General Brachistochrone Curve

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In history, the problem of Johann Bernoulli's Brachistochrone Curve (BBC) was assumed the case that the force of gravity on the falling body is constant, for example, the case of near the surface of the Earth. In this article, we will propose and solve a new problem of the General Brachistochrone Curve (GBC), in case of the body falling in a large space of Newton gravity field, or in stronger Newton gravity field, in which the force of gravity is inversely proportional to the square of the height above the center of the star or planet.

Now let us calculate the set of parametric equations of the General Brachistochrone Curve (GBC), and later compare between GBC and BBC.

As show in the Figure 1,

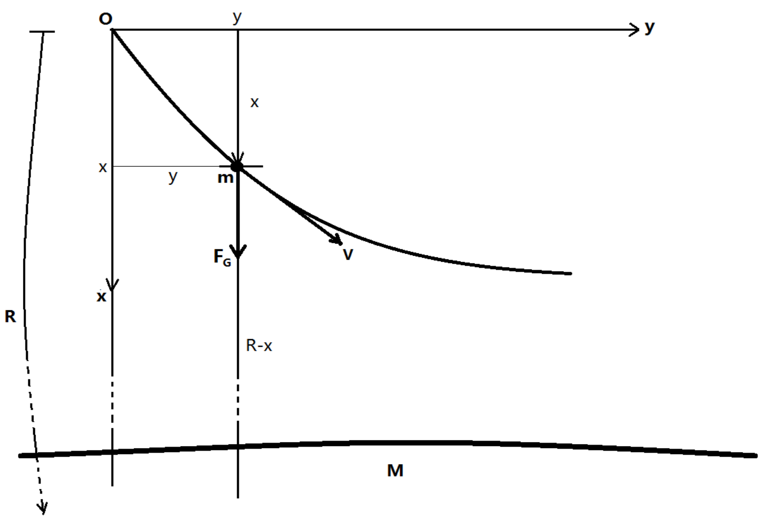

Figure 1

suppose a falling body acted only by the force of Newton gravity and the holding force which is always perpendicular to the moving direction of the body (in other words, perpendicular to the tangent of the falling curve).

Let the mass of the falling body is m, the mass of the Earth is M, (the Earth is sufficient wide), and R is the distance between their center. The falling height of the body from origin point O is x. The original velocity of the falling body at O point is 0. The force of gravity from the M acting on m is

$$F = G\frac{Mm}{(R-x)^2},$$

the work done by the gravity is equal to the increment of the kinetic energy:

$$\int_0^x F dx = \frac{1}{2}mv^2 - 0,$$

(where G, M, m and R are constants). Let $2GM = 1$, then we can obtain:

$$v = \sqrt{\frac{1}{R-x} - \frac{1}{R}},$$

the functional of the moving time is

$$t = \int_0^x \frac{\sqrt{1 + {y_x'}^2}}{\sqrt{\frac{1}{R-x} - \frac{1}{R}}} dx \tag{1}$$

the Lagrangian is

$$L = \frac{\sqrt{1 + {y_x'}^2}}{\sqrt{\frac{1}{R-x} - \frac{1}{R}}} \tag{2}$$

when the moving time t gets extreme value, the Euler-Lagrange equation must be satisfied

$$\frac{\partial L}{\partial y} = \frac{\partial^2 L}{\partial y_x' \partial x}. \tag{3}$$

since

$$\frac{\partial L}{\partial y} = 0,$$

then

$$\frac{\partial L}{\partial y_x'} = C$$

($C = const.$), thus we can obtain

$$y_x' = \pm\sqrt{\frac{x}{a - bx}} \tag{4}$$

where a and b are constants related with R and C by

$$\begin{cases} a &= (\frac{R}{c})^2 \\ b &= 1 + \frac{R}{c^2} \end{cases}$$

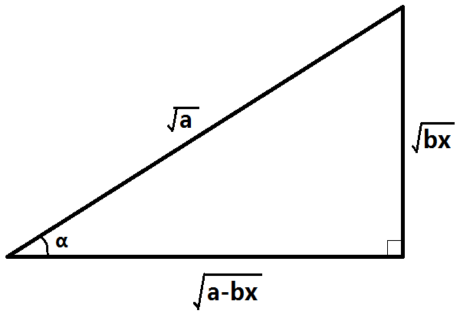

Figure 2

hence

$$y = \int_0^x \sqrt{\frac{x}{a - bx}} dx. \tag{5}$$

In order to solve the equation, suppose a right-triangle as in the Figure 2, then we have

$$\sin\alpha = \sqrt{\frac{bx}{a}}$$

that is

$$x = \frac{a}{b}\sin^2\alpha,$$

then

$$y = \int_0^x \frac{\tan\alpha}{\sqrt{b}} dx.$$

Since

$$\frac{dx}{d\alpha} = \frac{2a}{b}\sin\alpha\cos\alpha,$$

thus

$$y = \int_0^\theta \frac{\tan\alpha}{\sqrt{b}}\frac{2a}{b}\sin\alpha\cos\alpha d\alpha = \frac{2a}{b\sqrt{b}\int_0^\theta \sin^2\alpha d\alpha} = \frac{a}{b\sqrt{b}}(2\alpha - \sin^2 2\alpha),$$

and therefore

$$\begin{cases} x = \frac{a}{b}\sin^2\alpha \\ y = \frac{a}{2b\sqrt{b}}(2\alpha - \sin 2\alpha) \end{cases} \tag{6}$$

Let $\alpha = \frac{\theta}{2}$, then

$$\begin{cases} x = \frac{a}{2b}(1-\cos\theta) \\ y = \frac{a}{2b\sqrt{b}}(\theta - \sin\theta) \end{cases}$$

substituted in

$$\begin{cases} a = (\frac{R}{c})^2 \\ b = 1 + \frac{R}{c^2} \end{cases}$$

we have

$$\begin{cases} x = \frac{R^2}{2(R+c^2)}(1-\cos\theta) \\ y = \frac{R^2}{2(R+c^2)}(\theta - \sin\theta) \end{cases}$$

Let $r = \frac{R^2}{2(R+c^2)}$, that is $c^2 = R(\frac{R}{2r} - 1)$, or $\frac{1}{\sqrt{1+\frac{R}{c^2}}} = \sqrt{1 - \frac{2r}{R}} = b$, thus

$$\begin{cases} x = r(1-\cos\theta) \\ y = \sqrt{1-\frac{2r}{R}}[r(\theta - \sin\theta)] \end{cases} \tag{7}$$

(Obviously, here $2r < R$, that means $b = \sqrt{1-\frac{2r}{R}} < 1$). This is the set of parametric equations of the General Brachistochrone Curve. Now we can compare it with the set of parametric equations of Bernoulli's Brachistochrone Curve (BBC):

$$\begin{cases} X = r(1-\cos\theta) \\ Y = r(\theta - \sin\theta) \end{cases} \tag{8}$$

we can discover: when $2r << R$, there $\frac{2r}{R} \approx 0$, thus GBC becomes the limit of BBC.

We continue to compare other properties between GBC and BBC. When x and X have the same maximal value, $x_{max} = X_{max} = 2r$, we have $y(x_{max}) < Y(X_{max})$ (because $\sqrt{1-\frac{2r}{R}} < 1$), as shown in the Figure 3:

From another perspective, if we let

$$r_0 = \sqrt{1-\frac{2r}{R}}r, \quad \text{that is} \quad r = \frac{r_0}{\sqrt{1-\frac{2r}{R}}} = \frac{r_0}{b},$$

thus the set of parametric GBC equations is

$$\begin{cases} x = \frac{1}{b}[r_0(1-\cos\theta)] \\ y = r_0(\theta - \sin\theta), \end{cases}$$

compare it with BBC:

$$\begin{cases} X = r_0(1-\cos\theta) \\ Y = r_0(\theta - \sin\theta), \end{cases}$$

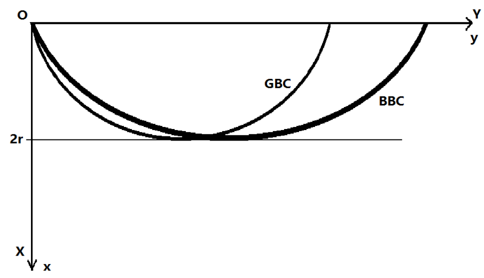

Figure 3

since $0 < b < 1$, that means $1/b > 1$, thus we can know: when x and X have the same minimal value, $x_{min} = X_{min} = 0$, (in other words, when $\theta = 2k\pi$, where k is a nonnegative integer), we have

$$y(x_{min}) = Y(X_{min}),$$

obviously, except the points of $x_{min} = X_{min}$, we always have:

$$x = \frac{X}{b} > X.$$

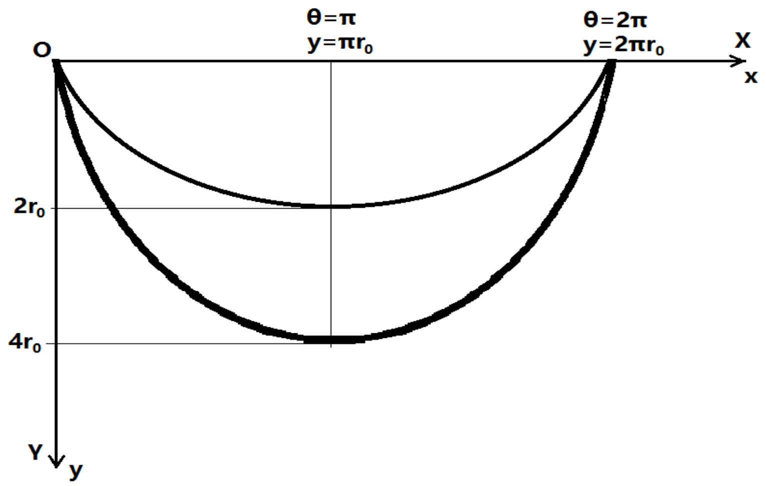

Figure 4

so, when the upper spans are the same, GBC is always below BBC, and BBC is the limit of GBC, as shown in Figure 4 and Figure 5.

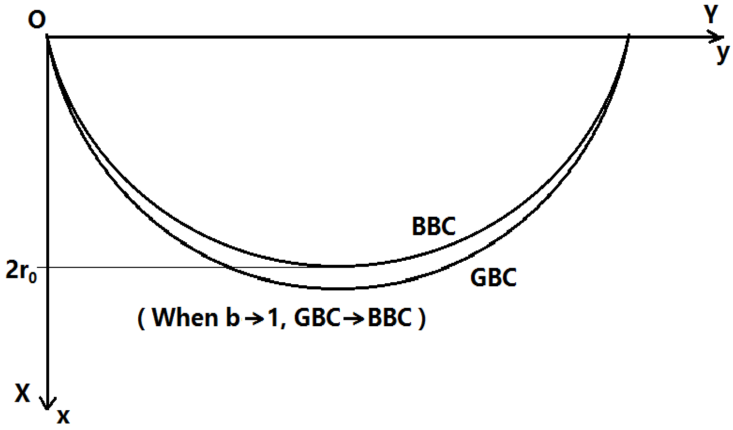

Figure 5

Finally, let we find how to get a image of GBC. Just like with rotating a picture of a circle we can get an image of a ellipse, (with the method of projection or photography), in the same way, we can rotate a picture of BBC to get an image of GBC, as shown in the Figure 6.

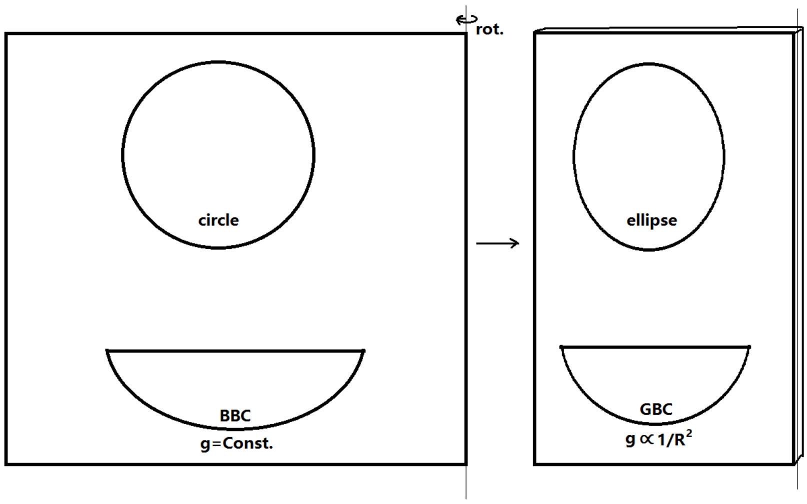

Figure 6

Reference:

(1) Alexander R. Klotz: *The Gravity Tunnel in a Non-Uniform Earth* *http://arxiv.org/abs/1308.1342v1*

(2)Christopher Grimm, John A. Gemmer: Weak and Strong Solutions to the Inverse-Square Brachistochrone Problem on Circular and Annular Domains arXiv:1605.01486v1 [math.OC] 5 May 2016

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