Visualizing $\zeta(n \geq 2)$ and Proving Its Irrationality

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Abstract

A number system is developed to visualize the terms and partials of $\zeta(n > 1)$. This number system consists of radii through dots on concentric circles that generate sectors. The sectors have areas corresponding to all rational numbers and can be added. Dots on the circles give an un-ambiguous cross reference to decimal systems in all bases. We show, in the proof section of this paper and using a modification of Cantor's diagonal method, that all $\zeta(n > 1)$ require a infinite decimal in all bases. This establishes the result.

1 Introduction

Apery's proof that

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

is irrational is difficult [1]. It doesn't generalize to show $\zeta(2n+1)$ is irrational for $n > 1$. Here we develop a much simpler proof that does so generalize.

The current state of affairs with proving z_n, n odd, is irrational is quite limited. It is known that there are infinitely many odd $n > 3$ that are irrational [12] and that at least one of 5, 7, 9, and 11 are irrational [17]. The proofs of these result uses group theory and complex analysis. Zudilin gives a literature review and develops both results in [16]. The even case follows easily from the transcendence of π [6, 11] and Bernoulli's famous formula:

$$\zeta(2n) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}.$$

This formula is derived from a trigonometric series expansion [2].

The attempts on the part of Zudilin and others reflect the combinatorial problem of the general case. One certainly senses that showing $\zeta(2)$ is irrational using Apery's ideas [5] is easier than showing $\zeta(3)$ and, closely reading [16] one sees Apery's ideas are generating an ever growing combinatorial puzzle. A way to *see* the similarities of all cases is our theme.

The visualization part of this article starts by exploring decimal representations. If some real number requires an infinite number of unambiguous digits in all basis, then it must be irrational. Hardy shows that all decimal representations of a rational number a/b in a given base d are finite, repeating, or mixed depending on the relationship between b and d [10]. The ambiguous case of say $.4\overline{9} = .5$ in base ten is not included as an infinite representation of the finite decimal .5, base 10. An irrational number in all bases is an infinite non-repeating decimal. The idea of the first part of the paper is to suggest that

$$\zeta(n) - 1 = z_n = \sum_{k=2}^{\infty} \frac{1}{k^n}, \tag{1}$$

$n > 1$, can't be represented by a finite decimal in any base.[1]

Our visualization involves a simple geometric construction that allows the terms of (1) to be given as sector areas and to be added. There is then some connection with circles on the plain, but our plain is not the complex plain, nor even the Cartesian plain – just concentric circles with sectors designated by a radius. If the radius goes through a point (we call it a dot) on a circle the sector area is given by a single decimal in a base associated with the circle. The construction allows for a clear visualization of the decimal representations of all terms, Section 2, and partial sums, Section 3, of (1) in all bases k^n, where k is a natural number greater than 1.[2]

In Section 4 we consider the limit of partials using $.\overline{1}$, base 4. The circles associated with this series generate finite decimals base 4, but no single circle or finite addition (finite decimal, base 4) can give the convergence value of 1/3. If this is generally true, then the convergence point of z_n must not reside as a dot on any of its term's circles, but its term's circles give all finite decimal representations in bases k^n. But this is all rational numbers between

[1]Henceforth, just z_n.

[2]Henceforth, just bases k^n.

0 and 1, so z_n must be irrational. We have some grounds to suspect the irrationality of z_n, all $n > 1$.

The proof part of the paper consists of two Sections. In Section 5, we show that partials of all z_n can't be expressed as finite decimals in any base d where d is the denominator of one of the partial's terms. The limiting case, then, is the rub [13]. Section 6 gives a proof that z_2 is irrational based on Cantor's classic proof that the real numbers are not countable [7]. Cantor's diagonal method consists of modifying a list of decimal numbers, supposed to be all reals in a fixed base with values between 0 and 1. Each digit down the diagonal of the list is modified, yielding a number that is not in the list, contradicting all reals have been enumerated. We first give a variation of this proof to show how it can be used to construct an irrational number. We then give a bolder modification of Cantor's technique. We list all rational numbers between 0 and 1 using sets with all bases k^2, single decimals. Using the partials of z_2, and the result of Section 5, we construct a number not associated with any rational number in any of the sets. The resultant infinite series, limit of the partials, we claim must be irrational.

In the conclusion, Section 7, we mention other series and argue that the result developed to show z_2 is irrational applies to other z_n, mutatis mudantis.

2 Term Visualization

The series z_2 is referenced in what follows, but any z_n can similarly be referenced.

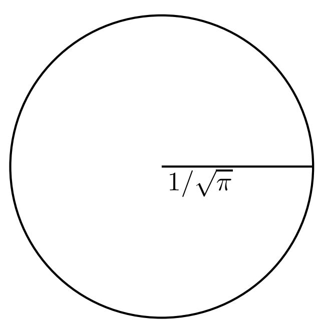

Figure 1: A circle with radius $1/\sqrt{\pi}$ has an area of 1.

We can visualize the first term, 1/4, of z_2 using a circle. In Figure 1 we have a circle of radius $\sqrt{1/\pi}$. The area of this circle is

$$\pi r^2 = \pi \cdot (\sqrt{1/\pi})^2 = 1.$$

In Figure 2, four equally spaced dots are placed around the circle, giving four equal sector areas. Each area must be 1/4 of the area of the circle or 1/4. Sector areas corresponding to these dots, between 0 and 1, are given by $x/4$, $x = 1, 2, 3$ or a single, non-zero decimal base 4. If a radius on the circle doesn't go through one of the dots, the sector area formed will require more than a single decimal in base 4: Figure 3. We will designate this circle with C_4.

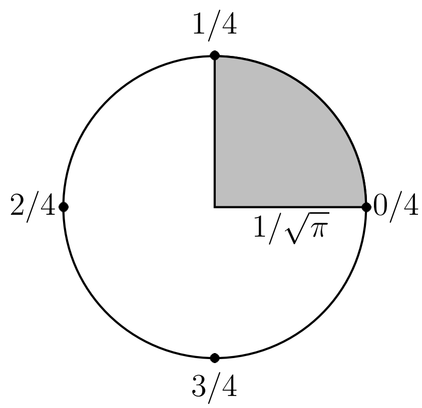

Figure 2: A circle with area 1 is divided up using 2^2. The area of the shaded sector is 1/4.

The next term is 1/9. The circle in Figure 4 has radius $\sqrt{2/\pi}$ with 9 equally spaced dots around it. Its area is 2:

$$\pi r^2 = \pi \cdot (\sqrt{2/\pi})^2 = 2.$$

We will designate this circle with C_9.

By making C_4 and C_9 concentric circles, Figure 5, the area of the annulus formed is 1: $2 - 1$. If a radius is drawn threw a dot on C_9, it will generate a sector area of $x/9$ on C_4. If a radius misses dots on both circles, then the sector area formed is not equal to a single decimal in either base 4 or base 9. It will require more than a single digit in either of these bases.

Figure 6 shows the first three terms of z_2 rendered with C_4, C_9, and C_{16}. Clearly, we can continue this process using equally spaced k^2 dots on circles

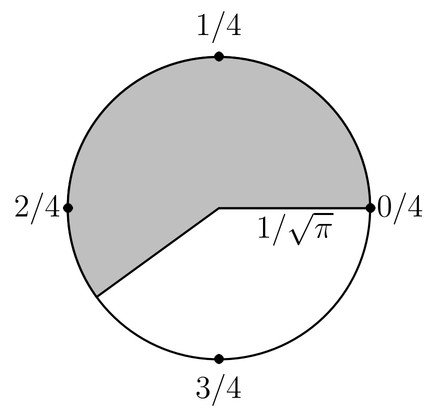

Figure 3: A radius that does not go through any dot generates a sector area that requires more than one decimal, base 4.

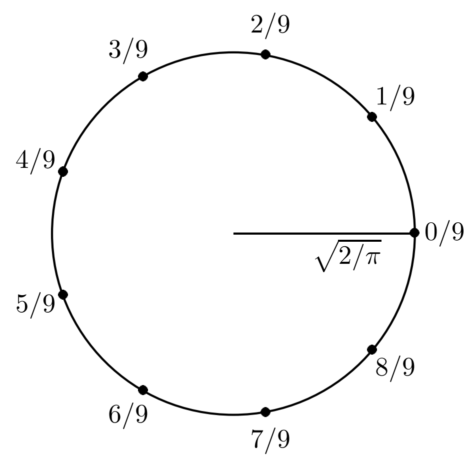

Figure 4: Nine equally spaced dots on a circle of radius $\sqrt{2/\pi}$: C_9.

of radius $\sqrt{(k-1)/\pi}$. If a given radius misses all dots on all such circles, the sector area associated with it must be irrational. This follows as the sector areas generated by radii through a given dot, say x, on the C_{k^2} circle will be given by $.x$ base k^2, a single decimal digit, and all rational numbers can be so designated; $km/k^2 = m/k$ with $m < k$.

This is a visualization of the terms of z_2. Next we will visualize adding these terms.

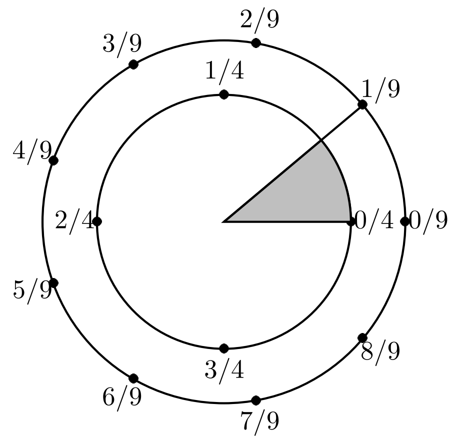

Figure 5: C_4 and C_9 as concentric circles. The area of the shaded sector is 1/9.

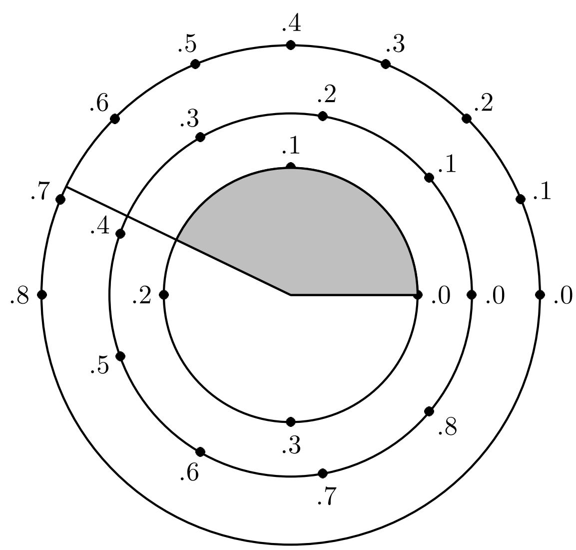

Figure 6: The shaded sector area is not a single decimal in base 4, 9, or 16.

3 Visualization of Partial Sums

Two sector areas can be added.

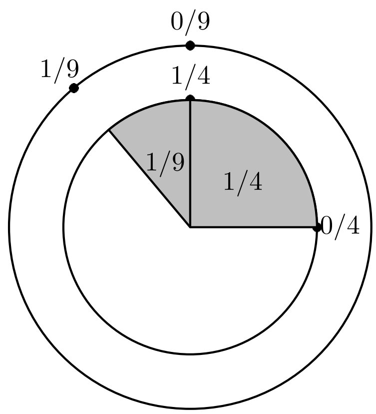

Figure 7: The addition of $1/4 + 1/9$ using C_4 and C_9 with C_9 offset.

In Figure 7, 1/4 is added to 1/9 by rotating C_9 in a counter-clockwise direction to line up with the 1/4 dot on C_4. This addition is somewhat analogous to the head to toe (here 1 to 0) method of vector addition. In Figure 8, 1/16 is added to $1/4 + 1/9$ using the same 1 (head) to 0 (toe) method. The resulting radius generates an area on all annuli and C_4's circle that corresponds to $1/4 + 1/9 + 1/16$. Clearly these additions can be used to form such radii for all partial sums of z_2.

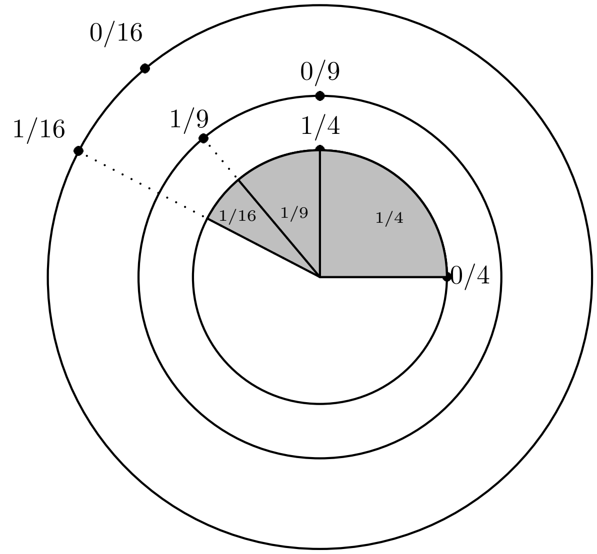

Figure 8: The addition of $1/4 + 1/9 + 1/16$ using C_4, C_9, and C_{16} with the offset method. The area of the shaded sector is the sum.

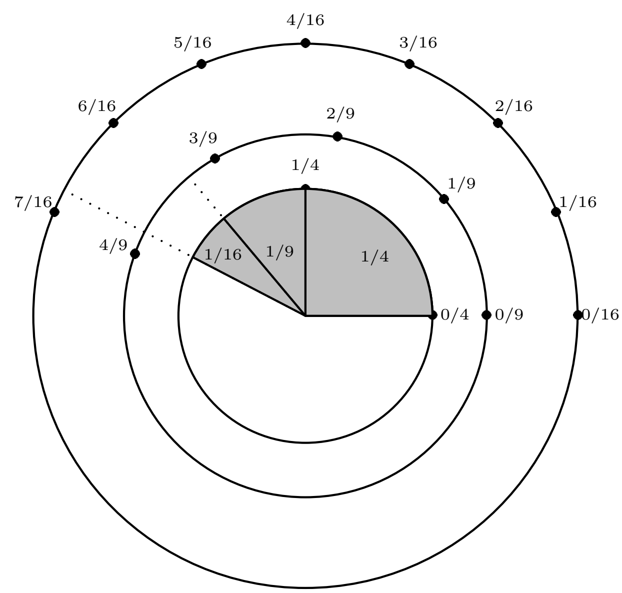

Figure 9: The radius associated with the sum $1/4 + 1/9 + 1/16$ misses all dots on C_4, C_9, and C_{16}.

Figure 7 and Figure 8 show rotations of C_9 and C_{16} to effect fraction additions. Figure 9 shows the resulting radius with the un-rotated versions of these circles. Designating partial sums,

$$s_k^m = \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{k^m},$$

Figure 9 accurately shows that the partial

$$s_4^2 = \sum_{k=2}^{4} \frac{1}{k^2}$$

generates a radius that does not go through any of its first few term's dotted circles. We can infer that the sum is not expressible as a single decimal digit in base 4, 9, or 16.

The denominators of z_2 are just all decimal bases squared. So, if a radius misses all dots on all C_{k^2} circles, then its associated sector area value must require more than one decimal in all k^2 bases. It must be irrational.

We can now visualize the problem of proving z_2 is irrational. We need to show that the limit radius generated by adding the terms of z_2 does not go through any of the dots on any of the circles defined by its terms. The difficulty is that radii can converge to a dot on a circle without a radius going through the dot. The geometric series gives guidance. We will analyze it next.

4 Geometric series

Infinite repeating decimals are really geometric series. For example, in base 4,

$$.\overline{1} = \sum_{k=1}^{\infty} \frac{1}{4^k}.$$

This geometric series has a convergence point of 1/3. All its terms occur in z_2, so we can use our dotted concentric circles to understand the relationship between the rotated C_{4^k}, $k > 1$, circles used to construct this sum and C_3, the un-rotated circle having a dot the sum converges to: that is, the unique radius for this convergence point. See Figure 10.

Here are some observations. Given any radius, representing a sector area's value, we can read from a system of dotted circles the decimal expansion

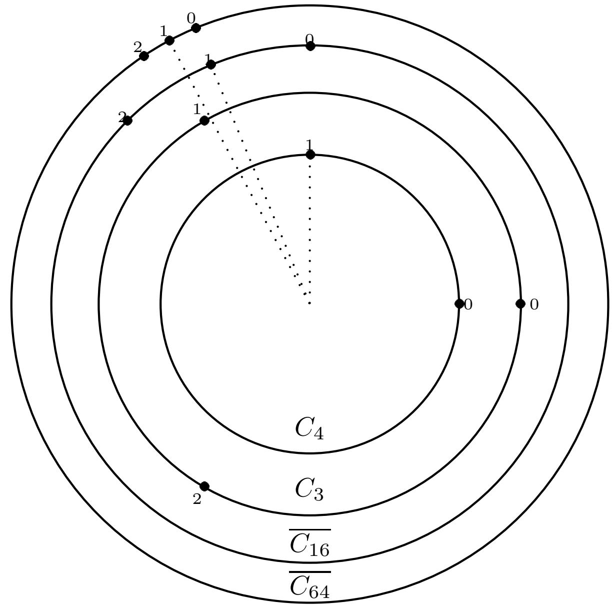

Figure 10: Circles C_4 with $\overline{C_{16}}$ and $\overline{C_{64}}$ rotated (indicated with overline) to generate the radius associated with .111 base 4. C_3, unrotated, has the convergence point for $.\overline{1}$: 1/3.

in a given base, like base 4; conversions via the modulus operator may be necessary to adjust the digits of the expansion. Also all convergent infinite series with terms of the form $1/a_k$ with a_k strictly increasing natural numbers have partial radii that rotate counter-clockwise around the circle and go through points on concentric circles farther and farther from the center. This forces series that converge to a rational number to have their convergence radius given by a radius going through an un-rotated *earlier* dot. We can see these patterns in Figure 10. As z_2 and generally z_k require rotations of all circles giving all rational numbers, there is no such earlier un-rotated circle having a rational point for these series to converge to. The additions of the terms perpetually offsets the radius formed from all rational numbers. This suggests that all z_k are irrational.

Another observation: there is only one radius for every area, rational and irrational. Unlike decimal representations where $.4\overline{9} = .5$, there is no ambiguity with reduced fractions and areas. For an irrational number, we can read the decimals from our figure and as the, note *the*, radius never goes through a point in all bases, it never terminates in all bases.

Also note that we observe trajectories, the radii of Figure 10, and how additions build new trajectories. It seems plausible that adjusting a trajectory with additions could cause new trajectories to miss all previous dots as well as the last term added's dots. That is we can perpetually adjust a trajectory to have it miss all dots. Think of a spaceship avoiding equally spaced meteorites arranged in concentric rings as the dots of C_{k^2} in front of us. We can avoid them all and we know, per convergence of z_n, that a single radius will emerge and be an irrational number.

Showing the radius for z_k never goes through a dot on the n^k system of concentric circles, shows that it must be irrational. In the next two sections we prove that the limit radius for z_2 does not go through any C_{k^2} dot.

5 Bertrand

Our aim in this section is to show that the reduced fractions that give the partial sums of z_n require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms.

The first lemma is a little more difficult than an exercise in Apostol's Introduction to Analytic Number Theory [2, p. 23, problem 30], its inspiration.

Lemma 1. *The reduced fraction, r/s giving*

$$s_k^m = \sum_{j=2}^{k} \frac{1}{j^m} = \frac{r}{s} \tag{2}$$

is such that 2^m divides s.

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^m, 3^m, \ldots, k^m\}$ will have a greatest power of 2, ma. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^m$ will have a greatest power of 2 exponent of mb. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + (k!)^m/3^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{3}$$

The term $(k!)^m/2^{ma}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $mb - ma$ for 2. As all other terms but this term will have more than an exponent of 2^{mb-ma} in their prime factorization, we have the numerator of (3) has the form

$$2^{mb-ma}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^m/2^{ma}$. The denominator, meanwhile, has the factored form

$$2^{mb}C,$$

where $2 \nmid C$. This leaves 2^{ma} as a factor in the denominator with no powers of 2 in the numerator, as needed. □

Lemma 2. *If p is a prime such that $k > p > k/2$, then p^m divides s in (2).*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^m}{(k!)^m}\sum_{j=2}^{k}\frac{1}{j^m} = \frac{(k!)^m/2^m + \cdots + (k!)^m/p^m + \cdots + (k!)^m/k^m}{(k!)^m}. \qquad (4)$$

As $(k, p) = 1$, only the term $(k!)^m/p^m$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^m/p^m$. As $p < k$, p^m divides $(k!)^m$, the denominator of r/s, as needed. □

Theorem 1. *If*

$$s_k^m = \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{k^m} = \frac{r}{s}, \qquad (5)$$

with r/s reduced, then $s > k^m$.

Proof. Bertrand's postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [10]. If k of (5) is even we are assured that there exists a prime p such that $k > p > k/2$. If k is odd $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even,

$p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^m p^m$ divides the denominator of (5) and as $2^m p^m > k^m$, the proof is completed. □

So, for z_2, we have the following.

Definition 1.

$$D_{k^2} = \{0, 1/k^2, \ldots, (k^2-1)/k^2\} = \{0, .1, \ldots, .(k^2-1)\} \text{ } \textit{base } k^2$$

Corollary 1.

$$s_n^2 \notin \bigcup_{k=2}^{n} D_{k^2}$$

Proof. Immediate. □

$+1/4$							
$+1/9$	$+1/4$	$+1/4$	$+1/4$	$+1/4$	$\ldots$	$+1/4$	
$\notin D_4$	$+1/9$	$+1/9$	$+1/9$	$+1/9$	$\ldots$	$+1/9$	
	$\notin D_9$	$+1/16$	$+1/16$	$+1/16$		$\vdots$	
		$\notin D_{16}$	$+1/25$	$+1/25$		$\vdots$	
			$\notin D_{25}$	$+1/36$		$\vdots$	
				$\notin D_{36}$			
						$+1/(k-1)^2$	
						$+1/k^2$	
						$\notin D_{k^2}$	
							$\ddots$

Table 1: A list of all rational numbers between 0 and 1 modified to exclude them all via partial sums of z_2.

Cantor

The result of applying Corollary 1 to all partial sums of z_2 is given in Table 1. The table shows that adding the numbers above each D_{k^2}, for all $k \geq 2$ gives results not in D_{k^2} or any previous rows such sets. So, for example, $1/4 + 1/9$ is not in D_4, $1/4 + 1/9$ is not in D_4 or D_9, $1/4 + 1/9 + 1/16$ is not in D_4, D_9, or D_{16}, etc.. Can we conclude that z_2 is irrational? The table should remind readers of Cantor's diagonal method. The catch with this conclusion is that we are not working with a single decimal system and verification via decimal notation is wanting. We can, however, build a proof using the property that this table indicates. The proof is a more complicated version of Sondow's geometric proof of the irrationality of e [14].

Theorem 2. *z_2 is irrational.*

Proof. We construct a sequence of lower and upper bounds using D_{k^2}. We will refer to the C_{k^2} circles, used in the visualization section, that represent D_{k^2} values as sector areas given by radii. Suppose the sector between 2/4 and 3/4 has infinitely many radii corresponding to partial sums going through it. Designate the lower bound L_{2^2} as the radius going through 2/4. Suppose the sector between 5/9 and 6/9 has infinitely many such radii going through it. Designate the upper bound U_{3^2} as the radius going through 6/9. Form the interval $I_1 = [2/4, 6/9]$. We know that each partial radii will either never reach a rational radius or cross a last radius when the upper limit of the partial is a unique n value. We can continue this nested interval building process with I_2 using C_{4^2} and C_{5^2} for the lower and upper bounds of the I_2 interval. Using Cantor's Intersection Theorem [3], there is a convergence point for the intersection of these intervals. It is z_2 and it can't equal any endpoint of the I_n intervals. Because all possible rationals are thus excluded, z_2 must be irrational. □

6 Conclusion

6.1 Other series

The telescoping series

$$\sum_{k=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1/2 - 1/3 + 1/3 - 1/4 + \cdots = 1/2$$

or

$$\sum_{k=2}^{\infty} \frac{1}{n(n+1)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = 1/2$$

shows the necessity of partials escaping terms. For example, the sum of the first three terms is 3/10 which can be expressed with 6/20 in D_{20}. Partial sums backtrack to earlier denominators thus preventing Cantor's diagonal process from being valid. The geometric series has partials that sum to fractions with denominators from the last term of the partial, but the term's denominators don't cover all pertinent rational numbers.

For both examples, placing them in a Cantor table, that is a table like Table 1, shows the necessity of partials escaping their terms and the terms *covering* the rationals.

6.2 General z_n

Do the ideas given here give a proof that $\zeta(n \geq 2)$? As all bases k^n have the same prime factors as k, the answer is yes: Table 1, in conjunction with Section 5, works when these other series are used.

If one assumes Corollary 1, does the proof distill to a geometric proof? Note that the denominator of the partial sums of a z_k series with upper bound n will be much larger than n^k, more like $(n!)^k$, so this theorem is highly plausible. Also simple number theory proofs show that $(n-1,n) = (n,n{+}1) = 1$, that the natural numbers are consecutively relatively prime. So one suspects such partial sums will have denominators that have chaotically occuring prime factors. This points to the central intuition about this series; the fractions added have denominators growing by one (to a power) and this marks how the series differs from the "spaciness" of the geometric and telescoping series. If one grants Theorem 1 as intuitively plausible is Figure 9 of Section 3 enough: the nudging of a trajectory by the terms (the addition of terms) of any $\zeta(n)$ builds a trajectory that never "hits" a rational dot; all rational sector areas are perpetually offset yielding a sector area that must be irrational – is that enough?

Finally, this paper suggests that one suspects a geometric self-similarity argument could be made: if one z_n is irrational, they all are. Hence, if this were true, via Apery or Bernoulli, all odd z_n are irrational by similarity with these known cases.

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