Grand Unification Equation

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Abstract

In my attempt to eliminate the Landau Pole from QED by "borrowing" asymptotic freedom from QCD, I was successful in uniting the coupling constants of the two, respectively. This equation, together with the already established electroweak unification forms a basis, within the Standard Model, to experimentally test Grand Unification. The part that can be tested experimentally is the value of the strong coupling constant for the energy value of the QCD integration parameter Λ, offering such a prediction for the first time. It should be also noted that I was successful in eliminating the Landau Pole.

I Introduction

The two dimensionless running constants $\alpha(Q)$ and $\alpha_S(Q)$ will be united in a single equation allowing us to both improve accuracy for the computing of $\alpha_S(Q)$ and remove the problem arising due to infinities in $\alpha(Q)$ known as the Landau pole [1] when on a very large value of Q the fine-structure constant $\alpha \to \infty$ therefore we will effectively make QED a mathematically complete theory. The Landau pole will be eliminated since QCD enjoys asymptotic freedom [2] offering us exactly what we need since β-expansion describes the asymptotic behavior of a denominator of convergents of continued fractions. The paper will also provide the first prediction for the value $\alpha_S(\Lambda_{QCD})$.

Let $\beta_i > 1$ be a real number and $T_{\beta_i}: [0,1) \to [0,1)$ be the β-transformation such that $T_{\beta_i}(x) = \beta_i x - [\beta_i x]$ where $[x]$ is the largest integer that does not exceed x. Every $x \in [0,1)$ can be uniquely expanded into a finite or infinite series:

$$x = \frac{\varepsilon_1(x)}{\beta_i} + \frac{\varepsilon_2(x)}{\beta_i^2} + \frac{\varepsilon_3(x)}{\beta_i^3} + \cdots + \frac{\varepsilon_n(x)}{\beta_i^n} + \cdots \qquad 1.1$$

where $\varepsilon_i(x) = [\beta_i x]$ and $\varepsilon_{n+1}(x) = \varepsilon_1\left(T_{\beta_i}^n x\right)$ for all $n \geq 1$. For any Borel set $A \subseteq [0,1)$ where C is a constant that depends on β_i such that $C^{-1}\lambda(A) \leq \mu(A) \leq \lambda(A)C$ where the measure of the length/energy scale μ is T_{β_i}-invariant meaning that β_i does not implicitly depend on μ and if A is Lebesgue measurable [3], then $\mu(A) = \sup\{\mu(k): k \subset A\}$ where k is compact.

If $x \in [0,1)$ then any irrational number can be written as $p_n(x)/q_n(x) = [c_1(x), c_2(x), \ldots, c_n(x), \ldots]$ where $c_1(x) = [x^{-1}]$ and $c_{n+1}(x) = a_1(T^n x)$ for any $n \geq 1$ is a convergent of the continued fraction expansion of x where $p_n(x)$ and $q_n(x)$ are relatively prime which allows us to study the asymptotic behavior of the denominator k_n

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{6 \log 2 \log \beta_i}{\pi^2} \qquad 1.2$$

For any $\varepsilon > 0$ there exists two positive constants A and α which depend on β_i thus for all $n \geq 1$:

$$\lambda\left\{x \in [0,1): \left|\frac{k_n}{n} - \frac{6 \log 2 \log \beta_i}{\pi^2}\right| \geq \varepsilon\right\} \leq A \cdot e^{-\alpha n} \qquad 1.3$$

If $\log \beta_i > \pi^2/6 \log 2$ then for λ almost all $x \in [0,1)$ exists a positive integer N_1 that depends on x, such that for all $n \geq N_1$:

$$\left|x - \frac{p_n(x)}{q_n(x)}\right| > x - x_n \qquad 1.4$$

In the opposite case when $\log \beta_i < \pi^2/6 \log 2$ we have an integer N_2 and for all $n \geq N_2$:

$$\left|x - \frac{p_n(x)}{q_n(x)}\right| < x - x_n \qquad 1.5$$

where the convergents of the β-expansion of x are $x_n = \varepsilon_1(x)/\beta_i + \varepsilon_2(x)/\beta_i^2 + \varepsilon_3(x)/\beta_i^3 + \cdots + \varepsilon_n(x)/\beta_i^n$.

II Running of $\alpha_S(Q)$ via β-expansion

The QCD Lagrangian is:

$$\mathcal{L}_{QCD} = \sum_f \bar{\Psi}_{f,a} \left(i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t_{ab}^C A_\mu^C - m_f \delta_{ab}\right) \Psi_{f,b} - \frac{1}{4} F_{\mu v}^A F^{A\,\mu v} \qquad (2.1)$$

where γ^μ are the Dirac γ-matrices, the $\Psi_{q,b}$ are quark-field spinors for a quark of flavor f and mass m_f, with a color-index a that runs from $a = 1$ to $N_C = 3$ and $g_s = \sqrt{4\pi\alpha_S}$ is the strong gauge coupling.

Let us remember that Lebesgue measure was used in the introduction to β-expansion. A generalization of the Lebesgue measure for any locally compact group is known as the Haar measure [4]. If we assume that the simple compact Lie group we need is $SU(N)$ one should see the paper by G. Nagy [5]. From the invariant metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ with a unitarity, provided that all eigenvalues λ_i different. For matrixes W and $W^\dagger$ we get $WdW^\dagger + dWW^\dagger = 0$ and in a given set of coordinates we attain the invariant measure:

$$d\mu(x) = \sqrt{\det g(x)} \prod_\alpha dx^\alpha \qquad 2.1$$

we parameterize the matrix W as $W = \exp(it_\alpha T_\alpha)$ therefore we conclude that:

$$\det g \sim (\det Q)^2 \prod_{i>j} \left|\lambda_i - \lambda_j\right|^4 \qquad 2.2$$

the distribution of eigenvalues on the unitary group is given by the invariant measure:

$$d\mu(\theta) = \prod_{i>j} \left|\exp(i\theta_i) - \exp(i\theta_j)\right|^2 \prod_i d\theta_i \quad 2.3$$

which is valid for U(N) groups. If we impose the constraint $\Sigma_i \theta_i = 0 \bmod 2\pi$ and implement it by a δ-function, the density distribution of the eigenvalues in SU(N) is given by the formula above as well. For the SU(3) group this would be of the form:

$$H(\theta) = \prod_{i>j} \left|\exp(i\theta_i) - \exp(i\theta_j)\right|^2 = \begin{vmatrix} 3 & \sum e_q & \sum e_q^2 \\ \sum \bar{e}_q & 3 & \sum e_q \\ \sum \bar{e}_q^2 & \sum \bar{e}_q & 3 \end{vmatrix} \quad 2.4$$

We must first describe the running of $\alpha_S(Q)$ and then proceed to connect it to the running of $\alpha(Q)$. Using β-functions for QCD, a SU(3) group theory we have for $i = 0, 1, 2, \dots, n$

$$2\beta(\alpha_S) = \frac{b_0}{2\pi}\alpha_S^2 - \frac{b_1}{4\pi^2}\alpha_S^3 - \frac{b_2}{64\pi^3}\alpha_S^4 - \cdots \quad 2.5$$

where the QCD β-function has a negative sign due to gluons carrying color charge which leads to self-interactions. At one-loop order we determine:

$$\alpha_S(\mu) = \frac{\alpha_S(\mu_0)}{1 - b_0\alpha_S(\mu_0)\ln\left(\frac{\mu}{\mu_0}\right)} \quad 2.6$$

Now we introduce the β-expansion for a real number q:

$$x = \sum_{n=0}^{\infty} c_n\, q^{-n} \quad 2.7$$

where for all $n \geq 0$ we have $0 \leq c_n \leq \lfloor q \rfloor$ where c_n doesn't have to be an integer and $\lfloor q \rfloor$ is a floor function [6].

We connect equations (2.6) and (2.7) which yields:

$$q^{-n} = \left(\frac{\alpha_S(Q^2)}{\pi}\right)^n + \mathcal{O}\left(\frac{\Lambda_{QCD}^4}{Q^4}\right) \quad 2.8$$

meaning that x from equation (2.7) equals $x = \delta_{QCD}$ which is the correction due to QCD effects and $\Lambda_{QCD} = 217(30)$ is the integration constant which coresponds to the scale where the perturbatively defined coupling would deverge and we use a value of 240 MeV but other values can be used as well. Knowing the values $c_1 = 1, c_2 = 1.9857 - 0.1152n_f, c_3 = -6.63694 - 1.20013n_f - 0.00518n_f^2 - 1.24\eta, c_4 = -156.61 + 18.77n_f - 0.797n_f^2 + 0.0215n_f^3 + C\eta$ where the coefficient C of the η-dependent piece in the α_S^4 term is yet to be determined and n_f is the number of flavors, allows us to establish that $\eta = \left(\sum e_q\right)^2/3\sum e_q^2$ is consistent with the Vandermonde eterminant of the Haar measure $H(\theta)$ for SU(3).

We introduce $v = 246$ GeV the vacuum expectation value of the Higgs field [7] that gives quarks mass and therefore we measure $\alpha_S(\Lambda_{QCD})$ for the first time being that no other work predicted the running value of α_S for the Λ_{QCD} parameter:

$$\alpha_S(\Lambda_{QCD}) = \frac{\ln\left(\frac{v}{\Lambda_{QCD}}\right)}{\pi^2} \qquad 2.9$$

This allows experimental tests to determine if the predicted value of $\alpha_S(\Lambda_{QCD})$ is accurate and therefore test the β-expansion method. The equation for the running value of $\alpha_S(Q)$ is then:

$$\alpha_S(Q) = \frac{\alpha_S(\Lambda_{QCD})}{\ln\left(\frac{Q}{\Lambda_{QCD}}\right)} \qquad 2.10$$

where it is evident that QCD enjoys asymptotic freedom. For $\alpha_S(Q^2)$ we add squared values for v, Λ_{QCD} and Q. The numerical results are provided in the table 1 bellow. All the values are in a great agreement with experimental values such as provided by [8], [9], [10], [11].

Particle Data Group [12] provides the world average $\alpha_S(M_z) = 0.1184(07)$. The method offered by β-expansion offers much desired [13] accuracy and simplicity. It might seem counterintuitive but instead of working within the U(1) group we remain within the SU(3) group with a connection to the $SU(2) \times U(1)$ symmetry through the necessary connection with the electroweak interactions and the Higgs mechanism.

Using the Lyapunov exponent for the rate of exponential divergence from the initial perturbed conditions, we have

$$\mathcal{L}(x) \coloneqq \lim_{n\to\infty} n^{-1} \log|(T^n)'(x)| \qquad 2.11$$

where the constant θ has the form $\theta = (\tau(\log \beta_i) - 1) \log \beta_i$ and $\tau(\gamma) \coloneqq \dim_H\{x \in [0,1): \mathcal{L}(x) = \gamma\}$ where we used the Housdorff dimension [14]. We eliminate the Landau pole in the running of $\alpha(Q)$ by establishing a link between $\alpha(Q)$ and $\alpha_S(Q)$ which eliminates the infinities since QCD enjoys asymptotic freedom:

$$\alpha^{-1}(Q) = \alpha^{-1}(0) - \left[\left(\frac{\pi^2}{\alpha_S(Q)}\right)^{\frac{1}{2}}\right] \tag{2.12}$$

where $\alpha^{-1}(0)$ is the value provided by NIST. The results are in good agreement with experimental values [15], [16].

Energy [GeV]	$\alpha_S(Q)$	$\alpha_S(Q^2)$	$\alpha^{-1}(Q)$	$\alpha^{-1}(Q^2)$
Λ_{QCD}	0.702404(06)	1.404808(07)		
1	0.492184(06)	0.492184(07)	132.557978(12)	132.557978(12)
10	0.188327(08)	0.188327(08)	129.796749(15)	129.796749(15)
M_z	0.118249(13)	0.118249(13)	127.90012(24)	127.90012(24)
$M_{GUT} = 10^{13}$	0.0224(21)	0.0224(21)	116.0454(33)	116.0454(33)
$M_{GUT} = 10^{15}$	0.0173(24)	0.0173(24)	113.151(26)	113.151(26)

Table 1: Measurments for the electromagneti and strong nuclear couplings for the values of $\Lambda_{QCD} = 240\ MeV$.

In order to prove the lack of Landau pole in my equations I provide the measurements albeit the Landau pole is not relevant for particle physics but is purely of academic interest. The value of the Landau pole is estimated roughly to be around 10^{286}eV, for this value $\alpha_S(Q) = 0.00112(24)$ and $\alpha^{-1}(Q) = 43.06(33)$ where the uncertainties are high since we are dealing with such a high energy level. We have thereby successfully eliminated the Landau pole.

III Conclusion and Debate

In the Standard Model, constants in the one-loop β-functions are given as:

$$b_i = \frac{2}{3}T(R_i)d(R_j)d(R_k) + \frac{1}{3}T(S_i)d(S_j)d(S_k) - \frac{11}{3}C_2(G_i) \tag{3.1}$$

we can summarize these results and insert the number of fermion generations $N_G = 3$ and Higgs doublets $N_H = 1$ obtaining us $b_1 = 41/10$, $b_2 = -19/6$ and $b_3 = -7$.

These three intersections point towards a range from $M_{GUT} = 10^{13}$GeV to $M_{GUT} = 10^{17}$GeV and correspond to a coupling ranging from $\alpha_{GUT}^{-1} = \alpha_S^{-1} \approx 40$ to 47 where we see an agreement for $M_{GUT} \approx 10^{13}$GeV in the table where $\alpha_{GUT}^{-1} = \alpha_S^{-1} = 44.64(18)$. This means that regardless of unification, the equations measure the running of the couplings in excellent agreement with both the experimental values and the theoretical predictions of the SM.

IV References

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