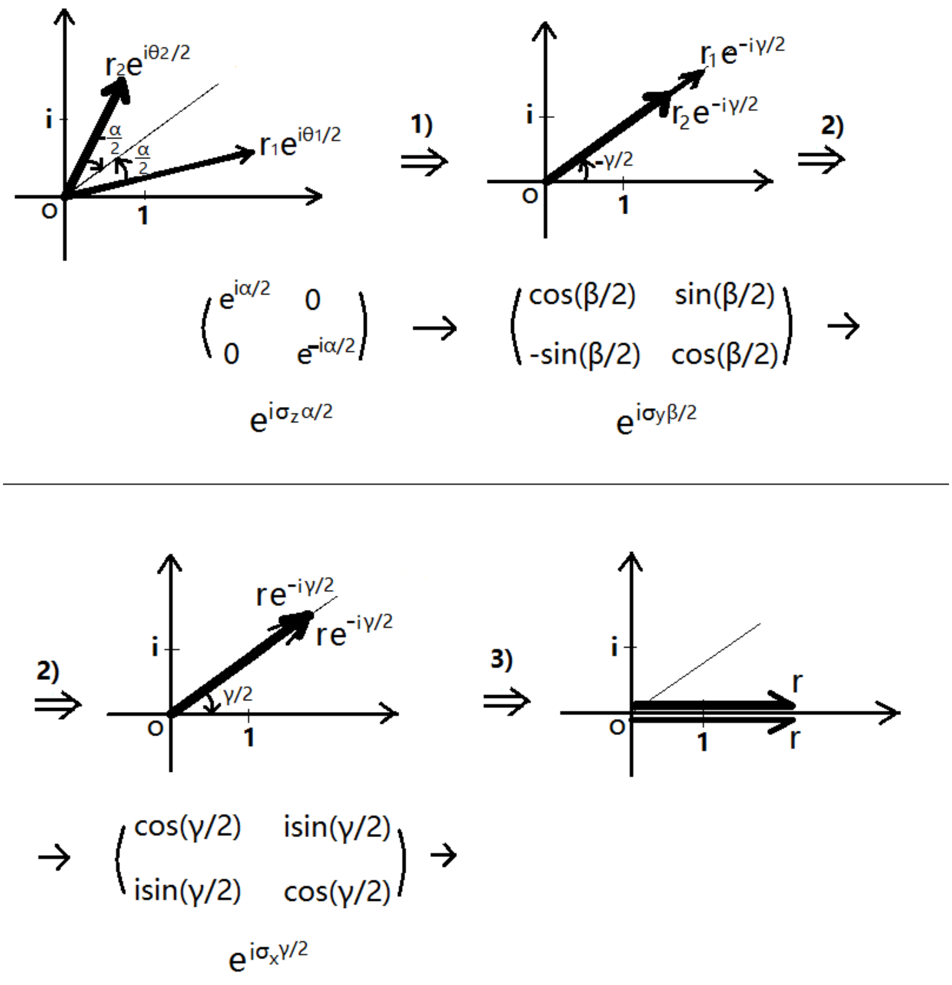

Figure 20

Any pair of complex numbers can be changed by the above three steps to following basic form: both are on the real axis, both lengths are equal. And next, with the reversing process, we can continue to change the basic form to another arbitrary form if we need: with any possible relative lengths, then with any possible common phase, then with any possible relative phases, and, satisfying the condition of what the determinant of the representation-matrix of the transformation is 1.

1) The 1st step can be carried out by:

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

Obviously this matrix is belong to SU_2, it can just change the phases of the two complex numbers reversely with $\pm\alpha/2$ angles at the same time.

Let suppose the relative angular from z_2 to z_1 is ϕ, and $\lambda/2$ is the phase-angle of their angular bisector, that means:

$$\frac{\phi}{2} = \frac{\theta_1/2 - \theta_2/2}{2}, \quad \frac{\lambda}{2} = \frac{\theta_1/2 + \theta_2/2}{2}$$

and when we select $\alpha = -\phi$, we have:

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} r_1 e^{i\theta_1/2} \\ r_2 e^{i\theta_2/2} \end{pmatrix} = \begin{pmatrix} r_1 e^{i\lambda/2} \\ r_2 e^{i\lambda/2} \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

can be represented by an exponential form. Suppose a matrix $A(\theta)$ with continuous parameter θ, assume $A = e^{iS\theta}$, where S is a constant matrix called the generator, ($detA = 1 \Leftrightarrow trS = 0$), and because $dA/d\theta = iSe^{iS\theta}$, let $\theta = 0$, we get

$$S = -i(dA/d\theta)\,|_{\theta=0}$$

in this way we can obtain:

$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} = e^{i\sigma_z \alpha/2},$$

$\sigma_z/2$ is the generator of this Lie group of transformation, σ_z is one of the Pauli matrices.

2) Now we will change the two modulus-lengths r_1 and r_2 in

$$\begin{pmatrix} r_1 e^{i\lambda/2} \\ r_2 e^{i\lambda/2} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} e^{i\lambda/2}$$

to an equal value r, maintaining their sum of square of their lengths, this means

$$2r^2 = (\sqrt{2}r)^2 = r_1^2 + r_2^2,$$

consider a right-triangle, the lengths of its three sides are r_1, r_2 and $\sqrt{2}r$, now maintain its hypotenuse but change the two rectangle-sides, or, relatively consider the rotation of the radius of a circle, recalling SO_2 in Section 7, we have:

$$\begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} e^{i\lambda/2} = \begin{pmatrix} r_1\cos\beta/2 + r_2\sin\beta/2 \\ -r_1\sin\beta/2 + r_2\cos\beta/2 \end{pmatrix} e^{i\lambda/2}$$

obviously the left matrix is also belong to SU_2, its exponential form is $e^{i\sigma_y\beta/2}$, where σ_y is one of the Pauli matrices, and $\sigma_y/2$ is the generator of this Lie group of transformation.

When we select a proper $\beta/2$, we can get:

$$r_1\cos\beta/2 + r_2\sin\beta/2 = -r_1\sin\beta/2 + r_2\cos\beta/2 = r$$

Now we obtain (through the two steps) that:

$$\begin{pmatrix} r_1 e^{i\theta_1/2} \\ r_2 e^{i\theta_2/2} \end{pmatrix} \longrightarrow \begin{pmatrix} r_1 e^{i\lambda/2} \\ r_2 e^{i\lambda/2} \end{pmatrix} \longrightarrow \begin{pmatrix} r e^{i\lambda/2} \\ r e^{i\lambda/2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} r e^{i\lambda/2}$$

respectively by the actions of $e^{i\sigma_z\alpha/2}$ and $e^{i\sigma_y\beta/2}$.

3) Finally we will rotate the two equal complex numbers onto x axis together. By using Euler's formula,

$$\begin{pmatrix} \cos\gamma/2 & i\sin\gamma/2 \\ i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} r e^{i\lambda/2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} r e^{i(\lambda/2+\gamma/2)}$$

when $\gamma = -\lambda$, we obtain the goal-vector, or the basic vector:

$$\begin{pmatrix} r \\ r \end{pmatrix}$$

We can also find:

$$\begin{pmatrix} \cos\gamma/2 & i\sin\gamma/2 \\ i\sin\gamma/2 & \cos\gamma/2 \end{pmatrix} = e^{i\sigma_x\gamma/2}$$

is belong to SU_2, where $\sigma_x/2$ is the generator of the Lie group of the transformation, σ_x is one of the Pauli matrices.

Thus, with the operations of 1), 2) and 3) orderly, in other words, with matrices $e^{i\sigma_z\alpha/2}$, $e^{i\sigma_y\beta/2}$ and $e^{i\sigma_x\gamma/2}$, we can change any of the ordered pair of complex numbers to the basic form, or by using the steps reversely can change the basic form to an arbitrary ordered-pair of complex numbers, and satisfy the conditions of SU_2.

Now we explain why using $\alpha/2$, $\beta/2$ and $\gamma/2$:

a)The state-vector $|r\rangle$ and its inverse $-|r\rangle = e^{i\pi}|r\rangle$ represent the same state of a physical system, that is to say, the phase (of the state-vector) rotating with

π, can be corresponding to 2π rotation of the physical system.

b)The generators of $e^{i\sigma_x\alpha/2}$, $e^{i\sigma_y\beta/2}$ and $e^{i\sigma_z\gamma/2}$ are $\sigma_x/2$, $\sigma_y/2$ and $\sigma_z/2$, their Lie algebras (or their commutation-relations) is as the same as of rotations of real 3D object. This will be manifested in the next section.

11. SO_3

SO_3 is not just a rotation of 3D vector, but a rotation of 3D rigid body in real space, so it always needs three parameters. SO_3 can be regarded as rotations orderly about three axes of a Descartes coordinates xyz which is going to be changed to $x'y'z'$. We wish the determinant of its representation-matrix is 1. The changing of the coordinates of a point $(x, y, z) \to (x', y', z')$ must be satisfied that:

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

We use the right-hand system of coordinates: when we rotate the 3D coordinate system 90° around z axis, the positive direction of x axis is just changed to the positive direction of the original y axis, and we denote this angle of rotation $-90°$, since it is corresponding to a vector in the coordinates rotating $-90°$.

As shown in Figure 21, we regard the $x'y'z'$ system as the goal system, or basic system, we will change another arbitrary system xyz to it by three rotating steps: 1)rotate α angle around the z axis, 2)rotate β around the new y axis, 3)rotate γ around the newer x axis, and when we select the proper values of these angles, we can change xyz system to the appointed $x'y'z'$. Of course, the reverse process can be used to change the basic system to any other systems, and all of these steps can satisfy the condition of SO_3 .

Notice that each of the three steps can be corresponding to the step about SU_2 in the section 8. The three steps can be represent by three 3D real matrices, they act on a vector successively, change it to another:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

it can be denoted in exponential form:

$$e^{iJ_x\gamma} e^{iJ_y\beta} e^{iJ_z\alpha} \vec{R} = \vec{R'}$$

the three matrices of generators J_z, J_y and J_x can be obtain with the method in section 9. We have :

$$J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

we can find

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y,$$

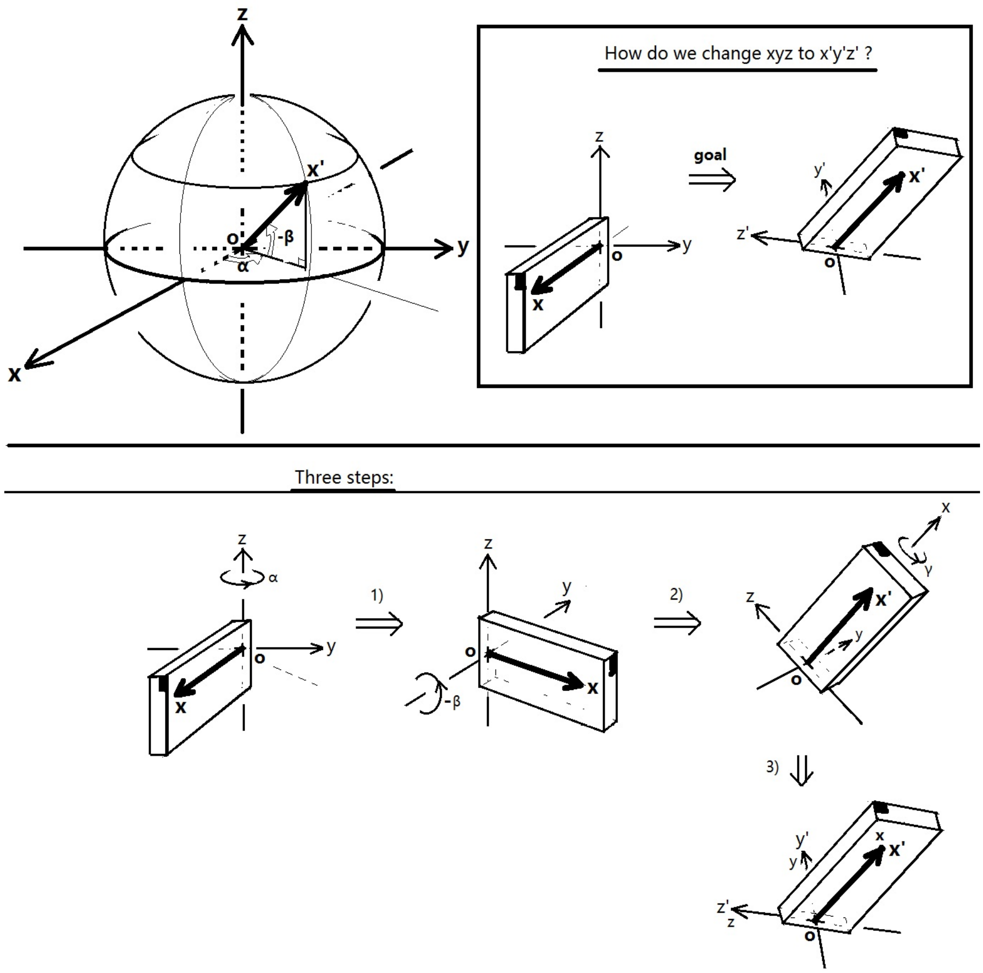

Figure 21

just as the same as:

$$[\frac{\sigma_x}{2},\frac{\sigma_y}{2}]=i\frac{\sigma_z}{2},\quad [\frac{\sigma_y}{2},\frac{\sigma_z}{2}]=i\frac{\sigma_x}{2},\quad [\frac{\sigma_z}{2},\frac{\sigma_x}{2}]=i\frac{\sigma_y}{2},$$

Notice

$$i\sigma_z=e^{i\frac{\sigma_z}{2}\pi}=e^{i\sigma_z\frac{\pi}{2}}\sim e^{iJ_z\pi},\qquad e^{i\frac{\sigma_z}{2}\alpha}=e^{i\sigma_z\frac{\alpha}{2}}\sim e^{iJ_z\alpha},$$

$$i\sigma_y=e^{i\frac{\sigma_y}{2}\pi}=e^{i\sigma_y\frac{\pi}{2}}\sim e^{iJ_y\pi},\qquad e^{i\frac{\sigma_y}{2}\beta}=e^{i\sigma_y\frac{\beta}{2}}\sim e^{iJ_y\beta},$$

$$i\sigma_x=e^{i\frac{\sigma_x}{2}\pi}=e^{i\sigma_x\frac{\pi}{2}}\sim e^{iJ_x\pi},\qquad e^{i\frac{\sigma_x}{2}\gamma}=e^{i\sigma_x\frac{\gamma}{2}}\sim e^{iJ_x\gamma},$$

and when we denote $Z=-iJ_z,\quad Y=-iJ_y,\quad X=-iJ_x,\quad$ we have

$$Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

we get three real matrices. They act on a vector $\mathbf{R}$ in 3D real space, will project $\mathbf{R}$ to the three plane xy, yz and zx respectively, get and then rotate $\mathbf{R}_{xy}$, $\mathbf{R}_{yz}$ and $\mathbf{R}_{zx}$ with 90° clockwise.

These can be seen in Figure 22 intuitively:

$$[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y,$$

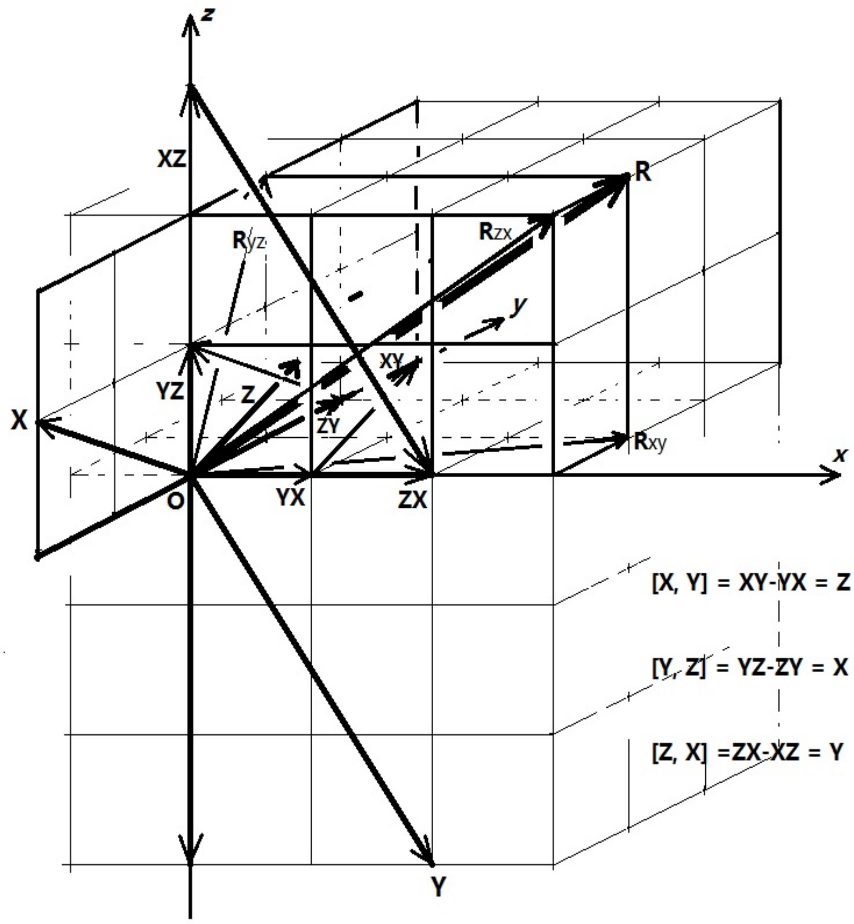

Figure 22

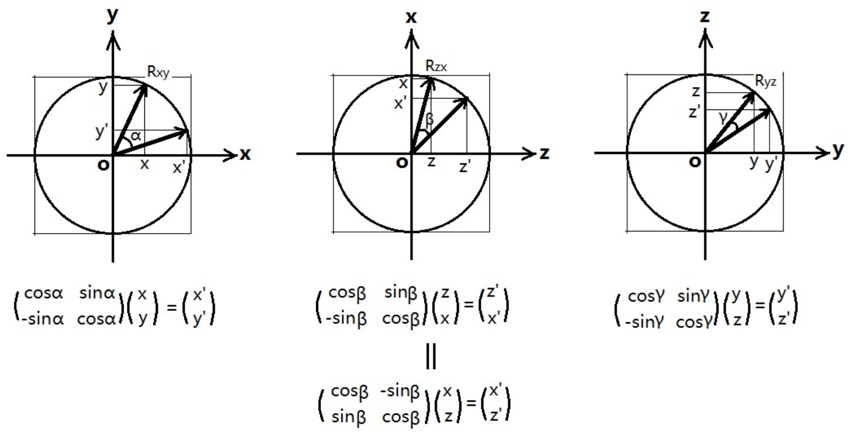

Figure 23

Finally, how do the three projections separately on the three plane xy, yz and zx rotate, when the coordinates rotate around z, y and x axis respectively? As shown in Figure 23 and following formulas, we compare them with SO_2 and SU_2.

$$e^{i\frac{\sigma_z}{2}(2\alpha)}\begin{pmatrix}0 & x-yi\\ x+yi & 0\end{pmatrix} = \begin{pmatrix}0 & x'-y'i\\ x'+y'i & 0\end{pmatrix} \sim \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x'\\ y'\end{pmatrix}$$

$$e^{i\frac{\sigma_y}{2}(2\beta)}\begin{pmatrix}z & x\\ x & -z\end{pmatrix} = \begin{pmatrix}z' & x'\\ x' & -z'\end{pmatrix} \sim \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}x\\ z\end{pmatrix} = \begin{pmatrix}x'\\ z'\end{pmatrix}$$

$$e^{i\frac{\sigma_x}{2}(2\gamma)}\begin{pmatrix}z & -yi\\ yi & -z\end{pmatrix} = \begin{pmatrix}z' & -y'i\\ y'i & -z'\end{pmatrix} \sim \begin{pmatrix}\cos\gamma & \sin\gamma\\ -\sin\gamma & \cos\gamma\end{pmatrix}\begin{pmatrix}y\\ z\end{pmatrix} = \begin{pmatrix}y'\\ z'\end{pmatrix}$$

12. How do we understand by sense that SU_2 can be corresponding to $SO3$?

SU_2 rotates the vectors in 2D complex space which is equal to 4D real space, but SO_3 is a group of rotation in 3D real space, why can they have a corresponding relation?

Actually, SU_2 rotates the 4D vector, but SO_3 rotates the 3D **rigid body** which carries a 3D coordinate system by itself. So they can both have the same three parameters.

Let we give an analogy for it with rotating a vector in 3D space and a corresponding operating in 2D space (a real plane).

As shown in Figure 24, Rotating an arbitrary 3D position-vector to the basic-vector $(1,0,0)$, can have two steps: 1) rotating it around z axis with a proper angle, 2) then rotating it around y axis with another proper angle.

$$\begin{pmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

This transformation group can be called VO_3.

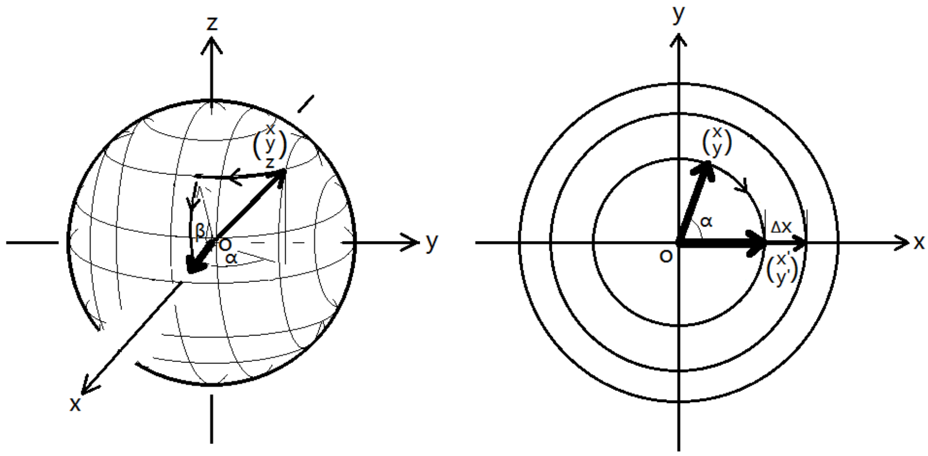

Figure 24

Correspondingly, their projections on the xy plane can give two steps of transformations, one is rotation around z axis as the same above, another is translation perpendicularly to y axis.

$$\begin{pmatrix}1 & 0 & -\Delta x\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha & 0\\ -\sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ 1\end{pmatrix}$$

where $|x| \leq r = |R\cos\alpha\cos\beta|$, R is the radius of the sphere, if $|x - \Delta x| > r$, it must be substituted: $x - \Delta x \to 2kr \pm (x - \Delta x)$, where k is an integer which can satisfy: $|2kr \pm (x - \Delta x)| \leq r$.

The transformation-group can be called EO_2, and what it acts on can be called a 2D **elastic body**.

The translation perpendicularly to y axis in xy 2D space is just corresponding to rotation around y axis in xyz 3D space. There is:

$$-\Delta x \sim x\cos\beta - z\sin\beta - x$$

Notice that when $z \to -z$, $\beta \to -\beta$, then $\Delta x \to \Delta x$, so, one transformation acting on the 2D elastic body is corresponding to two transformations acting on the 3D vector. By sense, this is because the upper and down hemisphere have the same projection onto the xy plane. Thus they can be analogous to the relation of SU_2 and SO_3.

$$SU_2 \cong \frac{SO_3}{Z_2}, \quad VO_3 \cong \frac{EO_2}{Z_2}$$

SU_2: rotation of 4D vector, SO_3: rotation of 3D rigid body,

VO_3: rotation of 3D vector, EO_2: transformation of 2D elastic body.

Z_2: 2 order cyclic group.

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