Neumann series seen as expansion into Bessel functions J_n based on derivative matching.

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Abstract

Multiplicative coefficients of a series of Bessel functions of the first kind can be adjusted so as to match desired values corresponding to a derivatives of a function to be expanded. In this way Neumann series of Bessel functions is constructed. Text presents known results.

Important notice

This text presents already known results [1]. Learning about their existence at the time when the text was almost finished, I refused to throw the text away. The ideas presented here maybe show the existing results from a somewhat different point of view. The following text was written from the perspective of presenting new results.

1 Introduction

Derivative matching is a natural idea in the domain of function approximation and function expansion. Two well-known realizations are Taylor series and Padé approximant[2]. The form of sum (of monomials) in the first case allows for easy term-by-term differentiation providing unique link between the n-th derivative of a function and the n-th monomial coefficient. For this reason the Taylor series is simple to construct. The procedure of derivative matching is much more complex for a rational function, yet the resulting approximation (the Padé one) may have in certain cases better convergence properties than Taylor series.

In this text I will apply the derivative matching procedure to the sum of Bessel functions of the first kind. Because of summation the derivative matching is feasible and an explicit formula linking coefficients and derivatives is found. Thus one gets an alternative expansion which might be a well suited alternative to Taylor polynomials under some circumstances. A very distinct feature of the proposed expansion is the behavior at infinity for a finite (i.e. cut-off) series: polynomials diverge whereas Bessel functions J_n converge to zero.

One should also notice that the series I introduce here are formally Neumann series [3, 1]. However, their construction is usually performed in a completely different way: in [1] the coefficients of Neumann series are determined as integrals of the function to be expanded weighted with Neumann polynomials. The approach presented in this text is completely independent, novel and, I believe, much less complicated from the technical point of view[1]. The full equivalence with Neumann series at the coefficient level seems to hold, yet it is not proven in this text.

Thanks to the formal resemblance of the presented expansion to the Neumann series one can borrow the motivation for the existence of such expansion: several useful applications of the Bessel function series with multiplicative coefficients are given in the introduction of [4].

2 Expansion into Bessel function of the first kind

A function $f(x)$ can be in the neighborhood of $x=0$ approximated by a series of Bessel functions of the first kind in the following way

$$f(x) \approx f(0)\, J_0(x) + \sum_{i=1}^{\infty} c_i J_i(x),$$

[1]No integration needed.

where

$$c_i = \sum_{j=0}^{2j\leq i} 2^{i-2j}\left[\left(\begin{array}{c} i-j-1 \\ j \end{array}\right) + 2\left(\begin{array}{c} i-j-1 \\ j-1 \end{array}\right)\right] d_{i-2j} \tag{1}$$

with

$$d_{i-2j} = \frac{d^{i-2j}}{dx^{i-2j}} f(x)|_{x=0}.$$

Here the symbol $\approx$ is meant only to indicate the derivative matching validity at any order and brackets $\binom{i}{j}$ stand for binomial coefficients. The series can be, of course, shifted to any other expansion point

$$f\left(x-x_0\right) \approx f\left(x_0\right) J_0\left(x_0\right) + \sum_{i=1}^{\infty} c_i J_i\left(x-x_0\right),\ c_i = c_i\left(\frac{d^{i-2j}}{dx^{i-2j}} f(x)|_{x=0}\right).$$

To prove the above statements one may start by examining the behavior of J_n functions when differentiated. One has

$$\frac{d}{dx} J_n\left(x\right) = \frac{1}{2} J_{n-1}\left(x\right) + \frac{1}{2} J_{n+1}\left(x\right).$$

This pattern leads, for higher-order derivatives, to a Pascal-like triangle

	J_{n-3}	J_{n-2}	J_{n-1}	J_n	J_{n+1}	J_{n+2}	J_{n+3}
				1			
$\frac{d}{dx}$			$+\frac{1}{2}$		$-\frac{1}{2}$		
$\frac{d^2}{dx^2}$		$+\left(\frac{1}{2}\right)^2$		$-2\left(\frac{1}{2}\right)^2$		$+\left(\frac{1}{2}\right)^2$	
$\frac{d^3}{dx^3}$	$+\left(\frac{1}{2}\right)^3$		$-3\left(\frac{1}{2}\right)^3$		$+3\left(\frac{1}{2}\right)^3$		$-\left(\frac{1}{2}\right)^3$

and gives the following higher-order derivative formula

$$\frac{d^m}{dx^m} J_n = \sum_{k=0}^{k=m} (-1)^k \left(\begin{array}{c} m \\ k \end{array}\right)\left(\frac{1}{2}\right)^m J_{n-m+2k}.$$

The expansion I propose is done at $x=0$ and for this value of x the only non-zero Bessel function is J_0. After m differentiations of the J_n function ($m \geq n$) two outcomes are possible:

- the J_0 function does not appear if $m-n$ is odd.
- the J_0 function appears at $k = \frac{m-n}{2}$ if $m-n$ is even.

Let me now focus on the whole series

$$f\left(x\right) = a_0 J_0\left(x\right) + a_1 J_1\left(x\right) + a_2 J_2\left(x\right) + \ldots = \sum_{l=0}^{\infty} a_l J_l\left(x\right)$$

at $x=0$. From each term one wants to extract, after m differentiation, the J_0 descendant. It is convenient to separate two scenarios

- Number of differentiations is even: the nonzero terms will be those from even terms because m and n need to have the same parity for $m-n$ to be even.

$$\begin{aligned}\frac{d^m}{dx^m} f\left(x\right)_{|x=0} &= \sum_{w=0}^{\frac{m}{2}+1} (-1)^{\frac{m-2w}{2}} \left(\begin{array}{c} m \\ \frac{m-2w}{2} \end{array}\right)\left(\frac{1}{2}\right)^m a_{2w} J_{2w-m+2\frac{m-2w}{2}}\left(0\right) \\ &= \sum_{w=0}^{\frac{m}{2}+1} (-1)^{\frac{m}{2}-w} \left(\begin{array}{c} m \\ \frac{m}{2}-w \end{array}\right)\left(\frac{1}{2}\right)^m a_{2w}.\end{aligned}$$

- Number of differentiations is odd: the nonzero terms will be those from odd terms because m and n need to have the same parity for $m-n$ to be even.

$$\begin{aligned}\frac{d^m}{dx^m} f\left(x\right)_{|x=0} &= \sum_{w=0}^{\frac{m-1}{2}} (-1)^{\frac{m-(2w+1)}{2}} \left(\begin{array}{c} m \\ \frac{m-(2w+1)}{2} \end{array}\right)\left(\frac{1}{2}\right)^m a_{2w+1} J_{2w+1-m+2\frac{m-(2w+1)}{2}}\left(0\right) \\ &= \sum_{w=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}-w} \left(\begin{array}{c} m \\ \frac{m-1}{2}-w \end{array}\right)\left(\frac{1}{2}\right)^m a_{2w+1}.\end{aligned}$$

The both rules can be written in a simple matrix form

$$\frac{d^m}{dx^m} f(x)_{|x=0} = \sum_{n \le m} D_{mn} a_n$$

with

$$D_{mn} = \begin{cases} (-1)^{\frac{m-n}{2}} \begin{pmatrix} m \\ \frac{m-n}{2} \end{pmatrix} \left(\frac{1}{2}\right)^m & \text{if } n \le m \text{ and } n-m \text{ is even} \\ 0 & \text{else} \end{cases}$$

where the indexing starts at zero. By construction, D is a lower triangular matrix which looks like ($m_{max}, n_{max} = 6$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{8} & 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{3}{8} & 0 & -\frac{1}{4} & 0 & \frac{1}{16} & 0 & 0 \\ 0 & \frac{5}{16} & 0 & -\frac{5}{32} & 0 & \frac{1}{32} & 0 \\ -\frac{5}{16} & 0 & \frac{15}{64} & 0 & -\frac{3}{32} & 0 & \frac{1}{64} \end{pmatrix}$$

where a simple rule applies

$$\begin{aligned} D_{00} &= 1, \\ D_{ij} &= \left(D_{i-1,j-1} - D_{i-1,j+1}\right)/2. \end{aligned}$$

To compute coefficients from derivatives one has to invert the matrix $C = D^{-1}$

$$\begin{aligned} a_i &= \sum_j \left(D^{-1}\right)_{i,j} \frac{d^j}{dx^j} f(x)_{|x=0} \\ &\equiv \sum_j C_{i,j} \frac{d^j}{dx^j} f(x)_{|x=0} . \end{aligned}$$

The matrix D can actually be inverted:

$$C_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ 2^j \left[\begin{pmatrix} \frac{i+j}{2} - 1 \\ j-1 \end{pmatrix} + 2 \begin{pmatrix} \frac{i+j}{2} - 1 \\ j \end{pmatrix} \right] & \text{else if } i \ge j \text{ and } i-j \text{ is even} , \\ 0 & \text{else} \end{cases} \tag{2}$$

where cases with negative arguments inside binomial coefficients are treated in a fully consistent way as proposed by Kronenburg [5] and this treatment is followed all along this text. The inverted matrix is lower triangular ($m_{max}, n_{max} = 6$)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 8 & 0 & 0 & 0 \\ 2 & 0 & 16 & 0 & 16 & 0 & 0 \\ 0 & 10 & 0 & 40 & 0 & 32 & 0 \\ 2 & 0 & 36 & 0 & 96 & 0 & 64 \end{pmatrix}$$

with the recurrence rule

$$\begin{aligned} C_{00} &= 1 \\ C_{ij} &= 2C_{i-1,j-1} + C_{i-2,j}. \end{aligned}$$

From 2 the main result 1 can be deduced.

In the proof of D and C being inverse one may write the latter in a different way

$$C_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ 2 & \text{else, if } j = 0 \text{ and } i \text{ is even} \\ 2^j \frac{i}{j} \left(\begin{array}{c} \frac{i+j}{2} - 1 \\ j - 1 \end{array} \right) & \text{else, if } i \geq j \text{ and } i - j \text{ is even} \\ 0 & \text{else} \end{cases}.$$

The equation to prove

$$\sum_{\gamma=0}^{\infty} D_{\alpha,\gamma} C_{\gamma,\beta} = \delta_{\alpha,\beta}$$

can be simplified, because D and C are lower triangular

$$\sum_{\gamma=0}^{\infty} D_{\alpha,\gamma} C_{\gamma,\beta} = \sum_{\gamma=\beta}^{\alpha} D_{\alpha,\gamma} C_{\gamma,\beta}.$$

The proof can be done in five steps:

1. Case $\alpha < \beta$. One has$(D \cdot C)_{\alpha,\beta} = 0$ because a product of lower triangular matrices is a lower triangular matrix.

2. Case $\alpha = \beta = 0$. One has$(D \cdot C)_{0,0} = 1$ by direct computation (single nonzero term).

3. Case $\alpha = \beta \neq 0$. One nonzero term appears

$$\begin{aligned} \left[\sum_{\gamma=\beta}^{\alpha} D_{\alpha,\gamma} C_{\gamma,\beta} \right]_{\alpha=\beta} &= D_{\alpha,\alpha} C_{\alpha,\alpha} \quad [\alpha - \beta = 0 \text{ is even}] \\ &= (-1)^{\frac{\alpha-\alpha}{2}} \left(\begin{array}{c} \alpha \\ \frac{\alpha-\alpha}{2} \end{array} \right) \left(\frac{1}{2} \right)^{\alpha} 2^{\alpha} \frac{\alpha}{\alpha} \left(\begin{array}{c} \frac{\alpha+\alpha}{2} - 1 \\ \alpha - 1 \end{array} \right) \\ &= \left(\begin{array}{c} \alpha \\ 0 \end{array} \right) \left(\begin{array}{c} \alpha - 1 \\ \alpha - 1 \end{array} \right) \\ &= 1 \end{aligned}$$

4. Case $\alpha > \beta \neq 0$.

$$\begin{aligned} \sum_{\gamma=\beta}^{\alpha} D_{\alpha,\gamma} C_{\gamma,\beta} &= \sum_{\gamma=0}^{\alpha-\beta} D_{\alpha,(\beta+\gamma)} C_{(\beta+\gamma),\beta} \\ &= \sum_{\gamma=0}^{(\alpha-\beta)/2} D_{\alpha,(\beta+2\gamma)} C_{(\beta+2\gamma),\beta} + \sum_{\gamma=0}^{(\alpha-\beta)/2} D_{\alpha,(\beta+2\gamma+1)} C_{(\beta+2\gamma+1),\beta} \\ &= \sum_{\gamma=0}^{(\alpha-\beta)/2} D_{\alpha,(\beta+2\gamma)} C_{(\beta+2\gamma),\beta} \end{aligned}$$

where the second term vanishes because its C matrix has indices with different parity ($\beta + 2\gamma + 1$ and β). Further, if α and β have different parity, then the result is zero, because indices of the D matrix of the first term have different parity. Thus two sub-cases appear

(a) α, β are both even, $\alpha = 2a,\ \beta = 2b$.

$$\begin{aligned} \sum_{\gamma=\beta}^{\alpha} D_{\alpha,\gamma} C_{\gamma,\beta} &= \sum_{\gamma=0}^{(2a-2b)/2} D_{2a,(2b+2\gamma)} C_{(2b+2\gamma),2b} + 0 \\ &= \left(\frac{1}{2} \right)^{2(a-b)} (-1)^{a-b} \sum_{\gamma=0}^{a-b} (-1)^{\gamma} \frac{b+\gamma}{b} \left(\begin{array}{c} 2a \\ a-b-\gamma \end{array} \right) \left(\begin{array}{c} 2b-1+\gamma \\ 2b-1 \end{array} \right) \\ &= \left(\frac{1}{2} \right)^{2(a-b)} (-1)^{a-b}\, \Omega^1_{a,b} \end{aligned}$$

where one looks to prove the last combinatorial expression to be equal to zero. This proof, which follows, was provided by *Markus Scheuer* [6]

$$\begin{aligned}
\Omega^1_{a,b} &= \sum_{\gamma=0}^{a-b} (-1)^\gamma \frac{b+\gamma}{b} \binom{2a}{a-b-\gamma} \binom{2b-1+\gamma}{2b-1} \\
&\qquad \left[\binom{p}{q} = \binom{p}{p-q},\ p = 2b-1+\gamma,\ q = 2b-1\right] \\
&= \sum_{\gamma=0}^{a-b} (-1)^\gamma \frac{b+\gamma}{b} \binom{2a}{a-b-\gamma} \binom{2b-1+\gamma}{\gamma} \\
&\qquad \left[(-1)^q \binom{p+q-1}{q} = \binom{-p}{q},\ p = 2b,\ q = \gamma\right] \\
&= \sum_{\gamma=0}^{a-b} \frac{b+\gamma}{b} \binom{2a}{a-b-\gamma} \binom{-2b}{\gamma} \\
&= \sum_{\gamma=0}^{a-b} \binom{2a}{a-b-\gamma} \binom{-2b}{\gamma} - 2\sum_{\gamma=0}^{a-b} \frac{\gamma}{-2b} \binom{-2b}{\gamma} \binom{2a}{a-b-\gamma} \\
&\qquad \left[\frac{q}{p}\binom{p}{q} = \binom{p-1}{q-1},\ p = -2b,\ q = \gamma\right] \\
&= \sum_{\gamma=0}^{a-b} \binom{-2b}{\gamma} \binom{2a}{a-b-\gamma} - 2\sum_{\gamma=0}^{a-b} \binom{-2b-1}{\gamma-1} \binom{2a}{a-b-\gamma} \\
&\qquad \left[\binom{s+t}{n} = \sum_{k=0}^{n} \binom{s}{k}\binom{t}{n-k},\ k = \gamma,\ n = a-b \geq 0,\ s = -2b,\ t = 2a\right] \\
&\qquad [\delta = \gamma - 1] \\
&= \binom{2a-2b}{a-b} - 2\sum_{\delta=-1}^{a-b-1} \binom{-2b-1}{\delta} \binom{2a}{a-b-1-\delta} \\
&\qquad \left[\binom{s+t}{n} = \sum_{k=0}^{n} \binom{s}{k}\binom{t}{n-k},\ k = \delta,\ n = a-b-1 \geq 0,\ s = -2b-1,\ t = 2a\right] \\
&= \binom{2a-2b}{a-b} - 2\binom{2a-2b-1}{a-b-1} \\
&\qquad \left[\binom{p-1}{q-1} = \frac{q}{p}\binom{p}{q},\ p = 2a-2b,\ q = a-b\right] \\
&= \binom{2a-2b}{a-b} - 2\frac{a-b}{2a-2b}\binom{2a-2b}{a-b} \\
&= 0
\end{aligned}$$

where Chu–Vandermonde identity was used twice.

(b) α, β are both odd, $\alpha = 2a+1,\ \beta = 2b+1$.

$$\begin{aligned}
\sum_{\gamma=\beta}^{\alpha} D_{\alpha,\gamma} C_{\gamma,\beta} &= \sum_{\gamma=0}^{a-b} D_{2a+1,(2b+1+2\gamma)} C_{(2b+1+2\gamma),2b+1} \\
&= \left(\frac{1}{2}\right)^{2(a-b)} (-1)^{a-b} \sum_{\gamma=0}^{a-b} (-1)^\gamma \binom{2a+1}{a-b-\gamma} \frac{2b+1+2\gamma}{2b+1} \binom{2b+\gamma}{2b} \\
&= \left(\frac{1}{2}\right)^{2(a-b)} (-1)^{a-b}\, \Omega^2_{a,b}
\end{aligned}$$

$$\begin{aligned}
\Omega^2_{a,b} &= \sum_{\gamma=0}^{a-b} \frac{2b+1+2\gamma}{2b+1} \binom{2a+1}{a-b-\gamma} (-1)^\gamma \binom{2b+1+\gamma-1}{\gamma} \\
&= \sum_{\gamma=0}^{a-b} \frac{2b+1+2\gamma}{2b+1} \binom{2a+1}{a-b-\gamma} \binom{-2b-1}{\gamma} \\
&= \sum_{\gamma=0}^{a-b} \binom{2a+1}{a-b-\gamma} \binom{-2b-1}{\gamma} - 2\sum_{\gamma=0}^{a-b} \frac{\gamma}{-2b-1} \binom{-2b-1}{\gamma} \binom{2a+1}{a-b-\gamma} \\
&= \sum_{\gamma=0}^{a-b} \binom{-2b-1}{\gamma} \binom{2a+1}{a-b-\gamma} - 2\sum_{\gamma=0}^{a-b} \binom{-2b-2}{\gamma-1} \binom{2a+1}{a-b-\gamma} \\
&= \binom{2a-2b}{a-b} - 2\sum_{\delta=-1}^{a-b-1} \binom{-2b-2}{\delta} \binom{2a+1}{a-b-1-\delta} \\
&= \binom{2(a-b)}{a-b} - 2\binom{2a-2b-1}{a-b-1} \\
&= \binom{2(a-b)}{a-b} - 2\frac{a-b}{2a-2b}\binom{2a-2b}{a-b} \\
&= 0
\end{aligned}$$

5. Case $\alpha > \beta = 0$. In this situation we use formula 2, which works also for $\beta = 0$. The reasoning from Case 4 remains valid and thus we need to study only the same-parity scenarios. By assumption β is even (equal to zero) therefore only one option remains, i.e. α has to be even also, $\alpha = 2a$

$$\begin{aligned}
\sum_{\gamma=\beta}^{\alpha} D_{\alpha,\gamma} C_{\gamma,\beta} &= \sum_{\gamma=0}^{(\alpha-\beta)/2} D_{\alpha,(\beta+2\gamma)} C_{(\beta+2\gamma),\beta} \\
&= \sum_{\gamma=0}^{a} D_{2a,2\gamma} C_{2\gamma,0} \\
&= D_{2a,0} C_{0,0} + \sum_{\gamma=1}^{a} D_{2a,2\gamma} C_{2\gamma,0} \\
&= D_{2a,0} + \sum_{\gamma=1}^{a} D_{2a,2\gamma} C_{2\gamma,0} \\
&= (-1)^a \binom{2a}{a} \left(\frac{1}{2}\right)^{2a} + (-1)^a \left(\frac{1}{2}\right)^{2a} \sum_{\gamma=1}^{a} (-1)^\gamma \binom{2a}{a-\gamma} \left[\binom{\gamma-1}{-1} + 2\binom{\gamma-1}{0}\right] \\
&= (-1)^a \left(\frac{1}{2}\right)^{2a} \left[\binom{2a}{a} + 2\sum_{\gamma=1}^{a} (-1)^\gamma \binom{2a}{a-\gamma}\right] \\
&= (-1)^a \left(\frac{1}{2}\right)^{2a} \Omega^3_a
\end{aligned}$$

$$
\begin{aligned}
\Omega_a^3 &= \binom{2a}{a} + 2\sum_{\gamma=1}^{a}(-1)^{\gamma}\binom{2a}{a-\gamma} \\
&\quad [\delta = a-\gamma] \\
&= \binom{2a}{a} + 2\sum_{\delta=a-1}^{0}(-1)^{a-\delta}\binom{2a}{\delta} \\
&= \binom{2a}{a} + 2\,(-1)^{a}\sum_{\delta=0}^{a-1}(-1)^{\delta}\binom{2a}{\delta} \\
&= \binom{2a}{a} + 2\,(-1)^{a}\,(-1)^{a-1}\binom{2a-1}{a-1} \\
&= \binom{2a}{a} - 2\binom{2a-1}{a-1} \\
&= \binom{2a}{a} - 2\frac{a}{2a}\binom{2a}{a} \\
&= 0
\end{aligned}
$$

The five cases cover all possibilities and therefore the proof is complete.

3 Examples

In Figure 1 I present approximations of four common elementary functions by derivative-matching series of Bessel functions together with the Taylor polynomial with 11 terms matched in both cases (function value and 10 derivatives). From what is seen on the picture, the Taylor polynomial perform somewhat better for the exponential function, the both series provide very similar results for the logarithm and the Bessel-function series outperform Taylor series in case of the cosine function.

It is known that Neumann series converge in the same domain as Taylor polynomials [1].

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URL `https://math.stackexchange.com/q/2425782`

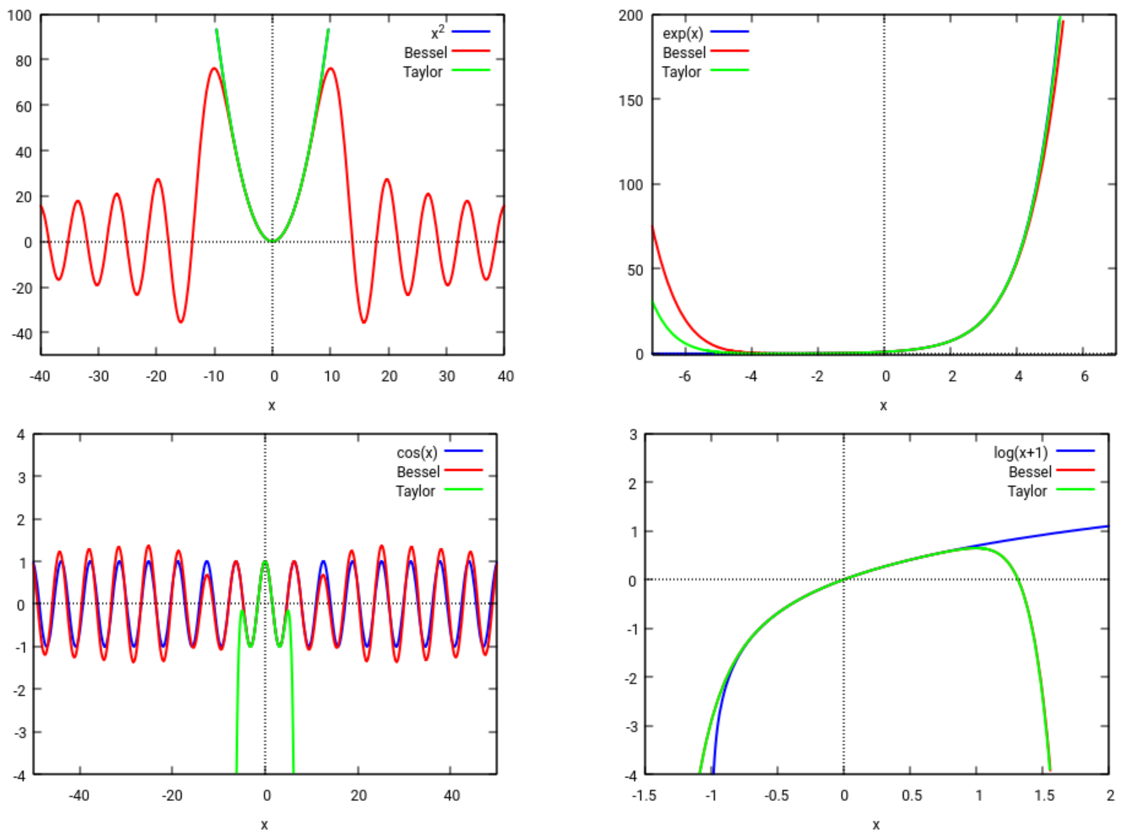

Figure 1: Approximation of selected elementary functions [$y = x^2$, $y = \exp(x)$, $y = \cos(x)$ and $y = \ln(x+1)$] by series of Bessel functions and by Taylor series with the value of the function and value of first ten derivatives matched.