A cousin of one of Ramanujan's Identities

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Abstract

This submission gives a closed form identity similar to one given by Ramanujan. A formula for infinitely many similar identities is presented here as well.

1 Definitions and Criteria on fields and algeabriac quantites in question

To start, consider p to be a prime of the form $4k+1$. Let ϵ be the fundamental unit of the real quadratic field with discriminant p with no concern on the embedding or sign of ϵ. Since p is a prime of the form $4k+1$, it just so happens that it can uniquely be written as the sum of two squares

$$p = a^2 + b^2$$

up to the order and sign of a and b. To slightly reduce confusion when it comes to the order, let b be even. Using the greek letter π, let it be defined as the algebraic quantity

$$\pi = a + b\imath$$

unless otherwise stated. Let ω be a solution in $\mathbb{F}_p$ to the equation

$$\omega^2 + 1 \equiv 0 \pmod{p}$$

Let K_1, K_2 be the fields defined as

$$K_1 = \mathbb{Q}(\sqrt{\epsilon})$$
$$K_2 = \mathbb{Q}(\sqrt{\pi})$$

and designate the basis of K_1's unit group as

$$\mathcal{O}_{K_1}^{\times} = < \sqrt{\epsilon}, \gamma >$$

Let $\gamma^{'}$ be the expression

$$\gamma^{'} = \left(\sqrt{\frac{\gamma^-}{\gamma^+}}\right)^{\mathrm{h}_{K_2}}$$

where the algebraic quantities $\gamma^{\pm}$ is are the positive/negative embeddings of γ into $\mathbb{R}$ such that

$$\mid \gamma^{'} \mid \leq 1$$

and the quantity h_{K_2} is the class number of K_2.

2 More definitions and tools

Still taking p to be the arbitrary $4k+1$ from the previous section, denote the quantity d as

$$d = 2^{\frac{1-\chi_8(p)}{2}}$$

where the character χ_8 is the real Dirichlet character taking the values

$$\chi_8(p) = (-1)^{\frac{p-1}{4}}$$

which is completely fine since p is of the form $4k+1$.
Let ψ be the non-trivial quadratic character on $\mathbb{F}_p$. Denote the quantity c as

$$c = \sum_{n \in \mathbb{F}_p^\times} \psi(n) \cdot n^2$$

Let ω be any integer solution in $\mathbb{F}_p^\times$ to the equation

$$\omega^2 + 1 \equiv 0 \pmod{p}$$

and take the related algebraic quantity τ as

$$\tau = \imath + \omega$$

where $\imath$ is $\sqrt{-1}$. The choice of ω is irrelevant to the purpose it serves; it does not matter whether ω or $\omega + p$ is used.

Let $(a; q)$ be a function of a and q defined by the product

$$(a; q) = \prod_{l \geq 0} \left(1 - a \cdot q^l\right)$$

This is the **q-Pochammer symbol**, $(a; q)_\infty$, but the $_\infty$ subscript has been ommitted out of convenience.

3 Unit Formula

By maintaining everything that was defined from before, with the exception that π is now the circle constant rather than a Gaussian prime, The quantity γ' can be given as the following product;

$$\gamma'^{d} = e^{-\frac{\pi \cdot c \cdot d}{p^2}} \cdot \prod_{l \in \mathbb{F}_p^\times} \left(\left(e^{\frac{2\pi \imath d \cdot \tau}{p} \cdot l}; e^{-2\pi d}\right) \cdot \left(e^{-\frac{2\pi \imath d \cdot \bar{\tau}}{p} \cdot l}; e^{-2\pi d}\right) \right)^{\psi(l)}$$

Alternatively, the quantity $-\dfrac{\pi \cdot c \cdot d}{p^2}$ can expressed differently as

$$-\frac{\pi \cdot c \cdot d}{p^2} = \frac{2\pi d}{p} \cdot L(\psi, -1)$$

where $L(\psi, s)$ is the Dirichlet L-function of the same character ψ defined earlier. If p is taken to be 5 and ζ_5 is the 5th root of unity

$$\zeta_5 = e^{\frac{2\pi \imath}{5}}$$

then by plugging in concrete quantities for all the undetermined quantities, this yields the curious identity

$$\frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} = e^{-\frac{8\pi}{25}} \prod_{l \geq 0} \frac{\left(1 - \zeta_5 e^{-(5l+1)\frac{4\pi}{5}}\right)\left(1 - \zeta_5^{-1} e^{-(5l+1)\frac{4\pi}{5}}\right)\left(1 - \zeta_5 e^{-(5l+4)\frac{4\pi}{5}}\right)\left(1 - \zeta_5^{-1} e^{-(5l+4)\frac{4\pi}{5}}\right)}{\left(1 - \zeta_5^{2} e^{-(5l+2)\frac{4\pi}{5}}\right)\left(1 - \zeta_5^{-2} e^{-(5l+2)\frac{4\pi}{5}}\right)\left(1 - \zeta_5^{2} e^{-(5l+3)\frac{4\pi}{5}}\right)\left(1 - \zeta_5^{-2} e^{-(5l+3)\frac{4\pi}{5}}\right)}$$

which resembles Ramanujan's identity

$$\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2} = e^{-\frac{2\pi}{5}} \cdot \prod_{l \geq 0} \frac{(1 - e^{-(5l+1)2\pi})(1 - e^{-(5l+4)2\pi})}{(1 - e^{-(5l+2)2\pi})(1 - e^{-(5l+3)2\pi})}$$

4 Remarks

A proof of the unit formula can be given by taking advantage of the residues of $\zeta_{K_1}(s)$ and $\zeta_{K_2}(s)$ at $s = 1$ while manually working out some quantities/properties of the fields (discriminant, Regulator, roots of unity, real and complex embeddings,...). A full proof is not difficult to follow but is relatively tedious. Producing a smooth and proper proof has been set aside for the distant future.