Proof of the Twin Prime Conjecture

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Abstract. I can proof that there are infinitely many twin primes. The twin prime counting function $\pi_2(n)$, which gives the number of twin primes less than or equal to n for any natural number n, is for $\lim n \to \infty$

$$\lim n \to \infty \qquad \pi_2(n) = 2\, C_2\, \frac{[\pi(n)]^2}{n}$$

where $\pi(n)$ is the prime counting function and C_2 is the so-called twin prime constant with $C_2 = 0{,}6601618\,...$ The prime numbers are equally distributed in the two progressions $(6\,i - 1)$ and $(6\,i + 1)$ as $n \to \infty$, respectively. Thus the prime counting function $\pi^-(n)$, which gives the number of primes less than or equal to n of the form $(6\,i - 1)$, and the prime counting function $\pi^+(n)$, which gives the number of primes less than or equal to n of the form $(6\,i + 1)$, are for $\lim n \to \infty$ both equal and one half of the prime counting function $\pi(n)$. To achieve both results I introduce a modified sieving method, based on the historical Eratosthenes sieve. This sieving method uses a shortened number line which is $\frac{1}{6}$ of the natural number line $\mathbb{N}$, where every integer i represents a possible twin prime of the form $(6\,i - 1, 6\,i + 1)$ and where every prime $p \geq 5$ generates two distinct infinite series $c_{1,2} + p\,i$ with $i = 1$ to ∞ and with $c_1 = \frac{p\pm1}{6}$ and $c_2 = p - \frac{p\pm1}{6}$, respectively, and thus two "starting values" $c_{1,2}$ within the interval $[1, p]$. According to Theorem 2.2 in conjunction with Theorem 1.4 I can prove that a number i on the shortened number line is not divisible ("sieveable") by any prime $p \geq 5$ from the ascending list of all primes is exactly $\left(1 - \frac{2}{p}\right)$ and all these "events" are independent of each other. Thus the proportion of two natural numbers n and $n + 2$ on the natural number line $\mathbb{N}$, both not divisible ("sieveable") by the ascending list of all primes $p \geq 5$ and $p \leq n$ can be described by the infinite product

$$\frac{1}{6} \prod_{\substack{p\ \text{prime} \\ p \geq 5}}^{p \leq n} \left(1 - \frac{2}{p}\right)$$

In comparison with the infinite product

$$\prod_{p\ \text{prime}}^{p \leq n} \left(1 - \frac{1}{p}\right)$$

which describes the proportion of natural numbers n not divisible by all the primes $p \leq n$ (according to the Eratosthenes sieve) I can proof that the Twin Prime Conjecture from Hardy and Littlewood is true.
With the modified sieving method I can also proof that a number i on the shortened number line, which represents either all the possible primes of the form $(6\,i - 1)$ or all the possible primes of the form $(6\,i + 1)$, is not divisible ("sieveable") by any prime p is $\left(1 - \frac{1}{p}\right)$ for $p \geq 5$ and all these "events" are independent of each other as well. This results in the fact that $\pi^-(n) = \pi^+(n) = \frac{1}{2}\,\pi(n)$ as $n \to \infty$.

1. Introduction:

For primes Euclid (ca. 300 B.C.) showed in his famous and well known proof that there are infinitely many primes. For twin prime pairs it was assumed that such a simple proof is very unlikely, because a great effort has already been made in the last 300 years by a big number of mathematicians all around the world (e.g. Yitang Zhang [1], Dan Goldston [2], Cem Yildirim and all their colleagues for modern times). The following proof rebuts the assumption and is based on the fact that the possible number of twin primes up to any primorial $n\#$, which are not "sieved" by all the primes less than n (see Theorem 2.2 in conjunction with Theorems 1.1 to 2.1), is described by the following equation, which is the so-called $\phi_2(n\#)$ function (see also table 1):

$$\phi_2(n\#) = \frac{1}{6}\, n\# \prod_{\substack{p\ \mathrm{prime} \\ p\geq 5}}^{p\leq n} \left(1 - \frac{2}{p}\right)$$

A comparison of the $\phi_2(n\#)$ function with the Euler $\phi(n\#)$ function for any primorial $n\#$, which gives the number of all possible primes up to any primorial $n\#$, which are not divided by all the primes less than n, with (see also table 1)

$$\phi(n\#) = n\# \prod_{p\ \mathrm{prime}}^{p\leq n} \left(1 - \frac{1}{p}\right)$$

finally shows that the twin prime counting function $\pi_2(n)$ is for $\lim n \to \infty$

$$\pi_2(n) = 2\, C_2\, \frac{[\pi(n)]^2}{n}$$

p	$n\#$	$\phi(n\#)$	$\phi_2(n\#)$
2, 3	6	2	1
5	30	8	3
7	210	48	15
11	2310	480	135
13	30030	5760	1485
17	510510	92160	22275
...	...	...	...

Table 1: values for the $\phi(n\#)$ and the $\phi_2(n\#)$ function up to p=17 and $n\# = 510510$

General Notations:

Let ℕ:={1, 2, 3, 4, ...} be the list of all positive integers, let a, c, g, i, j, k, m, q, s, z ∈ ℕ be positive integers, $n \in \mathbb{N}$ be used to denote natural numbers, ℙ:={2, 3, 5, 7, ...} be the ascending list of all primes p and their primorial $n\#$

$$n\# = \prod_{p\ \mathrm{prime}}^{p\leq n} p$$

be the product over all primes p less than or equal to any natural number n. Let the prime counting function $\pi(n)$ be the number of primes less than or equal to any natural number n with $\pi(n) =: \#\{p \in \mathbb{P} \mid p \leq n\}$ and the twin prime counting function $\pi_2(n)$ be the number of twin primes less than or equal to any natural number n with $\pi_2(n) =: \#\{(p, p+2) \in \mathbb{P} \mid p \leq n\}$.

Please note that the next four Theorems are based on each other step by step and Theorem 1.4 will be the fundamental theorem for the following modified sieving method for twin primes.

Theorem 1.1:

Let $P := \{p_1, p_2, \dots, p_g, \dots, p_s\}$ be the list of randomly generated primes $s \geq 1$ with $g = 1$ to s and their primorial s# be the product over all s primes (1.1)

$$s\# = \prod_{g=1}^{s} p_g$$

Please note that for $s = 1$ the "primorial s#" is the prime p_1 itself. Let the interval $[1, s\#]$ and all further infinite intervals $[m\ s\# + 1, (m+1)\ s\#\ ]$ on the natural number line $\mathbb{N}$ with $m = 1$ to ∞ be denoted by "all the infinite intervals s#".

Let further be the prime p_r a prime of the set $\mathbb{P} \neq P$ and the primorial $s_r\#$ be the product over all $(s + p_r)$ primes (1.2)

$$s_r\# = \prod_{g=1}^{s} p_g\ p_r$$

Let the interval $[1, s_r\#]$ and all further infinite intervals $[m\ s_r\# + 1, (m+1)\ s_r\#\ ]$ with $m = 1$ to ∞ be denoted by "all the infinite intervals $s_r\#$" and these intervals further be divided into p_r equal intervals, which are denoted by sections. These sections are defined by their terminal values $e_1, e_2, \dots, e_{p_r}$, whereupon $e_1 = 1\ s\#, e_2 = 2\ s\#, \dots, e_k = k\ s\#, \dots, e_{p_r} = p_r\ s\#$ with $k = 1$ to p_r and all these sections consist of s# values and are equal to all the intervals s# (1.3):

$$[1, e_1], [e_1 + 1, e_2], \dots, [e_{k-1} + 1, e_k], \dots, [e_{p_r - 1} + 1, e_{p_r}]$$

Let further be chosen the first value in all p_r sections and be indicated with E_1 and the second value in all p_r sections and be indicated with E_2 and so on until $j = s\#$. In general, let be chosen all values $j = 1$ to s# in all p_r sections and be indicated in all these sections with E_j.

Then: Within all the infinite intervals $s_r\#$ the prime p_r divides every integer E_j exactly one time.

Proof 1.1:

By definition the two integers s# (for $s = 1$ the prime p_1) and p_r are relatively prime $(s\# \perp p_r)$ and their greatest common divisor $\gcd(s\#, p_r) = 1$. Therefore the two integers s# and p_r divide themselves at their least common multiple $\text{lcm}(s\#, p_r) = s\#\ p_r$.

All integers E_j generate s# infinite series on the natural number line $\mathbb{N}$ and these s# series have the form $j + s\#\ (i - 1)$ with $j = 1$ to s# and $i = 1$ to ∞. The prime p_r also defines an infinite series of the form $p_r\ i$ with $i = 1$ to ∞ and therefore this two infinite series have a least common multiple of

$s\#\ p_r$ as well and expressed in numbers of sections a least common multiple of $\frac{s\#\ p_r}{s\#} = p_r$ sections, respectively.

If the prime p_r divides any integer E_j (for example E_1) in any of the sections k with $k = 1$ to p_r within the first interval $s_r\#$ the first time, thus the number of sections until the next integer E_j (for example E_1) is divided through the prime p_r is $(k + p_r)$ sections apart and this integer E_j (for example E_1) lies with $k \geq 1$ and $(k + p_r) > p_r$ out of the first interval $s_r\#$, namely in all the sections k of all further infinite intervals $s_r\#$. Therefore any integer E_j is divided through the prime p_r at most one time in every interval $s_r\#$ and thus the maximum number of divided integers E_j with $j = 1$ to $s\#$ through the prime p_r in any interval $s_r\#$ can only be $\sum_{j=1}^{s\#} j = s\#$. But at the same time $s\#$ is the actual amount of integers which are divided through the prime p_r in any interval $s_r\#$, because $\frac{s\#\ p_r}{p_r} = s\#$. Therefore the prime p_r must divide every integer E_j exactly one time within all the infinite intervals $s_r\#$. □

Theorem 1.2:

Theorem 1.2 is an extended form of Theorem 1.1 and therefore all the used notations in Theorem 1.1 are kept the same.

Let further be the infinite series of the prime p_r of the form $p_r\ i$ with $i = 1$ to ∞ not only be "started" at the integer p_r (as in Theorem 1.1) but also within the interval $[1, p_r]$. This p_r distinct infinite series will thus have the form $c + p_r\ (i - 1)$ with $i = 1$ to ∞ and $1 \leq c \leq p_r$.

Then: Within all the infinite intervals $s_r\#$ the prime p_r divides every integer E_j exactly one time, independently of the "starting value" c of the prime p_r within the interval $[1, p_r]$.
Please note that hereinafter it should be spoken of "sieved integers" if the "starting value" of the prime p_r is not equal to p_r itself.

Proof 1.2:

In Theorem 1.1 the starting value of the prime p_r within the first interval $s_r\#$ and all further intervals $s_r\#$ is the prime p_r itself and the terminal value is $s\#\ p_r$, thus within a range (distance) of $s\#\ p_r - p_r$ exactly $s\#$ integers are divided through the prime p_r. As long as the starting value c of the prime p_r plus the distance $s\#\ p_r - p_r$ is less than or equal to the total number of integers within any interval $s_r\#$

$$(s\#\ p_r - p_r + c) \leq s\#\ p_r$$

the "sieved" integers will be exactly $s\#$ as well and for $1 \leq c \leq p_r$ it is fulfilled.

According to Theorem 1.1 any integer E_j is divided through the prime p_r exactly one time in all the infinite intervals $s_r\#$ and it is independent in which section k the prime p_r sieves any integer E_j the first time. Thus the maximum number of divided integers E_j with $j = 1$ to $s\#$ through the prime p_r in all the infinite intervals $s_r\#$ can only be $\sum_{j=1}^{s\#} j = s\#$ as well. Therefore the prime p_r must divide every integer E_j with $j = 1$ to $s\#$ exactly one time within all the infinite intervals $s_r\#$, independently of the "starting value" c of the prime p_r within the interval $[1, p_r]$. □

Theorem 1.3:

Theorem 1.3 is an extended form of Theorem 1.1 and Theorem 1.2 and therefore all the used notations in Theorem 1.1 and Theorem 1.2 are kept the same.

Let further be $T \coloneqq \{t_1, t_2, \dots, t_a, \dots, t_q\}$ a set of randomly distributed "unsieved" integers t_a within the first interval s# with $a = 1$ to q and $1 \leq q < s\#$.

Let the set T generate an new set of "unsieved" integers $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ within the first interval $s_r\#$ and all further intervals $s_r\#$. Their distribution within all the infinite intervals $s_r\#$ corresponds, beginning at the starting values 1, $e_1, e_2, \dots, e_{p_r-1}$ of all p_r sections

$$[1, e_1], [e_1 + 1, e_2], \dots, [e_{k-1} + 1, e_k], \dots, [e_{p_r-1} + 1, e_{p_r}]$$

to the randomly distributed unsieved integers t_a within the interval $s\#$, because by definition any interval $s\#$ is equal to any sections k with its $s\#$ values E_j. The number of "unsieved" integers thus increases in all the infinite intervals $s_r\#$ to $q_n = q\, p_r$.

Then: Within all the intervals $s_r\#$ exactly q numbers from the set $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ are sieved from the prime p_r and it is independent from the "starting value" of the prime p_r within the interval $[1, p_r]$. The number of insieved integers of the set $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ is thus reduced from $q_n = q\, p_r$ to $q_r = q_n - q$ and $q_r = q\, p_r - q = q\,(p_r - 1)$, respectively.

Proof 1.3:

By definition, every unsieved integer t_a from the set T corresponds to a determined value E_j (by definition exactly q values E_j from the whole set of $s\#$ values E_j) and it is valid for all the $s\#$ sections within all the infinite intervals $s_r\#$.

According to Theorem 1.2 every integer E_j is sieved from the prime p_r exactly one time within all the intervals $s_r\#$ and it is independent of the "starting value" c of the prime p_r within the interval $[1, p_r]$. Therefore the unsieved integers t_a from the set T, which are by definition a subset of all $s\#$ values E_j, can only be sieved exactly one time from the prime p_r within all the intervals $s_r\#$ as well and it is independent of the "starting value" c of the prime p_r within the interval $[1, p_r]$. The number of insieved integers of the set $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ is thus reduced from $q_n = q\, p_r$ to $q_r = q_n - q$ and $q_r = q\, p_r - q = q\,(p_r - 1)$, respectively. □

Theorem 1.4:

Theorem 1.4 is an extended form of Theorem 1.1 to Theorem 1.3 and therefore all the used notations in Theorem 1.1 to Theorem 1.3 are kept the same.

Let further be z with $2 \leq z < p_r$ a selected number of p_r distinct infinite series of the prime p_r of the form $c + p_r\,(i - 1)$ with $i = 1$ to ∞ and with $1 \leq c \leq p_r$ and consider that these z distinct infinite series of the prime p_r run through all the infinite intervals $s_r\#$ synchronously and at the same time.

Then: Within all the intervals $s_r\#$ exactly q z numbers from the set $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ are sieved from the z distinct infinite series of the prime p_r of the form $c + p_r\,(i - 1)$ with $i = 1$ to ∞ and with

$1 \le c \le p_r$. The number of unsieved integers of the set $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ is thus reduced from $q_n = q\, p_r$ to $q_r = q_n - q\, z$ and $q_r = q\, p_r - q\, z = q\, (p_r - z)$, respectively.

Proof 1.4:

According to Theorem 1.3 each of the z infinite series of the prime p_r, which has by definition a distinct "starting value" c within the interval $[1, p_r]$, sieves exactly q integers of the set D and because these sieved integers are never at the same position within all the infinite intervals $s_r\#$, the number of sieved integers from the set D must be q z. The number of unsieved integers from the set $D \coloneqq \{d_1, d_2, \dots, d_{q_n}\}$ is thus reduced from $q_n = q\, p_r$ to $q_r = q_n - q\, z$ and $q_r = q\, p_r - q\, z = q\, (p_r - z)$, respectively. □

The next two Theorems describe the modified sieving method for twin primes.

Theorem 2.1:

a) Every possible twin prime, except the twin prime (3, 5), has the form $(6\,i - 1, 6\,i + 1)$ with $i = 1$ to ∞.

b) Any product of two integers of the form $(6\,i - 1)$ with $i = 1$ to ∞ gives an integer of the form $(6\,i + 1)$.

c) Any product of two integers of the form $(6\,i + 1)$ with $i = 1$ to ∞ gives an integer of the form $(6\,i + 1)$.

d) Any product of an integers of the form $(6\,i - 1)$ with an integers of the form $(6\,i + 1)$ with $i = 1$ to ∞ gives an integer of the form $(6\,i - 1)$.

Proof 2.1:

a) Every integer $n > 3$ can be written of the form $(6\,i - 2),\ (6\,i - 1),\ (6\,i),\ (6\,i + 1),\ (6\,i + 2),\ (6\,i + 3)$ with $i = 1$ bis ∞. Integers of the form $(6\,i - 2),\ (6\,i),\ (6\,i + 2)$ are divisible by 2 and thus cannot be prime. Integers of the form $(6\,i + 3)$ are divisible by 3 and thus cannot be prime as well. Therefore every twin prime except the twin prime (3, 5) must have the form $(6\,i - 1,\ 6\,i + 1)$ with $i = 1$ to ∞.

b) Any product of two integers of the form $(6\,i - 1)$ with $i = 1$ to ∞ is

$$(6\,i - 1)\,(6\,j - 1) = 6\,(6\,i\,j - j - i) + 1 \quad \text{with } i, j = 1 \text{ to } \infty$$

and thus an integer of the form $(6\,i + 1)$.

c) Any product of two integers of the form $(6\,i + 1)$ with $i = 1$ to ∞ is

$$(6\,i + 1)\,(6\,j + 1) = 6\,(6\,i\,j + j + i) + 1 \quad \text{with } i, j = 1 \text{ to } \infty$$

and thus an integer of the form $(6\,i + 1)$.

d) Any product of an integer of the form $(6\,i - 1)$ with an integer of the form $(6\,j + 1)$ with $i, j = 1$ to ∞ is

$$(6\,i - 1)\,(6\,j + 1) = 6\,(6\,i\,j - j + i) - 1 \quad \text{with } i, j = 1 \text{ to } \infty$$

and thus an integer of the form $(6\,\mathrm{i}-1)$. □

Theorem 2.2:

Let the number of all possible twin primes up to any primorial $\mathrm{n}\#$, which are not "sieved" by all the primes less than n of the form $(6\,\mathrm{i}-1, 6\,\mathrm{i}+1)$, except the twin prime (3, 5), be the so-called $\phi_2(\mathrm{n}\#)$ function.

Then: The so-called $\phi_2(\mathrm{n}\#)$ function for any primorial $\mathrm{n}\#$ is (2.1)

$$\phi_2(\mathrm{n}\#) = \frac{1}{6}\,\mathrm{n}\# \prod_{\substack{\mathrm{p\ prime} \\ \mathrm{p} \geq 5}}^{\mathrm{p} \leq \mathrm{n}} \left(1 - \frac{2}{\mathrm{p}}\right)$$

Proof 2.2:

As a first step we add the number 1 to any integer of the form $(6\,\mathrm{i}-1)$ and divide it through the number 6. As a second step we subtract the number 1 from any integer of the form $(6\,\mathrm{i}+1)$ and divide it through the number 6. Thus we receive two shortened number lines whereupon considered in conjunction every integer i with $\mathrm{i} = 1$ to ∞ represents a possible twin prime of the form $(6\,\mathrm{i}-1, 6\,\mathrm{i}+1)$. The shortened number line is $\frac{1}{6}$ of the natural number line $\mathbb{N}$.

Any product of the integers of the form $(6\,\mathrm{i}-1)$ and $(6\,\mathrm{i}+1)$ with $\mathrm{i} = 1$ to ∞ generates the following infinite series on the shortened number line:

a) Any product of two integers of the form $(6\,\mathrm{i}-1)$ with $\mathrm{i} = 1$ to ∞ generates (see also Proof 2.1.b):

$$6\,(6\,\mathrm{i\,j} - \mathrm{j} - \mathrm{i}) + 1 = \qquad |-1 \ \ |:6$$
$$6\,\mathrm{i\,j} - \mathrm{j} - \mathrm{i} =$$
$$(6\,\mathrm{j} - 1)\,\mathrm{i} - \mathrm{j} \qquad \text{with } \mathrm{i}, \mathrm{j} = 1 \text{ to } \infty$$

It relates to the following infinite series:

$$\mathrm{j} = 1: \quad -1 + 5\,\mathrm{i} \quad = 4, 9, 14, \dots \triangleq (5\,.\,5), (5\,.\,11), (5\,.\,17), \dots$$
$$\mathrm{j} = 2: \quad -2 + 11\,\mathrm{i} \quad = 9, 20, 31, \dots \triangleq (11\,.\,5), (11\,.\,11), (11\,.\,17), \dots$$
$$\mathrm{j} = 3: \quad -3 + 17\,\mathrm{i} \quad = 14, 31, 48, \dots \triangleq (17\,.\,5), (17\,.\,11), (17\,.\,17), \dots$$

.....

b) Any product of two integers of the form $(6\,\mathrm{i}+1)$ with $\mathrm{i} = 1$ to ∞ generates (see also Proof 2.1.c):

$$6\,(6\,\mathrm{i\,j} + \mathrm{j} + \mathrm{i}) + 1 = \qquad |-1 \ \ |:6$$
$$6\,\mathrm{i\,j} + \mathrm{j} + \mathrm{i} =$$
$$(6\,\mathrm{j} + 1)\,\mathrm{i} + \mathrm{j} \qquad \text{with } \mathrm{i}, \mathrm{j} = 1 \text{ to } \infty$$

It relates to the following infinite series:

$$\mathrm{j} = 1: \quad 1 + 7\,\mathrm{i} \quad = 8, 15, 22, \dots \triangleq (7\,.\,7), (7\,.\,13), (7\,.\,19), \dots$$
$$\mathrm{j} = 2: \quad 2 + 13\,\mathrm{i} \quad = 15, 28, 41, \dots \triangleq (13\,.\,7), (13\,.\,13), (13\,.\,19), \dots$$
$$\mathrm{j} = 3: \quad 3 + 19\,\mathrm{i} \quad = 22, 41, 60, \dots \triangleq (19\,.\,7), (19\,.\,13), (19\,.\,19), \dots$$

.....

c) Any product of an integer of the form $(6\,\mathrm{i}-1)$ with an integer of the form $(6\,\mathrm{i}+1)$ with $\mathrm{i}=1$ to ∞ generates (see also Proof 2.1.d):

$6\,(6\,\mathrm{i}\,\mathrm{j}-\mathrm{j}+\mathrm{i})-1=$ $\quad | +1 \;\; |:6$
$6\,\mathrm{i}\,\mathrm{j}-\mathrm{j}+\mathrm{i}=$
$(6\,\mathrm{j}+1)\,\mathrm{i}-\mathrm{j}$ $\quad$ with $\mathrm{i},\mathrm{j}=1$ to ∞

It relates to the following infinite series:

$\mathrm{j}=1$: $\quad -1+7\,\mathrm{i} \quad = 6, 13, 20, \ldots \triangleq (7\,.\,5), (7\,.\,11), (7\,.\,17), \ldots$
$\mathrm{j}=2$: $\quad -2+13\,\mathrm{i} \quad = 11, 24, 37, \ldots \triangleq (13\,.\,5), (13\,.\,11), (13\,.\,17), \ldots$
$\mathrm{j}=3$: $\quad -3+19\,\mathrm{i} \quad = 16, 35, 54, \ldots \triangleq (19\,.\,5), (19\,.\,11), (19\,.\,17), \ldots$
.....

or

$(6\,\mathrm{i}-1)\,\mathrm{j}+\mathrm{i}$ $\quad$ with $\mathrm{i},\mathrm{j}=1$ to ∞

It relates to the following infinite series:

$\mathrm{i}=1$: $\quad 1+5\,\mathrm{j} \quad = 6, 11, 16, \ldots \triangleq (5\,.\,7), (5\,.\,13), (5\,.\,19), \ldots$
$\mathrm{i}=2$: $\quad 2+11\,\mathrm{j} \quad = 13, 24, 35, \ldots \triangleq (11\,.\,7), (11\,.\,13), (11\,.\,19), \ldots$
$\mathrm{i}=3$: $\quad 3+17\,\mathrm{j} \quad = 20, 37, 54, \ldots \triangleq (17\,.\,7), (17\,.\,13), (17\,.\,19), \ldots$
.....

The latter two infinite series $(6\,\mathrm{j}+1)\,\mathrm{i}-\mathrm{j}$ and $(6\,\mathrm{i}-1)\,\mathrm{j}+\mathrm{i}$, respectively, represent for $\mathrm{i},\mathrm{j}=1$ to ∞ the same integers on the shortened number line, but as for the following modified sieving method for twin primes any prime p must be considered successively from the ascending list of all primes p, both infinite series will be successively taken into account.

If we look at the infinite series from above in greater detail, we will see that every prime p generates on the shortened number line two distinct infinite series of the form $\mathrm{c}_{1,2}+\mathrm{p}\,\mathrm{j}$ with $\mathrm{j}=1$ to ∞. For example, prime 5 generates two infinite series $1+5\,\mathrm{j}$ and $-1+5\,\mathrm{j}$, prime 7 generates two infinite series $1+7\,\mathrm{j}$ and $-1+7\,\mathrm{j}$, prime 11 generates two infinite series $2+11\,\mathrm{j}$ and $-2+11\,\mathrm{j}$ and so on… . All positive starting values from the distinct infinite series from above represent the corresponding prime p on the natural number line $\mathbb{N}$ and for the following sieving method every prime p will be considered as well. Thus the two distinct infinite series for every prime p will have a starting value, expressed through the corresponding prime p, with $\mathrm{c}_1=\frac{\mathrm{p}\pm1}{6}$ and $\mathrm{c}_2=\mathrm{p}-\frac{\mathrm{p}\pm1}{6}$, respectively, and therefore two starting values $\mathrm{c}_{1,2}$ within the interval $[1,\mathrm{p}]$.

Modified sieving method for twin primes:

Step 1: On the shortened number line, the first prime $\mathrm{p}=5$ generates two infinite series, namely
$1+5\,\mathrm{j} \quad = 1, 6, 11, 16, \ldots \triangleq (5\,.\,1), (5\,.\,7), (5\,.\,13), (5\,.\,19), \ldots$ and
$-1+5\,\mathrm{j} \quad = 4, 9, 14, \ldots \triangleq (5\,.\,5), (5\,.\,11), (5\,.\,17), \ldots$
The sieved integers from the two infinite series from above are apart from each other by the value +3 (1-4; 6-9; 11-14; …), thus the prime $\mathrm{p}=5$ sieves on the shortened number line $\frac{2}{5}$ of all integers and remain $\frac{3}{5}$ of all integers unsieved.

Step 2: The next prime $p = 7$ generates also two infinite series, namely

$1 + 7\,j \quad = 1, 8, 15, 22, \dots \triangleq (7\,.\,1), (7\,.\,7), (7\,.\,13), (7\,.\,19), \dots$ and

$-1 + 7\,j \quad = 6, 13, 20, \dots \triangleq (7\,.\,5), (7\,.\,11), (7\,.\,17), \dots$

According to Theorem 1.4 with $s\# = p_1 = 5$, $s_r\# = 5\,.\,7 = 35$, $p_r = 7$, $q = 3$ and $z = 2$ the prime $p_r = p = 7$ sieves $q\,z = 3\,.\,2 = 6$ integers within all the infinite intervals $s_r\#$ and lets $q_r = q\,(p_r - z) = 3\,(7 - 2) = 15$ integers within all the infinite intervals $s_r\#$ be unsieved.

Thus the number of unsieved integers up to the primorial $i\# = 5\,.\,7 = 35$ is (2.2)

$$\phi_2(35) = 35 \prod_{\substack{p\ \mathrm{prime} \\ p \ge 5}}^{p \le 7} \left(\frac{p-2}{p}\right) = 15$$

Step 3: The next prime $p = 11$ generates also two infinite series, namely

$2 + 11\,j \quad = 2, 13, 24, 35, \dots \triangleq (11\,.\,1), (11\,.\,7), (11\,.\,13), (11\,.\,19), \dots$ and

$-2 + 11\,j \quad = 9, 20, 31, \dots \triangleq (11\,.\,5), (11\,.\,11), (11\,.\,17), \dots$

According to Theorem 1.4 with $s\# = 5\,.\,7 = 35$, $s_r\# = 35\,.\,11 = 385$, $p_r = 11$, $q = 15$ and $z = 2$ the prime $p_r = p = 11$ sieves $q\,z = 15\,.\,2 = 30$ integers within all the infinite intervals $s_r\#$ and lets $q_r = q\,(p_r - z) = 15\,(11 - 2) = 135$ integers within all the infinite intervals $s_r\#$ be unsieved.

Thus the number of unsieved integers up to the primorial $i\# = 385$ is (2.3)

$$\phi_2(385) = 385 \prod_{\substack{p\ \mathrm{prime} \\ p \ge 5}}^{p \le 11} \left(\frac{p-2}{p}\right) = 135$$

Step 4 to ∞: All these steps can be applied for the ascending list of all primes p and because all primes p generate two infinite series $c_{1,2} + p\,i$ with two distinct starting values $c_{1,2}$ within the interval $[1, p]$, the number of unsieved integers on the shortened number line up to any primorial $i\#$ is (2.4)

$$\phi_2(i\#) = i\# \prod_{\substack{p\ \mathrm{prime} \\ p \ge 5}}^{p \le 6\,i \pm 1} \left(\frac{p-2}{p}\right)$$

and expressed for the natural number line ℕ and thus for the primorial n#, which represents all the possible twin primes, which are not “sieved” by all the primes less than n, except the twin prime (3, 5), it is (see also table 1) (2.5)

$$\phi_2(n\#) = \frac{1}{6}\,n\# \prod_{\substack{p\ \mathrm{prime} \\ p \ge 5}}^{p \le n} \left(\frac{p-2}{p}\right) = \frac{1}{6}\,n\# \prod_{\substack{p\ \mathrm{prime} \\ p \ge 5}}^{p \le n} \left(1 - \frac{2}{p}\right)$$

□

2. Proof of the Twin Prime Conjecture

Theorem 3:

There are infinitely many twin primes.

Proof 3:

Let the prime counting function $\pi(n)$ be the number of primes less than or equal to n. The prime number theorem [3] states that as $n \to \infty$ (3.1)

$$\lim n \to \infty \qquad \frac{\pi(n)}{\frac{n}{\ln(n)}} = 1$$

and thus the prime number density function for $\lim n \to \infty$ is (3.2)

$$\lim n \to \infty \qquad \frac{\pi(n)}{n} = \frac{1}{\ln(n)}$$

Let the twin prime counting function $\pi_2(n)$ be the number of twin primes less than or equal to n and the twin prime number density function be (3.3)

$$\frac{\pi_2(n)}{n}$$

Let further be $\phi(n\#)$ the Euler ϕ function (Euler totient function) for any primorial $n\#$ (3.4)

$$\phi(n\#) = n\# \prod_{p \text{ prime}}^{p \le n} \left(1 - \frac{1}{p}\right) = n\# \prod_{p \text{ prime}}^{p \le n} \left(\frac{p-1}{p}\right)$$

where the product extends over all primes p dividing $n\#$. This function states the number of all possible primes up to any primorial $n\#$, which are not divided by all the primes less than n, and includes all the actual primes within the interval $[n, n^2]$.

Mertens' 3rd Theorem [4] describes the connection between the infinite product $\prod_{p \text{ prime}}^{p \le n} \left(\frac{p-1}{p}\right)$ and the natural logarithm ln(n) as $n \to \infty$ (3.5)

$$\lim n \to \infty \qquad \ln(n) \prod_{p \text{ prime}}^{p \le n} \left(\frac{p-1}{p}\right) = e^{-\gamma}$$

and with the Euler-Mascheroni constant $\gamma = 0{,}57721 \ldots$ the equation from above (3.5) in conjunction with (3.4) gives (3.6)

$$\lim n \to \infty \qquad \prod_{p \text{ prime}}^{p \le n} \left(\frac{p-1}{p}\right) = \frac{1}{\ln(n)} e^{-\gamma} = \frac{1}{\ln(n) * e^{\gamma}} = \frac{1}{\ln(n^{e^{\gamma}})} = \frac{\phi(n\#)}{n\#}$$

Thus the Euler $\phi(n\#)$ function for any primorial $n\#$ divided through the primorial $n\#$ is equal to one divided through the natural logarithm $\ln(n^{e^{\gamma}})$ with $n^{e^{\gamma}} = n^{1,7810720\ldots}$ as $n \to \infty$.

As for $\lim n \to \infty \ \frac{1}{\ln(n^{e^\gamma})}$ is also equal to the prime number density function $\frac{\pi(n^{e^\gamma})}{n^{e^\gamma}}$ (see equation 3.2), thus the Euler $\phi(n\#)$ function not only includes all the actual primes within the interval $[n, n^2]$ but also describes the prime number density function $\frac{\pi(n^{e^\gamma})}{n^{e^\gamma}}$ with $n < n^{e^\gamma} = n^{1,7810720\ldots} < n^2$ if it is divided through the primorial n# (3.7):

$$\lim n \to \infty \qquad \frac{\phi(n\#)}{n\#} = \prod_{p\ prime}^{p\leq n} \left(1 - \frac{1}{p}\right) = \frac{\pi(n^{e^\gamma})}{n^{e^\gamma}}$$

Let $\phi_2(n\#)$ be the so-called ϕ_2 function for any primorial $n\#$ (see Theorem 2.2) with (3.8)

$$\phi_2(n\#) = \frac{1}{6}\ n\# \prod_{\substack{p\ prime \\ p\geq 5}}^{p\leq n} \left(\frac{p-2}{p}\right)$$

According to Theorem 2.2 this function states the number of all the possible twin primes up to any primorial $n\#$, which are not divided or "sieved" by all the primes less than n, except the twin prime (3, 5), and includes all the actual twin primes within the interval $[n, n^2]$.

With the so-called twin prime constant C_2 [5], which is defined as (3.9)

$$\lim n \to \infty \qquad C_2 = \prod_{\substack{p\ prim \\ p\geq 3}}^{p\leq n} \left(\frac{p\,(p-2)}{(p-1)^2}\right) = 0{,}6601618 \ldots$$

where the products extends over all primes except the prime 2, and with (3.10)

$$\prod_{\substack{p\ prime \\ p\geq 5}}^{p\leq n} \left(\frac{p-2}{p}\right) = \left[\prod_{\substack{p\ prime \\ p\geq 5}}^{p\leq n} \left(\frac{p-1}{p}\right)\right]^2 \prod_{\substack{p\ prime \\ p\geq 5}}^{p\leq n} \left(\frac{p\,(p-2)}{(p-1)^2}\right) =$$

$$= \left[\frac{2}{1}\ \frac{3}{2}\right]^2 \left[\prod_{p\ prime}^{p\leq n} \left(\frac{p-1}{p}\right)\right]^2 \frac{4}{3} \prod_{\substack{p\ prim \\ p\geq 3}}^{p\leq n} \left(\frac{p\,(p-2)}{(p-1)^2}\right)$$

we obtain from (3.8), (3.10) and (3.4) for $\lim n \to \infty$ (3.11)

$$\phi_2(n\#) = \frac{1}{6}\ n\#\ 9 \left[\prod_{p\ prime}^{p\leq n} \left(\frac{p-1}{p}\right)\right]^2 \frac{4}{3}\ C_2 =$$

$$\frac{\phi_2(n\#)}{n\#} = 2\ C_2 \left[\prod_{p\ prime}^{p\leq n} \left(\frac{p-1}{p}\right)\right]^2 =$$

$$\lim n \to \infty \qquad \frac{\phi_2(n\#)}{n\#} = 2\ C_2 \left[\frac{\phi(n\#)}{n\#}\right]^2$$

The equation from above states that there is a fixed relation between the possible primes and the possible twin primes up to any primorial $n\#$ as $n \to \infty$. We further know that the Euler $\phi(n\#)$ function for primes and the so-called $\phi_2(n\#)$ function for twin primes also include the actual primes and the actual twin primes within the interval $[n, n^2]$. As the Euler $\phi(n\#)$ function for any primorial $n\#$ divided through the primorial $n\#$ describes the prime number density function $\frac{\pi(n^{e^\gamma})}{n^{e^\gamma}}$ for $\lim n \to \infty$ (3.7), the equation from above also states the relation between the prime number density function $\frac{\pi(n^{e^\gamma})}{n^{e^\gamma}}$ and the twin prime number density function $\frac{\pi_2(n^{e^\gamma})}{n^{e^\gamma}}$ for $\lim n \to \infty$ and in general the relation between the prime number density function $\frac{\pi(n)}{n}$ and the twin prime number density function $\frac{\pi_2(n)}{n}$ as $n \to \infty$ (3.12):

$$\lim n \to \infty \qquad \frac{\pi_2(n)}{n} = 2\,C_2 \left[\frac{\pi(n)}{n}\right]^2$$

As the prime number density function $\frac{\pi(n)}{n}$ can never become zero, the twin prime number density function $\frac{\pi_2(n)}{n}$ can never become zero as well and thus there are infinitely many twin primes. The twin prime counting function $\pi_2(n)$ for $\lim n \to \infty$ is (3.13)

$$\lim n \to \infty \qquad \pi_2(n) = 2\,C_2 \frac{[\pi(n)]^2}{n}$$

□

An estimate for $n \gg 0$ is (3.14)

$$\pi_2(n) \cong 2\,C \frac{[\pi(n)]^2}{n} \quad \text{with } C \cong C_2$$

and (3.15)

$$\Delta\pi_2(n) \cong 2\,C \frac{[\Delta\pi(n)]^2}{\Delta n} \quad \text{with } C \cong C_2$$

respectively (see Table 2 for the equation 3.15 with the actual primes and the actual twin primes from 10^6up to 10^{16} and the calculated constant $C \cong C_2 = 0{,}66016\ldots$).

	$\pi(n)$	$\pi_2(n)$	Δn	$\Delta\pi(n)$	$\Delta\pi_2(n)$	C
10^6	78.498	8.169				
10^7	664.579	58.980	$10^6 - 10^7$	586.081	50.811	0,66566
10^8	5.761.455	440.312	$10^7 - 10^8$	5.096.876	381.332	0,66055
10^9	50.847.534	3.424.506	$10^8 - 10^9$	45.086.079	2.984.194	0,66062
10^{10}	455.052.511	27.412.679	$10^9 - 10^{10}$	404.204.977	23.988.173	0,66070
10^{11}	4.118.054.813	224.376.048	$10^{10} - 10^{11}$	3.663.002.302	196.963.369	0,66058
10^{12}	37.607.912.018	1.870.585.220	$10^{11} - 10^{12}$	33.489.857.205	1.646.209.172	0,66050
10^{13}	346.065.536.839	15.834.664.872	$10^{12} - 10^{13}$	308.457.624.821	13.964.079.652	0,66044
10^{14}	3.204.941.750.802	135.780.321.665	$10^{13} - 10^{14}$	2.858.876.213.963	119.945.656.793	0,66040
10^{15}	29.844.570.422.669	1.177.209.242.304	$10^{14} - 10^{15}$	26.639.628.671.867	1.041.428.920.639	0,66037
10^{16}	279.238.341.033.925	10.304.195.697.298	$10^{15} - 10^{16}$	249.393.770.611.256	9.126.986.454.994	0,66034

Table 2: actual numbers for $\pi(n)$, $\pi_2(n)$, Δn, $\Delta\pi(n)$, $\Delta\pi_2(n)$ and C

3. Prime Number Theorem for Arithmetic Progressions

Theorem 4:

The prime counting function $\pi^{-}(\mathrm{n})$, which gives the number of primes less than or equal to n of the form $(6\ \mathrm{i}-1)$, and the prime counting function $\pi^{+}(\mathrm{n})$, which gives the number of primes less than or equal to n of the form $(6\ \mathrm{i}+1)$, are for $\lim \mathrm{n} \to \infty$ both equal and one half of the prime counting function $\pi(\mathrm{n})$:

$$\lim \mathrm{n} \to \infty \qquad \pi^{-}(\mathrm{n}) = \pi^{+}(\mathrm{n}) = \frac{1}{2}\ \pi(\mathrm{n})$$

Proof 4:

As a first step we add the number 1 to every integer of the form $(6\ \mathrm{i}-1)$ and divide it through the number 6. We thus receive a shortened number line where any integer i represents a possible prime of the form $(6\ \mathrm{i}-1)$. The shortened number line is $\frac{1}{6}$ of the natural number line $\mathbb{N}$.

The same sieving method according to Theorem 2.2 in conjunction with Theorem 2.1.d and Theorem 1.3 is applied for the two infinite series $(6\ \mathrm{i}-1)\ \mathrm{j}+\mathrm{i}$ and $(6\ \mathrm{j}+1)\ \mathrm{i}-\mathrm{j}$ with $\mathrm{i}, \mathrm{j} = 1$ to ∞, which generates the following infinite series

$1 + 5\ \mathrm{j} \quad = 1, 6, 11, \ldots$
$-1 + 7\ \mathrm{j} \quad = 6, 13, 20, \ldots$
$2 + 11\ \mathrm{j} \quad = 2, 13, 24, \ldots$
$-2 + 13\ \mathrm{j} \quad = 11, 24, 37, \ldots$
….

and we obtain a so-called $\phi^{-}(\mathrm{i}\#)$ function for any primorial i# (4.1)

$$\mathrm{i}\# = \prod_{\substack{\mathrm{p\ prime} \\ \mathrm{p} \geq 5}}^{\mathrm{p} \leq 6\ \mathrm{i} \pm 1} \mathrm{p}$$

(4.2)

$$\phi^{-}(\mathrm{i}\#) = \mathrm{i}\# \prod_{\substack{\mathrm{p\ prime} \\ \mathrm{p} \geq 5}}^{\mathrm{p} \leq 6\ \mathrm{i} \pm 1} \left(\frac{\mathrm{p}-1}{\mathrm{p}}\right) = \mathrm{i}\# \prod_{\substack{\mathrm{p\ prime} \\ \mathrm{p} \geq 5}}^{\mathrm{p} \leq 6\ \mathrm{i} \pm 1} \left(1 - \frac{1}{\mathrm{p}}\right)$$

The equation from above (4.2) expressed for the natural number line $\mathbb{N}$ and thus for the primorial $\mathrm{n}\#$ and the product of all primes less then n gives (4.3)

$$\phi^{-}(\mathrm{n}\#) = \frac{1}{6}\ \mathrm{n}\#\ \frac{2}{1}\ \frac{3}{2} \prod_{\mathrm{p\ prime}}^{\mathrm{p} \leq \mathrm{n}} \left(1 - \frac{1}{\mathrm{p}}\right) =$$

$$\phi^{-}(\mathrm{n}\#) = \frac{1}{2}\ \mathrm{n}\# \prod_{\mathrm{p\ prime}}^{\mathrm{p} \leq \mathrm{n}} \left(1 - \frac{1}{\mathrm{p}}\right)$$

This equation states the number of all possible primes of the form $(6\,i - 1)$ up to the primorial $n\#$, which are not divided or "sieved" by all the primes less than n, and includes all the actual primes of the form $(6\,i - 1)$ within the interval $[n, n^2]$.

As a second step we subtract the number 1 from every integer of the form $(6\,i + 1)$ and divide it through the number 6. We thus receive a shortened number line where any integer i represents a possible prime of the form $(6\,i + 1)$.

The same sieving method from above is applied for the two infinite series $(6\,j - 1)\,i - j$ and $(6\,j + 1)\,i + j$ with $i, j = 1$ bis ∞, which generates the following infinite series

$-1 + 5\,j \quad = 4, 9, 14, \ldots$

$1 + 7\,j \quad = 1, 8, 15, \ldots$

$-2 + 11\,j \quad = 9, 20, 31, \ldots$

$2 + 13\,j \quad = 2, 15, 28, \ldots$

....

and we receive a so-called $\phi^+(n\#)$ function for any primorial $n\#$, which is identical to equation (4.3) (4.4)

$$\phi^+(n\#) = \frac{1}{2}\, n\# \prod_{p\ prime}^{p \le n} \left(1 - \frac{1}{p}\right)$$

As we know that the Euler $\phi(n\#)$ function for any primorial $n\#$ divided through the primorial $n\#$ is equal to the prime number density function $\frac{\pi\left(n^{e^\gamma}\right)}{n^{e^\gamma}}$ for $\lim n \to \infty$ (3.7):

$$\lim n \to \infty \qquad \frac{\phi(n\#)}{n\#} = \prod_{p\ prime}^{p \le n} \left(1 - \frac{1}{p}\right) = \frac{\pi\left(n^{e^\gamma}\right)}{n^{e^\gamma}}$$

we see in comparison with equation (4.2) and (4.4), respectively, that the prime counting function $\pi^-(n)$ and the prime counting function $\pi^+(n)$ is exactly one half of the prime counting function $\pi(n)$ for $\lim n \to \infty$ (4.5)

$$\lim n \to \infty \qquad \pi^-(n) = \pi^+(n) = \frac{1}{2}\ \pi(n)$$

□

4. Conclusion:

I studied Civil Engineering at the Technical University of Vienna and I am examiner at the Austrian Patent Office. It was my unbiased look at twin prime pairs, which induced me to investigate the so-called $\phi(n\#)$ function from scratch and with an open mind. Theorems 1.1 to 2.2 are the results of this unbiased look. Especially Theorem 1.4 can be a helpful step for further investigation in number theory and especially in prime number theory.

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