A new Lagrangian of the simple harmonic oscillator

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Abstract
A new Lagrangian functional of the simple harmonic oscillator has been proposed. The derived equation of motion is *almost* same as that of the conventional Lagrangian functional. The equation of motion is derived from Euler-Lagrange equation by performing partial derivatives on the Lagrangian functional of the second variation of the calculus of variations. The new Hamiltonian functional of the simple harmonic oscillator has also been derived.

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Introduction
The simple harmonic oscillator model is very important in physics (Classical and Quantum). Harmonic oscillators occur widely in nature and are exploited in many manmade devices, such as clocks and radio circuits. They are the source of virtually all sinusoidal vibrations and waves.

Discussion
(1) First variation of the Calculus of Variation
It is known that the Euler-Lagrange equation resulting from applying the first variation of the Calculus of Variations of a Lagrangian functional $L(t, q(t), \dot{q}(t))$ of a single independent variable $q(t)$, its first derivative $\dot{q}(t)$ of following action

$$I[q(t)] = \int L(t, q(t), \dot{q}(t))\ dt$$

when varied with respect to the arguments of integrand and the variation are set to zero, i.e.

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$$
\begin{aligned}
0 = \delta\ I[q(t)] &= \delta \int L(t, q(t), \dot{q}(t))\ dt \\
&= \int \delta[L(t, q(t), \dot{q}(t))]\ dt \\
&= \int \left\{ \frac{\partial L}{\partial t}\delta t + \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \right\} dt
\end{aligned}
$$

is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

Provided that the variation δq vanishes at the end points of the integration and the Lagrangian function doesn't depend explicitly on time (i.e. $\frac{\partial L}{\partial t} = 0$).

Defining the generalized momentum p as

$$p = \frac{\partial L}{\partial \dot{q}}$$

Then, the Euler-Lagrange equation may be written as

$$\frac{\partial L}{\partial q} = \dot{p}$$

Defining the generalized force F as

$$F = \frac{\partial L}{\partial q}$$

Then, the Euler-Lagrange equation has the same mathematical form as Newton's second law of motion:

$$F = \dot{p}$$

(i) The Lagrangian functional of simple harmonic oscillator

The Lagrangian functional of simple harmonic oscillator in one dimension is written as:

$$L = -\frac{1}{2}k\ x^2 + \frac{1}{2}m\ \dot{x}^2$$

The first term is the potential energy and the second term is kinetic energy of the simple harmonic oscillator.

The equation of motion of the simple harmonic oscillator is derived from the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

To give

$$-kx - m\ddot{x} = 0$$

This is the same as the equation of motion of the simple harmonic oscillator resulted from application of Newton's second law to a mass attached to spring of spring constant k and displaced to a position x from equilibrium position.

Solving this differential equation, we find that the motion is described by the function

$$x(t) = x_0 \cos(\omega_0\, t - \varphi),$$

where $x_0 = x(t = t_0)$ and $\omega_0 = \sqrt{\dfrac{k}{m}} = \dfrac{2\pi}{T}$.

(ii) The first Hamiltonian functional of simple harmonic oscillator

The Hamiltonian functional $H = H(q, p)$ is derived from the first Lagrangain with the use of the Legendre transfom;

$$H = p\dot{q} - L$$

and defining $p = \dfrac{\partial L}{\partial \dot{q}}$ as the generalized momentum. Calculating the right hand side in the equation defining the Hamiltonian, we get

$$H = -\frac{1}{2}k\ x^2 + \frac{1}{2m}p^2$$

(2) Second Variations of the Calculus of Variations

It is known that the Euler-Lagrange equation resulting from applying the second variations of the Calculus of Variations of a Lagrangian functional $L(t, q(t), \dot{q}(t), \ddot{q}(t))$ of a single independent variable $q(t)$, its first and second derivatives $\dot{q}(t)$, $\ddot{q}(t)$ of following action

$$I[q(t)] = \int L(t, q(t), \dot{q}(t), \ddot{q}(t))\ dt$$

When varied with respect to the arguments of integrand and the variation are set to zero, i.e.

$$\begin{aligned}
0 &= \delta\ I[q(t)] = \delta \int L(t, q(t), \dot{q}(t), \ddot{q}(t))\ dt \\
&= \int \delta[L(t, q(t), \dot{q}(t), \ddot{q}(t))]\ dt \\
&= \int \left\{ \frac{\partial L}{\partial t}\delta t + \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) \right]\delta q + \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \left(\delta q \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right) + \left(\frac{\partial L}{\partial \ddot{q}}\delta \dot{q}\right) \right] \right\} dt
\end{aligned}$$

is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0$$

Provided that the variations δq and $\delta \dot{q}$ vanish at the end points of the integration.

The Model

(1) The Lagrangian functional of the simple harmonic oscillator

The new Lagrangian functional of the simple harmonic oscillator in one dimension is written as

$$L = -\frac{1}{2}kx^2 - mx\ddot{x}$$

The equation of motion is derived from Euler-Lagrange equation by performing the partial derivatives on the Lagrangian functional $L(x(t), \dot{x}(t), \ddot{x}(t))$:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0$$

With the terms calculated as follows

$$\frac{\partial L}{\partial x} = -kx - m\ddot{x};$$

$$\frac{\partial L}{\partial \dot{x}} = 0;$$

$$\frac{\partial L}{\partial \ddot{x}} = -mx.$$

The equation of motion is

$$-kx - m\ddot{x} - m\ddot{x} = 0$$

Or,

$$kx + 2m\ddot{x} = 0$$

It differs from the equation of motion of the simple harmonic oscillator derived from the first variation method by the factor 2 in the second term.

(2) The Hamiltonian functional of the simple harmonic oscillator

The Hamiltonian functional $H = H(q(t), \dot{q}(t), p, \frac{\partial L}{\partial \ddot{q}})$ of the simple harmonic oscillator in the second variation can obtained form the Euler-Lagrange equation of the second variation as follows:

First, define the generalized momentum in the second variation as

$$p \equiv \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}$$

Then, the Euler-Lagrange equation may be written as

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} \\ &= \frac{\partial L}{\partial x} - \frac{d}{dt}[\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}}] \\ &= \frac{\partial L}{\partial x} - \frac{d}{dt}[p] \\ &= \frac{\partial L}{\partial x} - \dot{p} \end{aligned}$$

This yield

$$\dot{p} = \frac{\partial L}{\partial x}$$

This has the same mathematical form as of the Euler-Lagrange equation of the first variation and the Newton's second law of motion.

The corresponding Legendre transformation in the second variations is written as:

$$H = p\dot{q} + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - L$$

Substituting the corresponding variables of the simple harmonic oscillator

$$\begin{aligned} p &= \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}} \\ &= 0 - \frac{d}{dt}(-mx) \\ &= m\dot{x} \end{aligned}$$

This implies

$$\dot{x} = \frac{p}{m}$$

in the Legendre transformation to obtain the Hamiltonian of the SHO

$$\begin{aligned} H &= p\dot{q} + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - L \\ &= p(\frac{p}{m}) + \ddot{x}(-mx) - (-\frac{1}{2}kx^2 - mx\ddot{x}) \\ &= \frac{p^2}{m} + \frac{1}{2}kx^2 \end{aligned}$$

which is off the Hamiltonian obtain in the first variation by a factor of $\frac{1}{2}$ in the first term.

Conclusion:

The second variation of the method of calculus of variation is rich in its applicability than the first variation. Although there were *no* kinetic energy term (first derivative) in the new Lagrangian of the simple harmonic oscillator we *almost* obtained the correct equation of motion similar to those of the first variation and of the Newton's second law of motion. The second variation of the calculus of variations is promising in constructing Lagrangian of dynamical system which were difficult to construct by following the first variation. It is possible to construct the long looked for: *The Lagrangian of the damped harmonic oscillator* using the second variation of the method of the calculus of variations.

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