The Origin of Gravity and Newton's law.

An attempt to answer this question with the help of existing concepts.

By: Hubert J. Veringa.
Emeritus Professor at Eindhoven University, The Netherlands
veringa48@planet.nl

Abstract.
In this document an attempt is made to explain the origin of gravity. The basis for the analysis is a merger of Quantum theory and Relativity. Nowhere in the analysis there is any need to deviate from well proven and successful concepts of both theories and rules of calculation, and no exotic new particles will have to be introduced. By doing so it is demonstrated that, next to its local interactions of a multi-particle system, the Schrödinger equation leads to pairs of two and only two members. This solution is used as the invariant term in the quantized Einstein energy equation which finally leads to gravitational interactions between members of the pairs. With this particular solution for the quantum-mechanical wave function it is found that gravity is a second order effect operating over a long range. In this document it is tried to give a complete and consistent account of all steps that have been taken in the derivation of the classical Newton's law. Further the document emphasizes precise justification of some of the basic assumptions made and how it works out on a cosmological scale. It is also found that the generator of gravity is contributing mass to particles that have gravitational interaction.

Keywords.
Gravity, Quantum Physics, Special Relativity, Dynamic mass, Cosmology.

1. Introduction.

In our daily life, gravity is experienced everywhere and at all moments. Without gravity the world as an entity would not exist, the Sun would not shine, water waves would not run, etc. Even if we would evaluate the consequences of a small change in the gravitational interaction, the universe would look different from how it is now. It is accepted as an inescapable force that keeps our existence together. However, where we have some basic understandings of the processes around us, there was not exist a suitable explanation for this force at a microscopic level.

Gravitation interaction manifests itself where other forces are not the determining factor. Therefore, in our real world, we see that our direct vicinity has structures that are changing over short distances like mountains, cities, sky scrapers, boats, forests etc. At larger distances, of the order of 100 kilometers, the gravity becomes the dominant factor and bodies begin to take spherical shapes. Obviously, the smaller the gravity is, so to speak at smaller planets than earth, the structural variability will become larger. That the electromagnetic interaction becomes insignificant in shaping the environment is not due to the form of the electrostatic interaction, which has basically the same shape as the gravitational interaction, but it is due to

the fact that positive and negative charges balance and compensate for their interaction. The influence of electromagnetism is becoming insignificant already at short distances.
Now the general belief is that any suitable theory should include, or will be, a merger of classical quantum theory and relativity, but until now no theory that is widely accepted has been proposed. Following earlier papers [9] and [10], the present document will give an updated scheme of analysis for the mutual interaction between particles that have some exchange with respect to time and space, The remarkable thing is that, apparently for more than one reason, particles will be interacting in groups of two and only two and can give rise to gravitational exchange. This pair formation is described quantum mechanically. Either starting from the classical Schrödinger equation or the relativistic Einstein energy equation, but this latter formulated in a quantum mechanical setting known as the "Klein Gordon" (KG) equation, results in the same wave function describing pairs of particles. Since this wave function represents a pair potential, a relativistic mass can be attributed to it which is used in the KG-equation to derive an interaction field between the members that form the ensemble. It is found that the right form of Newton's gravity law emerges by consequently working through the proposed schemes of both quantum mechanics and the basic equations of relativity theory as expressed by the quantum mechanical equivalent of the Einstein energy equation [6], [7].

2. Groups of particles and sub-spaces.
Gravity is an attractive force between two bodies, or, at a microscopic level, two particles and therefore any theory will have to account for multi-particle systems. For the development of the theory we have therefore to modify the Hamiltonian for such a multi-particle system.
The most simple expression for the kinetic energy in the Hamiltonian for a group of particles numbered by k is given by:

$$\hat{p}^2/2m = \sum_k \widehat{p_k}^2/2m_k \; . \tag{2.1}$$

This expression does, however, not clearly enough describe the behaviour of particle interaction as members of a group, but it will be shown that an alternative representation is possible in which still the total kinetic energy remains the same. The first step is to write equation slightly differently:

$$\hat{p}^2/2m = \sum_k \widehat{p_k}^2/2m_k = \sum_k \left(\widehat{p_k}/\sqrt{2m_k}\right)^2 . \tag{2.2}$$

This equation does not look so special, but it shows that, if we want to modify the kinetic energy in the Hamiltonian, we will have to perform our analysis in the $\widehat{p_k}/\sqrt{2m_k}$ – space. For reasons that will become clear later we will now modify the Hamiltonian for the two-particle ensemble (ij) and refer to Figure 1.
In this Figure 1 particles m_i and m_j are moving with momenta p_i and p_j. But we are interested in their behaviour in the space as seen from point O_2 and therefore we apply the cosine-rule to both triangles "1" and "2". Knowing that:
$\cos\delta_2 = -\cos(180 - \delta_2)$ and taking $p_{ij}/\sqrt{2m_i} = p_{ji}/\sqrt{2m_j}$, it follows that:

$$\widehat{p_i}^2/2m_i + \widehat{p_j}^2/2m_j = \widehat{p_g}^2/2m_g + \widehat{p_{ij}}^2/2m_i + \widehat{p_{ji}}^2/2m_j . \tag{2.3}$$

In this modified kinetic energy part of the Hamiltonian the first term at the right hand is the kinetic energy of the group, identified with label g, consisting of m_i and m_j with mass $m_g = m_i + m_j$ and moving as one single entity. The second term and third term left are the kinetic energies in the sub-space. The group momentum vector $\widehat{p_g}/\sqrt{2m_g}$ is not equal to any of the other ones so that $\widehat{p_g}$ has to be defined separately. As the interaction between the two particles is only within the sub-space r_{ij}, we will not have to bother about this first term at the right hand. This is fortunate because it depends on the angle between $\widehat{p_i}/\sqrt{2m_i}$ and $\widehat{p_j}/\sqrt{2m_j}$ which would severely complicate the problem.

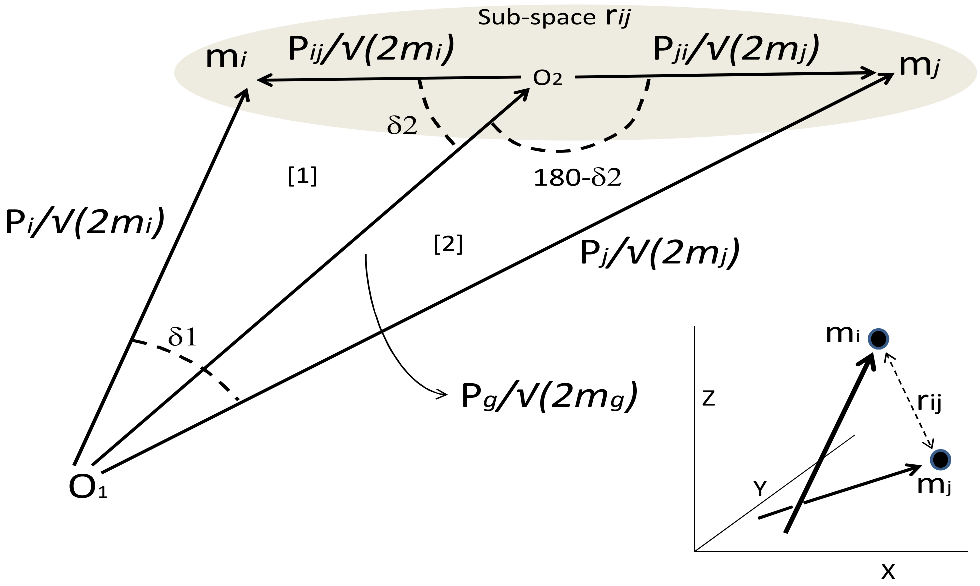


Figure 1: *The relation (2.3) found by applying the cosine-rule to both triangles [1] and [2] if the lengths of the arrows $p_{ij}/\sqrt{2m_i}$ and $p_{ji}/\sqrt{2m_j}$ are the same. In this view vectors and operators are treated as equivalent. Note that, in this two-dimensional momentum space, the velocity or the momentum vectors for particles always have the same origin.*

It can also be seen that this modification of the Hamiltonian only works well for two particles as the geometrical argument is confined to one plane. More particles would compel us to perform the analysis in many more different planes and would not give a tractable solution. Another important observation is that because $p_{ij}/\sqrt{2m_i} = p_{ji}/\sqrt{2m_j}$ the sub-space is symmetric from the point of view of an observer in O_2. This issue of symmetry will come back in the solution of the Schrödinger equation with the modified Hamiltonian in the sub-space.

We will now extend the modified Hamiltonian equation obtained by substituting equation (2.3) into (2.2) for more than two particles, but all of them interacting in groups of two and only two:

$$\sum_k \widehat{p_k}^2/2m_k = 1/N(\sum_g \widehat{p_g}^2/2m_g + 1/2\sum_{i\neq j}(\widehat{p_{ij}}^2/2m_i + \widehat{p_{ji}}^2/2m_j)). \tag{2.4}$$

The *N*-factor, the number of particles, is necessary as in the summation each particle is counted *N* times. The pairs are counted by the *g*- index. Later, when the analysis brings us to the final result, we will come back to the group momentum and evaluate the consequence of its dependence on the momenta of m_i and m_j.
For completeness we will now derive this dependence and come back to it later. For this we apply the cosine-rules from the corner O_1 for the triangles “1” and “2” separately and together: “1+2”. the equations are:

$$\left(\widehat{p_{ij}}/\sqrt{2m_i} + \widehat{p_{ji}}/\sqrt{2m_j}\right)^2 = \widehat{p_i}^2/2m_i + \widehat{p_j}^2/2m_j - 2\widehat{p_i}\widehat{p_j}cos\delta_1/\sqrt{4m_im_j}, \tag{2.5a}$$

$$\widehat{p_g}^2/2m_g = \widehat{p_i}^2/2m_i + \widehat{p_j}^2/2m_j - \widehat{p_{ij}}^2/2m_i - \widehat{p_{ji}}^2/2m_j. \tag{2.5b}$$

This cosine factor showing up complicates the analysis, but in the end it will not trouble our analysis as it can be circumvented.

Considering relativity the analysis will have to be repeated starting from the equation (2.4), but as it is only dealing with the momenta, the result of the previous analysis can be used if we simply replace the vector $p_k/\sqrt{2m_k}$ by $cm_k\sqrt{\gamma_k^2 - 1}$ with the *k*- label representing *i, j, g* and p_{ij}, p_{ji} unchanged. The symbol γ_k equals $(1 - v_k^2/c^2)^{-1/2}$. The analysis will be continued in paragraph 7.

3. The sub-space in more detail.
The total wave function describing a particle or a larger entity under its local influences, $\Psi_{loc}(r_{loc}, t)$, and its extension in outer space, $\Psi_{inf}(r_{inf}, t)$, is given by: $\Psi_{tot} = \Psi_{loc}\Psi_{inf}$. The coordinate r_{loc} is the position of the centre-of-mass of the particle inside the atom or nucleus or a solid object or, eventually of a body as a whole, and the coordinate, r_{inf}, is the position of this entity from the point of view of an outside observer. They, therefore, can be considered as mutually independent. In the same way we define, as before, the Hamilton operator as: $\widehat{H}_{tot} = \{\hat{p}^2\}_{loc}/2m_{loc} + \{\hat{p}^2\}_{inf}/2m_{inf} + V_{loc}(r_{loc}) + V_{inf}(r_{inf})$. The masses m_{loc} and m_{inf} are not necessarily the same. The m_{inf} is the mass to be connected to the particle as it can move freely around whereas m_{loc} is the mass of the particle under the influence of the local interactions, sometimes called “reduced mass”. It follows that:

$$\widehat{H}_{tot}\Psi_{tot} = (\{\hat{p}^2\}_{loc}/2m_{loc} + \{\hat{p}^2\}_{inf}/2m_{inf} + V_{loc} + V_{inf})(\Psi_{loc}\Psi_{inf}) =$$

$$= (\{\hat{p}^2\}_{loc}/2m_{loc} + V_{loc})\Psi_{loc}\Psi_{inf} + (\{\hat{p}^2\}_{inf}/2m_{inf} + V_{inf})\Psi_{loc}\Psi_{inf}. \tag{3.1}$$

Separating the local effect from the surroundings we can set:

$$(\{\hat{p}^2\}_{loc}/2m_{loc} + V_{loc})\,\Psi_{loc} = E_{loc}\Psi_{loc} \text{ and:} \tag{3.2a}$$

$$(\{\hat{p}^2\}_{inf}/2m_{inf} + V_{inf})\,\Psi_{inf} = E_{inf}\Psi_{inf}. \tag{3.2b}$$

The first equation (3.2a) is the Schrödinger equation describing the behaviour of the entity in its local environment like in the nucleus or a solid or, again, as the entity as a whole where it has its individual interactions. The second equation (3.2b) describes its movement or presence in the outer space in which the particle, or as a larger entity or part of it, can move around. By taking V_{inf} as a constant it is assumed that the behaviour out of its local influences is taken into consideration. This second equation is the starting point in the development of the theory in the next paragraphs. The splitting up as in equation (3.2a) and (3.2b) disconnects the local interactions, as is normally done in quantum mechanics, from the movement or presence of the particle or the entity individually.
In what follows we will only consider the second equation as this gives the generator for the gravitational interaction. Because we are interested in the effects of masses outside the local interactions we will from now on take for the mass m_{inf} the quantity m, as it will also be the case for the coordinates.
Starting from the unmodified Hamiltonian, the general solution of a wave equation describing independent particles in spherical symmetry is initiated by the operator:

$\hat{p}^2/2m = \sum_k \hat{p}_k^2/2m_k$, and reads:

$$\Psi = ..\Psi_i\Psi_j \ldots \Psi_l = ..\left(\frac{\alpha_i}{r_i}\right) e^{i\beta_i r_i} \text{x} \left(\frac{\alpha_j}{r_j}\right) e^{i\beta_j r_j} \text{x....x} \left(\frac{\alpha_l}{r_l}\right) e^{i\beta_l r_l} = \prod_k \left(\frac{\alpha_k}{r_k}\right) e^{i\beta_k r_k}. \tag{3.3}$$

We have regrouped the kinetic energy contribution to the Hamiltonian for the same set of particles as:

$$\hat{p}^2/2m = \sum_k \hat{p}_k^2/2m_k \ = 1/N(\sum_g \hat{p}_g^2/2m_g \ + 1/2\sum_{i\neq j}(\hat{p}_{ij}^2/2m_i \ + \hat{p}_{ji}^2/2m_j\,)),$$

and first we will only consider the second part of it at the right hand side, to start with the group (ij) of two particles only, thus we restrict ourselves to the sub-space with coordinates r_{ij}.

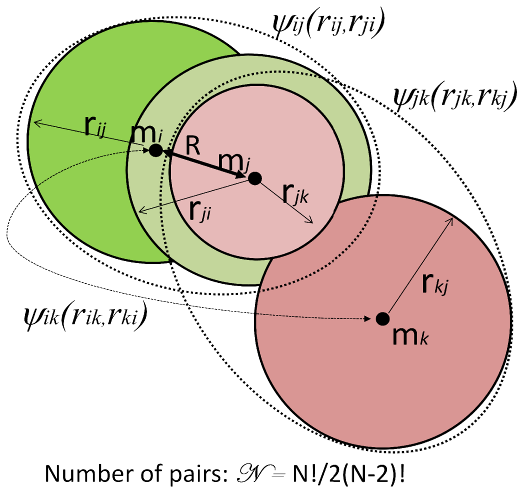


Figure 2*: Forming and describing of 𝒩=N!/2(N-2)! Pairs. In this example the number of groups is three.*

Per group there are two independent particles. For the group under consideration, like in Figure 2, it is indicated by the masses m_i and m_j. and they experience some force reflected by the potential V_i and V_j. Spherical symmetry is next adopted and the only boundary condition is that the wave function is zero at infinity. We have per pair one coordinate system r_{ij} around m_i and one r_{ji} around m_j. In this way an observer at a distance r_{ij} from particle m_i and at r_{ji} from particle m_j will see that the total wave equation of the individual pair (ij) is defined as follows [4], [5]:

$$\widehat{H_{ij}}\Psi_{ij,t} = i\hbar\frac{\partial}{\partial t}\Psi_{ij,t} = -\left(\frac{\hbar^2}{2m_i}\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{\hbar^2}{2m_j}\frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}}\right)\Psi_{ij,t} + \left(V_i + V_j\right)\Psi_{ij,t} \quad (3.4)$$

$\Psi_{ij,t}$ is the time and space dependent wave function. The time dependence can be removed by replacing the time dependent wave function $\Psi_{ij,t}$ by $\Psi_{ij}e^{iE_{ij}t/\hbar}$. Further, define $V_i + V_j$ by $V_{ij}$and we get:

$$(E_{ij} - V_{ij})\Psi_{ij} + \frac{\hbar^2}{2m_i}\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}}\Psi_{ij} + \frac{\hbar^2}{2m_j}\frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}}\Psi_{ij} = 0. \quad (3.5)$$

To simplify the equation replace $E_{ij} - V_{ij}$ by ε_{ij} to propose a solution that is valid in areas where the V_{ij} is not of great influence anymore as follows:

$$\Psi_{ij} = \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)e^{i\beta_{ij}r_{ij} + i\beta_{ji}r_{ji}}, \quad (3.6)$$

where $\alpha_{ij}$and β_{ij} are constants independent of space coordinates and time. This solution means that we consider the wave function outside the surroundings where the potential energy with all its peculiarities has a very minor effect on the shape of the wave function. The only interaction that can play a role will then be based solely on gravitational interaction. By substituting the solution in equation (3.5) the following relation is found:

$$-\frac{\hbar^2 i}{r_{ij}r_{ji}}\left(\frac{\alpha_{ij}\beta_{ji}}{m_j} + \frac{\alpha_{ji}\beta_{ij}}{m_i}\right)e^{i\beta_{ij}r_{ij} + i\beta_{ji}r_{ji}} - \frac{\hbar^2}{2}\left(\frac{\beta_{ij}^2}{m_i} + \frac{\beta_{ji}^2}{m_j}\right)x$$

$$x\left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)e^{i\beta_{ij}r_{ij} + i\beta_{ji}r_{ji}} + \varepsilon_{ij}\left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)e^{i\beta_{ij}r_{ij} + i\beta_{ji}r_{ji}} = 0. \quad (3.7)$$

The complex first term at the left hand side is to be set to zero and in a pair-wise process $\alpha_{ij}\beta_{ji}/m_j + \alpha_{ji}\beta_{ij}/m_i = 0$ and $\beta_{ij}^2\hbar^2/2m_i + \beta_{ji}^2\hbar^2/2m_j = \varepsilon_{ij} = \sigma(m_i + m_j)$ so that for every value of the energy there will be a value for σ and the $\beta's$ can adapt themselves. Therefore, whatever is the situation in which m_i and m_j find themselves, there is always a β_{ji} and a β_{ij} and they have no influence on the $\alpha's$ as long as $\alpha_{ij} = \alpha_{ji}$. It means, that the interaction occurs in the sub-space with a pair to be considered as one single entity with a mass of $(m_i + m_j)$ and, apart from the separation between the members of the pair (R), independent of the situation these members are in. Further, it has to be noticed that the Schrödinger equation based on the modified Hamiltonian only is possible for groups of two and only two particles. This conclusion has already been drawn in a slightly different way in the previous paragraph where the geometrical argument in momentum space is only possible for two partices with momenta vectors in one plane.
We already came across the fact that the sub-space r_{ij} in momentum space for the observer in O_2 in Figure 1 is symmetric and therefore the solution $\Psi(\alpha_{ij}, \alpha_{ji})$ is symmetric, meaning, again, that $\alpha_{ij} = \alpha_{ji}$.
At the moment not much is known about the $\alpha's$, but one requirement to be imposed on the wave function is that it represents a pair of particles. For the time being it can be said that:
i. The $\alpha's$ cannot depend on the running variables in the wave equation: r_{ij} or t. It will be a

constant that can only depend on fundamental nature constants and the particle masses.
ii. It should make no difference for the outside world how one member sees its partner or whether and how we see the two members of the pair. It means that we can say: $\alpha_{ij} = f(m_i)f(m_j)$.
iii. There is no pair if either m_i or m_j equals zero so that $f(m_i) = 0$ for $m_i = 0$ and the pair potential should increase linearly with both participating masses in the pair.

To sum up also the movement of the group as one entity and the fact that there are N particles and $\mathcal{N}=N!/2(N-2)!$ pairs, leads to a total wave function as:

$$\Psi = \prod_{ij}\left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} \prod_g \left(\frac{\alpha_g}{r_g}\right) e^{i\beta_g r_g}. \qquad (3.8)$$

The second product is due to the first contribution to the momentum-based energy term in equation (2.4) and, as already mentioned, it generates no gravitational interaction. The index g is identified by the pair (ij) as indicated in figure 1. The first term in the product (3.8) gives gravitational interaction in the case of two, and only two members in an ensemble where the sum is taken over all possible and unique pairs (ij). As the pairs are to be considered in their own unique coordinate system r_{ij}, there is no reason to consider all the pairs together but only the behaviour of a single pair. In the end we will add up all the contributions of the pairs as shown schematically in Figure 4 in paragraph 5.
There is freedom in the choice of the particles m_i, m_j, ---, m_l, ---. It can actuallly be anything like elementary particles, nuclei or even larger entities if, at least, we can describe such an entity by a single wave function in its own coordinate system and solve the equation to form a pair with another entity.

Later it will be confirmed that, as before and for the sake of symmetry in the mutual gravitational interaction, the two $\alpha's$ should be equal. It also means that the β's have opposite signs and fixed values and, by taking the $\alpha's$ equal, we make their values independent of the masses and the energies of the members of the pair. The ε_{ij} could have been split into two separate quantities as ε_{ij} and ε_{ji} to dedicate the β_{ij}^2 and β_{ji}^2-values to the separate energies of the two particles.
It is also interesting to notice that the solution of the wave equation for the pairs like in equation (3.8) looks different from a solution for a single particle on the basis of the unmodified Hamiltonian as in equation (3.3). For instance, if we take a look at the r_{ij}-dependence in the solution (3.8), we see that there is an extra r_{ij}-dependent factor in the exponential term. This latter term is insufficient to make such a solution applicable for the operator working on r_{ij}. For it to be sufficient we need the total pre-exponential factor as given in equation (3.8).

An alternative approach is possible by taking the KG-equation as the starting point. In this way we guarantee full co-variance throughout the entire analysis. The Einstein energy equation is the basis of the KG- equation [5] and reads:

E^2 - $p^2c^2 = m_0^2c^4$ or expressed alternatively: $E^2/m_0^2c^4$ - $p^2/m_0^2c^2 = 1$,

and translated into quantum mechanical language for an ensemble of two particles [6]:

$$(E_{ij}^2/m_{0i}^2c^4 - E_{ji}^2/m_{0j}^2c^4)\Psi_{ij} - ((\widehat{p_{ij}})^2/m_{0i}^2c^2 - (\widehat{p_{ji}})^2/m_{0j}^2c^2))\Psi_{ij} = 0. \tag{3.9}$$

Where $\widehat{p_{ij}}^{\,2}$ is the square of the momentum operator in spherical coordinates as in equation (3.3) and m_{0i}, m_{0j} the rest mass of the particle i, j in the ensemble (*ij*). Also in this case it immediately can be seen that, with the solution of the form as in equation (3.6), the same interpretation as before can be given. It is even possible to show that the solution of equation (3.9) in the limit of c to infinity becomes identical to equation (3.6). So there is not much news in this alternative, but a wave equation with zero masses starting from:

$$(E_{ij}^2 + E_{ji}^2)\Psi_{ij} - c^2((\widehat{p_{ij}})^2 - (\widehat{p_{ji}})^2)\Psi_{ij} = 0 \tag{3.10}$$

has a non constant solution in space and time coordinates. This is remarkable as a zero mass particle like a photon can result in a mass-like presence in open space. It may well be that this is the basis for the fact that in the Friedmann cosmological equations also energy related gravitational pull has to be adopted [3].

Now we come to the central transition point from quantum mechanics to quantum-based relativity.
The wave function as derived gives the presence of an entity to which a rest mass, m_{0ij}, can be dedicated. In quantum mechanical language this rest mass becomes an operator and therefore it has to be multiplied by the wave function and its conjugated function: $\psi_{ij}^* m_{0ij}^2 \psi_{ij}$ and we get:

$$\psi_{ij}^* c^4 m_{0ij}^2 \psi_{ij} = c^4 m_{0ij}^2 \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right)^2. \tag{3.11}$$

This equation says that there is a probability that this pair potential can be found anywhere in the free space, but it is obviously concentrated between and around the two particles forming the pair. So there is a space coordinate dependent probability to find it somewhere, but this probability is not connected to other quantities like energy.
At this moment it is obvious already that m_{0ij} will be proportional to the masses of both participating particles in the interaction. But this is for the time being only a temporary conclusion. It will be justified later as it is of great importance for the final derivation of the gravity law.
Another very important thing is that from $\alpha_{ij}\beta_{ji}/m_j + \alpha_{ji}\beta_{ij}/m_i = 0$ with $\alpha_{ij} = \alpha_{ji}$ it follows that there is for the α-values some freedom in choosing its dependence on relativistic parameters such that the right hand side of equation (3.11) becomes an invariant as it should be, but also it is important for the conclusion that the form of the gravity law is independent of the mutually interacting masses of macroscopic bodies.

4. Relativistic interaction.

Now, in the next step, the pair is considered as essentially one entity and the problem can be analysed in the relativistic four dimensional space. We will draw up the KG-equation remembering the rules of adding up four-vectors and subsequently the formation of the invariant out of this sum. In this representation, however, the rest mass due to the interacting particles in the pair (*ij*), m_{0ij}, is to be considered as an entity that is completely independent of all the other rest masses formed.

But the most important difference from the treatment before is that we will be working in the momentum based sub-space r_{ij} where the group is seen as one single entity. The energy reflects the energy of the two particles together as well as masses and momenta like: $p^2 = (\widehat{p_{ij}} + \widehat{p_{ji}})^2$ and: $E^2 = \left(\widehat{E_i} + \widehat{E_j}\right)^2$ with: $E^2 - p^2c^2 = m_0^2c^4$.

Again we will have to translate this equation into the appropriate quantum mechanical language for pairs as one entity and therefore make the following transformations:

$$-p^2c^2\varphi_{ij,t}\varphi_{ji,t} = (m_0^2c^4 - E^2)\varphi_{ij,t}\varphi_{ji,t}, \quad E^2 = \left(\widehat{E_i} + \widehat{E_j}\right)^2 = -\hbar^2\frac{\partial^2}{\partial t^2} \text{ and:}$$

$$p^2 = (\widehat{p_{ij}} + \widehat{p_{ji}})^2 = -\hbar^2\left(\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ij}}\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ji}}\frac{\partial}{\partial r_{ij}}\right).$$

The last expression is, as different from earlier, a mixed sum of the momenta. This representation is a consequence of the fact that the particles have been treated only in pairs and that spherical symmetry remains to be adopted.

Referring to Figure 2 the total relativistic KG-equation for a number of pairs ($\mathcal{N}$) now will be set up. There are *N* particles which make a total of $\mathcal{N}=N!/2(N-2)!$ pairs, each of which are described by a wave function as a solution of the initial Schrödinger equation. As before, the α-values accommodate all necessary multiplication factors.

Adding up for all pairs, treating them as mutually independent and taking into account the basic rules of quantum mechanics and four-vector algebra lead to:

$$c^2\hbar^2\sum_{ij}\left(\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ij}}\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ji}}\frac{\partial}{\partial r_{ij}}\right)\prod_{ij}\varphi_{ji,t}\varphi_{ij,t} =$$

$$= \sum_{ij} m_{0ij}^2\left(\frac{\alpha_{ij}^2}{r_{ij}^2} + 2\frac{\alpha_{ij}}{r_{ij}}\frac{\alpha_{ji}}{r_{ji}} + \frac{\alpha_{ji}^2}{r_{ji}^2}\right)\prod_{\boldsymbol{ij}}\varphi_{ji,t}\varphi_{ij,t} - \sum_{ij}(\widehat{E_i} + \widehat{E_j})^2\prod_{ij}\varphi_{ij,t}\varphi_{ji,t} \tag{4.1}$$

$$\text{with: } \prod_{ij}\varphi_{ij,t}\varphi_{ji,t} = F(t)\prod_{ij}\varphi_{ij}\varphi_{ji} = \prod_{ij}\varphi_{ij}\varphi_{ji}\prod_1^{\mathcal{N}}\left(\frac{\alpha_g}{r_g}\right)e^{i(k_gr_g - \omega_gt)}. \tag{4.2}$$

m_{0ij} is the rest mass to be dedicated to the interaction field created by the masses m_i and m_j. This factor also accommodates the c^2 as in equation (3.11).The pairs in both products, in total $\mathcal{N}=N!/2(N-2)!$ are numbered by *g,* if there are *N* particles. The term $e^{i(k_gr_g-\omega_gt)}$ expresses a wave propagating in radial direction representing the moving of individual groups, but with reducing amplitude, or, rather probability, as it progresses. If there is no interaction between members of the pairs ($\alpha_{mn} = 0$) we get the movement of the individual particles outside their local influence.

This set-up has a very delicate interpretation. It shows that an observer from outside sees a pair creating a sub-space but cannot determine its structure inside. In the space inside, expressed by the coordinates r_{ij} and r_{ji}, gravitational interactions are occurring. Our observer only sees the separate interacting members of the pair with an energy due to this interaction as is shown schematically in Figure 3. It is as if we see two persons who have made a secret agreement and are, by acting as a pair, exchanging information. We can see both persons but we cannot explain why they behave as they behave.

As before, the time dependences can be removed by setting:

$$\varphi_{ij,t}\varphi_{ji,t} = \varphi_{ij}\varphi_{ji}e^{i(E_{ij}+E_{ji})t/\hbar}, \tag{4.3}$$

so that:

$$\textstyle\sum_{ij}(\hat{E}_i+\hat{E}_j)^2\varphi_{ij,t}\varphi_{ji,t}=\sum_{ij}(E_{ij}+E_{ji})^2\varphi_{ij}\varphi_{ji}. \tag{4.4}$$

If all $\alpha's$ would have been equal to zero, a propagating wave $\varphi_{ij,t}\varphi_{ji,t}$ extending in the radial direction with the light velocity would have resulted. Non zero values of α reduce this speed and, as a consequence, give mass to the field $\varphi_{ij,t}\varphi_{ji,t}$.

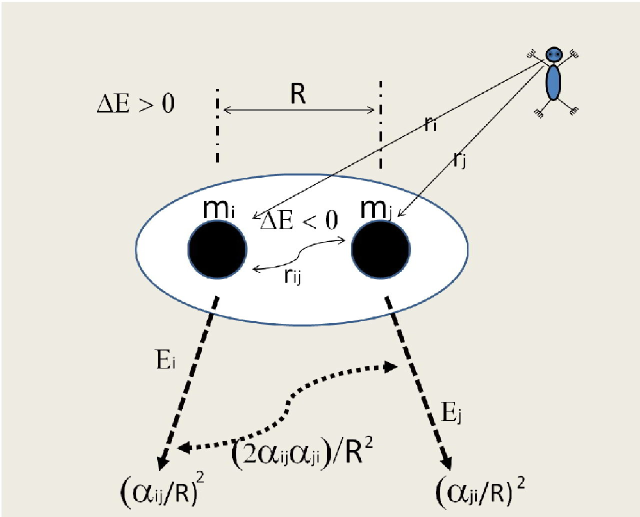


***Figure 3**: Energy transfer from the pair to the surroundings and the sub-space (white area) with internal exchanges as observed from far away.*

The proposed solution will be *):

$$\varphi_{ij} = \gamma_{ij}r_{ij}^{-m_{0ij}\alpha_{ij}/\hbar c}, \tag{4.5}$$

γ_{ij} is the amplitude, not to confuse with the relativity factor γ_k.
From the boundary condition that $\varphi_{ij}(r_{ij},\alpha_{ij}) = 0$ for r_{ij} to infinity a fourth condition on the $\alpha's$ can be derived:
iv. α_{ij} is positive under all circumstances.
Equation (4.5) is inserted into:

$$\textstyle\sum_{ij}(E_{ij}+E_{ji})^2\prod_{ij}\varphi_{ji}\varphi_{ij} - \sum_{ij} m_{0ij}^2\left(\frac{\alpha_{ij}^2}{r_{ij}^2} + 2\frac{\alpha_{ij}}{r_{ij}}\frac{\alpha_{ji}}{r_{ji}} + \frac{\alpha_{ji}^2}{r_{ji}^2}\right)\prod_{ij}\varphi_{ji}\varphi_{ij} +$$

$$+c^2\hbar^2\textstyle\sum_{ij}\left(\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}}r_{ij}^2\frac{\partial}{\partial r_{ij}} + \frac{1}{r_{ji}^2}\frac{\partial}{\partial r_{ji}}r_{ji}^2\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ij}}\frac{\partial}{\partial r_{ji}} + \frac{\partial}{\partial r_{ji}}\frac{\partial}{\partial r_{ij}}\right)\prod_{ij}\varphi_{ji}\varphi_{ij} = 0, \tag{4.6}$$

and some algebra needs to be done during which it will be found that many terms on the left hand side are equal to the ones at the right hand side and therefore disappear **). We get:

$$(E_{ij}^2 + 2E_{ij}E_{ji} + E_{ji}^2)\varphi_{ij}\varphi_{ji} - c\hbar m_{0ij}\left(\frac{\alpha_{ij}}{r_{ij}^2} + \frac{\alpha_{ji}}{r_{ji}^2}\right)\varphi_{ji}\varphi_{ij} = 0. \tag{4.7}$$

At this point a remark has to be made: removing the term α_{kl}^2/r_{kl}^2 means that some basic interaction occurs between the gravitational field and the particle. Obviously, for this separate term, a KG-equation can be formulated that shows that an entity with some relativistically derived mass operates and leaves behind a contribution to the interaction energy in the equation (4.7). So already at this point there is direct interaction between the pair and the field around. Also removing the term with $\alpha_{ij}\alpha_{ji}/r_{ji}r_{ij}$ means that there is a third interaction between the fields and the pair. It is schematically represented in Figure 3.
Taking all these interactions into account it is seen that all α-terms, as they occur in equation (4.6), have disappeared. This has a profound meaning: in this model gravity is due to second order effects of the peculiarities of the spherical symmetry in a relativistic setting. The effect is weak and operates over a long range.

The contributions can now be redistributed, but first multiply all terms by $r_{ij}r_{ji}$ and observe that the proposed solution is the only one that gives a sharp value for the quantity $E_{ij}\, r_{ij}$ and $E_{ji}\, r_{ji}$:

$$(E_{ij}^2 r_{ji} r_{ij} + E_{ij}E_{ji}r_{ji}r_{ij})\varphi_{ij}\varphi_{ji} - c\hbar m_{0ij}\alpha_{ij}\frac{r_{ji}}{r_{ij}}\varphi_{ij}\varphi_{ji} = 0, \tag{4.8a}$$

$$(E_{ji}^2 r_{ji} r_{ij} + E_{ij}E_{ji}r_{ji}r_{ij})\varphi_{ij}\varphi_{ji} - c\hbar m_{0ij}\alpha_{ji}\frac{r_{ij}}{r_{ji}}\varphi_{ij}\varphi_{ji} = 0. \tag{4.8b}$$

Cutting the equation (4.7) into two separate ones as given in equations (4.8a) and (4.8b) looks like arbitrary, as any cut between terms can be made. But if we now come back to the original suggestion and shown in Figure 1, we see that in the sub-space the gravitational interaction becomes symmetric. The gravitational energy of particle *i* is equal to the gravitational energy of particle *j*. It also reflects the point that a pair has to be seen as one entity. The observer cannot distinguish between the separate members of the pair.
It is also important to notice that the operators $\widehat{E_k}$ and r_l commute. It means that "*Er*" is the quantity that has a sharp value, meaning that *E* has sharp value if *r* is well defined.

*) The solution proposed, but more general: $\varphi_{ij} = \gamma_{ij} r_{ij}^n$, can also be applied to a KG-equation not involving pair formation so that $\varphi_i = \gamma_i r_i^n$ and giving a solution similar to $E_i\, r_i$. The exponent *n* remains undefined so that $E_i\, r_i$ can adopt any arbitrary value. It can also be chosen to be zero to give a reference point to the system.

**) This solution (4.5) reduces all invariant and momentum terms in equation (4.6) but brings back a single gradient term. No other solution has better performance than the one proposed in (4.5) so it is to be considered as the most appropriate one. It is actually the spherical symmetry that is responsible for this remaining gradient term. In this model, therefore, the whole story is reduced to one simple statement: Gravity is a consequence of the three dimensional space with spherical symmetry and nothing else.

5. Law of gravity.

Most important for finding out how the members of a pair see each other is to consider the equations (4.8a) and (4.8b) from the viewpoint of an observer who sees the particle m_i at a distance of r_{ij} and particle m_j from a distance r_{ji}. They already know that $\alpha_{ij} = \alpha_{ji} = \alpha$, $r_{ij} = r_{ji} = R$, and $E = E_{ij} + E_{ij}$ with $E_{ij} = E_{ij}$ so that $E_{ij} = E/2$. There are also no operators anymore in equation (4.7a) and (4.7b). In this conclusion a geometrical factor is established in momentum space as in Figure 1 which will need some more justification to be given later in this paragraph. Obviously an electron and a proton forming a pair will have mutual interaction which are the same although their masses differ by some factor of about 1800. The result is a simple relation:

$$2(E/2)^2R^2 = c\hbar\alpha m_0. \tag{5.1}$$

Because both particles in the pair change their energy by the same amount, it follows for the two members of the ensemble together that:

$$ER = \sqrt{2c\hbar\alpha m_0}, \tag{5.2}$$

and the gravitational force is given by: $-\partial E/\partial R = constant/R^2$.

Now it is important to see how pairs consisting of particles of different masses present themselves in α and m_0. It looks like both parameters are tightly glued together in, for instance equation (3.11), but they stem from different places. α is derived from the quantum mechanical considerations whereas m_0 comes from the relativity theory. They have in common that both reflect the fact that pairs of obviously undefined mass units are responsible for the interaction. The most simple conclusion, which needs some justification, would be that: $m_{0kl}\alpha_{kl} = \sigma_1(m_k m_l).\sigma_2(m_k m_l)$.

In accordance with the rules of four-vector algebra we build up two bodies m_1 and m_2 composed of n_1 and n_2 by their mass identical building blocks identified by m_i in m_1 and m_j in m_2 such that they in total make up the mass of m_1 and m_2. The process is shown in Figure 4. These building blocks can be anything like elementary particles, collection of atoms, as long as their masses are the same.

According to equation (5.2) the interaction between the two bodies m_1 and m_2 will be given by $(ER)_{12} = \sqrt{2c\hbar\alpha_{12}m_{012}}$ in which the factor $\alpha_{12}m_{012}$ stems from the composite solution as in equation (4.3). This composite solution reads:

$$\varphi_{12} = \gamma_{12}r_{12}^{-m_{012}\,\alpha_{12}/\hbar c} = \gamma_{12}r_{12}^{-(m_{012}/m_{0ij})(\,\alpha_{12}m_{0ij})/\hbar c}. \tag{5.3}$$

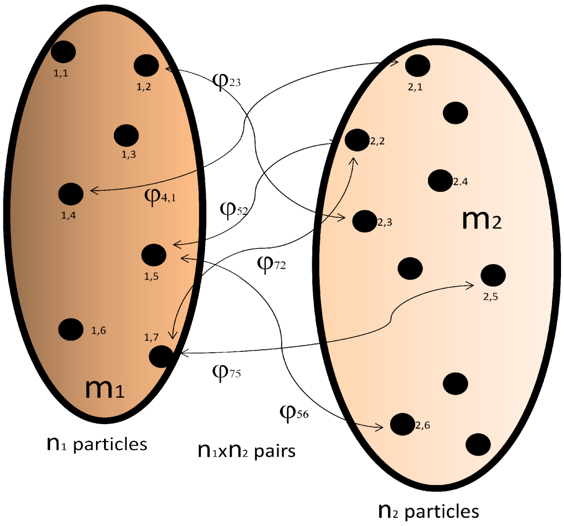
First we consider the rest-mass carrying the gravitational interaction.

It is found that the pair (ij) has gravitational energy, say ε and thus a mass ε/c^2. In the interaction space between m_1 and m_2 there are n_1n_2 pairs carrying the interaction between m_1 and m_2 and so we can conclude that $m_{012} = n_1n_2m_{0ij}$.

The function φ_{ij} for the pair (ij), occurs in φ_{12} n_1n_2 times and so we get in the multiplication of $\varphi_{ij}\varphi_{ji}$ over all pairs between m_1 and m_2: $n_1n_2m_{0ij}(\alpha_{ij} + \alpha_{ji}) = m_{0ij}(\alpha_{12} + \alpha_{21})$.

Combining the two arguments we finally get: $m_{012}\alpha_{12} = n_1^2 n_2^2 m_{0ij}\alpha_{ij}$ and, remembering that m_1 and m_2 are composed of identical building blocks, we can conclude that $m_{012}\alpha_{12} = \sigma m_1^2 m_2^2$, which leads to the result:

$$E_{12} = E_{21} = \sqrt{2\sigma c\hbar}.(m_1 m_2)/R. \tag{5.4}$$



The distance between the masses is due to the R_{ij}-values which, obviously, are different for each building block. It means that R is to be considered as the center-of-mass- distance between m_1 and m_2.

Figure 4: *Interaction between masses by building up structures from equal mass units.*

In view of this equation (5.4) we conclude that also the gravitational interaction is proportional to both masses of the participating particles in the pair.
In the Figure 4 a more simple argument is given by the summation starting from the equation (5.4): all pairs that have been formed are acting independently so that we can add all the contributions of different masses at their individual locations together and in this way constitute bodies in the real world without any interference. This latter argument, however, violates causality. It starts from the assumption that the interaction is proportional to the product of the masses, and builds up the interaction between larger bodies. It is an easy argument and it says that we can start from any size of building blocks and build up the macroscopic structure such that at all sizes and steps in its building up the equation (5.4) remains valid. The basic reason that this is possible is due to the freedom of choice for α_{ij} as it came as a conclusion from equation (3.7). This adding up of all the interactions between particles, which in part see each other at different distances, is a problem that has already been solved in the formulation of the classical theory of electrostatics [8]. In this way, finally, Newton's gravitation law is obtained which, in vector notation reads: $div\boldsymbol{g} = 4\pi\rho G$ in which $\boldsymbol{g}$ is defined as a gravitational vector field around an entity constituting a space coordinates dependent mass density ρ. G is the well known gravitational constant equal to: 6.673x10^{-11} $m^3kg^{-1}sec^{-2}$ [3].
In accordance with the theory of electrostatics the gravity law can also be given in vector representation for bodies M_1 and M_2 which have their centres of gravity at a separation of R:

$$\boldsymbol{F_{12}} = (GM_1M_2/R^3)\,\boldsymbol{R}. \tag{5.5}$$

From the equations (5.4) and (5.5) an explicit expression for the parameter σ can be derived and also, with the help of these equations the small mass to be attributed to the gravitational interaction can be found. This σ parameter is equal to 2.7x10^2 Jm/kg^4.

In Figure 1 a sub-space is presented in momentum space with an observer in point O_2 or, eventually, observers on the two particles. They see that the total energy of the pair (ij) is lower than the energy of the particles separately. But since $E_{ij} = E_{ji}$ they must conclude that $r_{ij} = r_{ji}$. The locations where $r_{ij} = r_{ji}$, have in this respect, a specific meaning.

If we form the operator $\widehat{p_{ij}} - \widehat{p_{ji}}$ and apply it to the wave function: $\psi_{ij} = \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}}$ we get the "eigenvalue" equation:

$$\frac{\hbar}{i}\left(\frac{\partial}{\partial r_{ij}} - \frac{\partial}{\partial r_{ji}}\right)\psi_{ij} = \left\{\hbar\left(\frac{\alpha_{ij}\beta_{ij}}{r_{ij}} - \frac{\alpha_{ji}\beta_{ji}}{r_{ji}}\right) - \frac{\hbar}{i}\left(\frac{\alpha_{ij}}{r_{ij}^2} - \frac{\alpha_{ji}}{r_{ji}^2}\right)\right\} e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}}. \quad (5.6)$$

In the earlier argument as shown in Figure 7 we have the mass units $m_1 = m_2$, or for this argument $m_i = m_j$, and so we have $\beta_{ij} = -\beta_{ji}$, and we get a pure "eigenvalue" equation for the operator $\widehat{p_{ij}} - \widehat{p_{ji}}$ if $r_{ij} = r_{ji}$. So the observer at point O_2 in Figure 1 with $r_{ij} = r_{ji} = R/2$, will see a sharp and well defined value for the difference between the momenta of the mass entities m_i and m_j. This observer can evaluate equations (4.8a) and (4.8b) but in that case with $E = E_{ij} = E_{ij}$ leading to the same result as equation (5.1):

$$2(E/2)^2R^2 = 2E^2(R/2)^2 = c\hbar\alpha m_0. \quad (5.7)$$

The other option is two observers, one on m_i and one on m_j, so that and $r_{ij} = r_{ji} = R$, but now with $E = E_{ij} + E_{ij}$ and the result of equation (5.7).

6. Transfer of energy and mass.

In the analysis going from equation (4.6) to (4.8) terms are disappearing due to the solution proposed in equation (4.3). But this has to be interpreted with caution. The pair function $\varphi_{ij}\varphi_{ji}$ in equation (4.1) represents a field carrying the gravitational energy. Therefore, the disappearance of the generator at the left hand side of equation (4.6), $(\alpha_{ij}/r_{ij} + \alpha_{ji}/r_{ji})^2$, involves exchange of energy from the pair to the surrounding space which is equal to the energy given in equation (5.4). As a consequence, when the positive value for the energy is taken, the energy of the pair itself is reduced by the same amount. In that case the interaction between the members of the pair is attractive. The process is schematically shown in Figure 3. The opposite situation in which the energy of the pair is positive, which in principle is allowed by the Einstein energy, is not possible when we assume that the energy of the vacuum, to be taken as the reference point, is zero. In this interpretation the interaction between mass and the surroundings is a means to transfer mass related energy (mc^2) to gravitational energy. This transfer changes the rest masses of the pair but does not create new mass.

If, however the vacuum state is, as it is generally believed, a non-zero energy state there might be energy available which increases with the interaction area, the white area in Figure 3, that can be transferred to the pair. The situation could be such that, when the distance between the members of the pair increases, the energy needed is reducing whereas the energy, or number of fluctuations carrying sufficient energy is increasing. It means that at some separation distance of the members of the pair the interaction can become repulsive as the Einstein equation allows both negative and positive values for the interaction energy.

A solution for the Schrödinger equation of a pair of particles for an observer at distances r_{ij}

and r_{ji} from particle i and j is given in equation (3.6). Now if we put our observer close by particle i, the second term in equation (3.6) becomes negligible against the first term:

$$\Psi_{ij} = \left(\frac{\alpha_{ij}}{r_{ij}} + \frac{\alpha_{ji}}{r_{ji}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} \cong \left(\frac{\alpha_{ij}}{r_{ij}}\right) e^{i\beta_{ij}r_{ij}+i\beta_{ji}r_{ji}} \text{ and: } \Psi^*_{ij}\Psi_{ij} \cong \left(\frac{\alpha_{ij}}{r_{ij}}\right)^2. \tag{6.1}$$

The KG-equation in operator language now reads:

$$-\hbar^2\left(\frac{\partial^2}{\partial t^2} - c^2\frac{1}{r_{ij}^2}\frac{\partial}{\partial r_{ij}} r_{ij}^2\frac{\partial}{\partial r_{ij}}\right)\varphi_{ij,t} = m_{0ij}^2\left(\frac{\alpha_{ij}}{r_{ij}}\right)^2 \varphi_{ij,t}. \tag{6.2}$$

Setting the right hand side to zero, a mass-less particle, we see an equation for a travelling wave at the speed of light. To get rid of the singularity we set $\alpha_{ij}/r_{ij} = \alpha_{ij}/r_{ip}$ for $r_{ij} < r_{ip}(= r_p)$, and removing the first term on the left hand side gives the London Equation which explains the shielding of the inside of a superconducting material from the outside magnetic field: the "Meissner" effect [2]. A similar thing can be imagined in this case with the $\varphi_{ij,t}$-field for $r_{ij} < r_p$. The distance r_p can be identified as the distance from the centre to where local influences have no impact.

We can solve the equation (6.2) with in the right hand term r_p for r_{ij}, but it is not necessary as it can immediately be seen that it dedicates mass to the field in the vicinity of the particle which is equal to $m_p = m_{0i}\ \alpha_{ij}/r_p c^2$. As this is the mass to be attributed to the i^{th} particle, due to another particle somewhere in the surroundings, we will have to add up over all particles which can make a pair with our particle, so with $m_p = m_i$:

$$m_p = \sum_j m_{0ij}\alpha_{ij}/r_p c^2 = (\sigma/_{r_p c^2})\sum_j m_p^2 m_j^2. \tag{6.3}$$

The consequence is that either $m_p = 0$, a mass-free particle, or:
$m_p = r_p c^2/\sigma\sum_j m_j^2$, with, as shown, $m_{0ij}\alpha_{ij} = \sigma m_i^2 m_j^2$. First the equation allows that there are mass-free particles like a photon which makes no pairs according the theorem based on the Schrödinger equation, but it can, according to the KG-equation (3.10). It could generate gravity as it is argued in Chapter 9: Cosmography of W.D. Heacox's book on the expanding Universe [8]. Second, the other solution is that there is a mass carrying particle whose mass becomes higher when r_p increases and, most important, it is all the mass in the surroundings that generate the mass of the i^{th} particle. It is actually mass due to the field, but since the singularity moves with the particle the observer nearby can only interpret it as a mass contribution to the particle he is looking at. The conclusion taken here corresponds to Mach's ideas about the effect of all physical entities in the universe.

It would be tempting to evaluate m_p but, as we know already from observation, it is better to estimate the size or the extension of the particle if only this effect is responsible for the mass. The analysis concerns incredibly large and small numbers but leads to a surprising outcome. Starting from $m_p = r_p c^2/\sigma\sum_j m_j^2$ and assuming that the mass of the universe is basically due to protons and neutrons with almost the same mass, so $m_p = m_j$, and assuming there are N particles in the whole universe giving it a total mass of M_u we can set:

$$M_u = Nm_j = Nr_p c^2/\sigma\sum_j m_j^2 = Nr_p c^2/\sigma N m_j^2 = r_p c^2/\sigma m_j^2. \tag{6.4}$$

Estimates of the size of the universe on the basis of the inverse Hubble constant and the fact that the average intergalactic density is 1000 hydrogen atoms per cubic meter tells us that the total mass of the universe is of the order of 10^{55} kg. σ is calculated in paragraph 5 at $2.7\text{x}10^{2}$ Jm/kg^4 and the proton mass is $1.7\text{x}10^{-27}$ kg [11]. It leads to an estimate for the r_p-value in the order of 10^{-15} m, which is about the size of a proton (0.8 femtometers) [1]. An electron which is 1840 times lighter than the proton will, according to equation (6.3), see the same surrounding as the proton, so its size would be smaller by the same factor.
Although the correspondence with measured data is surprisingly good, it is still a rough estimate and not without speculation.
Even a discrepancy by a factor of 10 would already be acceptable for the outcome of this analysis. For instance, the sub-space due to the generator $m_{0i}^2\ (\alpha_{ij}/r_{ij} + \alpha_{ji}/r_{ji})^2$ would be a quantum-mechanical reality, but it says nothing about its internal structure and interactions. The mass of the universe is rather uncertain in view of the discussion about dark matter, and the proton size, or how to define it, is not so obvious.
The surprising, and at the same time bizarre, conclusion of the analysis given is that, apparently, each single particle has interaction with all other particles in the cosmos. It means that in the universe an unimaginable number of pair-wise interactions exists with greatly varying intensity and extensions and which depend on the masses of the members of the pair. It is difficult to comprehend, but it follows unambiguously from the equations describing the behaviour of the pairs.
As a last remark for this paragraph, causality is of importance to keep in mind. The model starts from the fact that there are masses, and it is seen that they can form pairs and generate gravity. It yields numerical data about the masses following gravitational parameters. The strength of the model is the consistency of the data with what we observe in reality. On the other hand one can say that the mass can be introduced into the Schrödinger equation as an unknown quantity and the theory comes back with a numerical value for it if the size of the particle is known.

7. Gravity depending on dynamical masses.

In paragraph 2 at the end it is mentioned that a group as a whole, identified with the label *g,* has kinetic energy and therefore a relativistic mass equal to $\gamma_g m_g$. Although the present theory is only concerned with the situation in the sub-space r_{ij} where gravity originates, it still is of interest to know the dynamic mass of the group because the Hamiltonian operator has been modified. The momentum of the group, $\widehat{p_g}/\sqrt{2m_g}$, will determine the dynamic mass to be dedicated to the members of the group. To see this we come back to equation (5.4), now renumbered to (7.1):

$$E_{ij} = \sqrt{2\sigma c\hbar}.\,(m_i m_j)/R. \qquad (7.1)$$

This equation is valid for the sub-space in which the gravitational energy is independent of the momenta of the particles in the group. But by inspection we see a problem.
For an observer outside the sub-space the second term left in equation (4.6) should be invariant under Lorentz transformation. However, the r_{kl} transforms as a member of a four-vector. Therefore, the parameters α_{kl} or, in the case of equation (7.1), σ should transform in

the same way as r_{kl}, but apparently it would make left and right hand side in equation (5.2) transform differently, which cannot be the case. We should, however, notice that the Planck's constant, h, is invariant, but $\hbar = h/2\pi$ is not.
Make the following "thought-experiment". Consider a pair flying away from us at a speed v such that the separation vector of the members of the pair is aligned in the direction of v. Due to the fact that π transforms just like $1/r_{kl}$ the result is that the interaction energy of the pair we measure becomes invariant. There is invariance throughout if the alignment perpendicular to the speed. So the conclusion is that the interaction energy in the pair is invariant and independent of the alignment towards the observer. We can see the pair moving by and, whatever alignment they have, we will see the same interaction energy. But for the observer outside the sub-space, actually in point O_1 in figure 1, the group as a whole is moving which gives a dynamical mass to the particles in the group, but with the same γ_g- factor.
Knowing this we can from equations (2.5a) and (2.5b), in principle, give the value for this group momentum if the replacement of the vectors $\widehat{p_k}/\sqrt{2m_k}$ by their relativistic equivalents has been done. However, there remains a disturbing $cos\delta_1$-term making a general solution inappropriate. But the purpose of an endeavour in which such a group related dynamic mass is significant makes only sense where gravity is important and speeds are approaching the speed of light. So it is not relevant outside the realm of cosmology.
In this respect the main problem of the incompatibility between quantum theory and relativity, however, comes to the surface. We therefore have to carefully replace the vectors in Figure 1 by the relativistically relevant ones which are to be derived from the equations (2.5a) and (2.5b) leading to the transitions:

$p_a/\sqrt{2m_a} \mapsto cm_a\sqrt{\gamma_a^2 - 1}$ with $a = i, j$ and g.

Now we can put our observer on one of the interacting particles, say m_i in the group (ij), and consider the surroundings from this point of view so that $p_i = 0$. In this case $cos\delta_1 = -1$, but because $p_i = 0$ the $cos\delta_1$- factor has no influence anymore. We end up in a rather complicated situation if we want to know the mass and γ_g- values for the group and we find non relativistic:

$$v_g^2/c^2 = \frac{m_j}{2(m_i+m_j)} v_j^2/c^2, \text{ and relativistic: } {\gamma_g}^2 - 1 = \frac{m_j^2}{2\eta^2(m_i+m_j)^2}\left({\gamma_j}^2 - 1\right). \qquad (7.2a, 7.2b)$$

The extra parameter η complicates the situation. If our observer is on mass m_i which is much smaller than m_j: $\eta = 1$, and both equations are identical. But in paragraph 5, equation (5.3) we have constructed our bodies with building blocks of masses $m_i$which all are identical in their masses. We should, therefore, start from the case of $m_i = m_j$, so that: $\eta = \sqrt{1/2}$.
The result is:

$$ {\gamma_g}^2 - 1 = \left({\gamma_j}^2 - 1\right)/4. \qquad (7.3)$$

With the aid of the definition of γ it is easily changed into the relation:

$$v_g^2/c^2 = v_j^2/(4c^2 - 3v^2). \qquad (7.4)$$

This gives the mass to be allotted to both members of the group. At low velocities (v << c), The mass of the group particles is determined by half the speed of the moving particle. When the speed of the moving particle approaches the light velocity, both speeds become equal. This result is similar to the velocity addition rule for relativistic velocities on the basis of standard relativity theory [7], but in this case arrived at in way involving gravity.
At low speeds we have to dedicate dynamic mass to both particles and the equation will read:

$$\boldsymbol{F_{12}} = \boldsymbol{R}G\big(M_{01}/\sqrt{(1-v^2/4c^2}\,M_{02}/\sqrt{(1-v^2/4c^2)}\big)/R^3. \tag{7.5}$$

When speeds are approaching the speed of light, of course, the speeds of both particles are still the same and opposite, but at the value $v_{j.}$ An alternative way of interpreting equation (7.5) is to place the observer in the sub-space in the middle between the two particles so that the observer sees their speeds $v' = v/2$ and opposite and the distance $R' = R/2$. In that case the equation becomes:

$$\boldsymbol{F_{12}} = \boldsymbol{R}'G\big(M_{01}/\sqrt{(1-v'^2/c^2}\,M_{02}/\sqrt{(1-v'^2/c^2)}\big)/4R'^3. \tag{7.6}$$

This interpretation has to be considered as an alternative interpretation of equation (7.5) and not of the real situation of two particles moving away from the observer at equal but opposite speeds. This is because the equations are derived for the case that we have taken the momentum of one of the group members as zero. It, however, allows a remarkable interpretation. It looks like a “mirror” mass shows up at a distance of R from the moving mass that moves at the same speed as the moving one. Far from the light speed the relative speed between the two masses is double the speed seen by the observer but when it approaches c, the relative speed becomes c as well.

In conclusion it can be said that particles in a group in the sub-space have gravitational interaction have masses which must be corrected with the relativistic transformation factor γ_g as defined by equation (7.2a and -b).
The kinetic energy of the group remains to be defined by the value:

$$E_{kin} = T_k = M_{01}(\gamma_1 - 1)c^2 + M_{02}(\gamma_2 - 1)c^2. \tag{7.8}$$

8. Discussion and conclusions.
An attempt is made to find an explanation for the gravity law, or Newton’s third law starting from well established and proven theorems: Special Relativity and Quantum Mechanics. Although these two theories cannot be readily combined, it is possible to use the outcome of quantum mechanical considerations as starting point for further analysis by taking into account the rules of specific relativity. If we apply the two concepts in those areas where they have their applicability it was proved possible to derive the gravity law as it has been established already more than three hundred years ago. The main issues in the analysis are:
1. We can separate the local behaviour of particles in its direct environment like a gas, liquid or a solid from its behavior in free space as a member of a larger entity.
2. We can modify the Hamiltonian of a set of two individual particles, or tightly connected

entities, such that for two and only two of such entities, characterized by their masses, the Hamiltonian is represented by a group kinetic energy operator and a second part which is the direct interaction in a separate momentum based sub-space.
3. From this, a group wave function and a wave function representing the members in the group emerge, the first one is found to be responsible for the dynamic masses to be allotted to the particles in the gravity law and the second one is responsible for the gravitational interaction.
4. The two particle wave function is then recognized as a pair potential in a sub-space between the members of the group and is taken as the relativistically invariant rest mass in the Klein Gordon field equation.
5. By solving the Klein Gordon field equation for the pair represented as a single entity we finally arrive at the right form of Newton's third law of gravity. Also by adding up the basic functions for single group of particles, or groups of particles, the right form of the gravity law between large bodies is obtained.
6. Considering the dynamics of the group as a whole also the influence of the dynamic relativistic mass in the gravity equation is derived.
7. The Klein Gordon equation is also found to be applicable at the level of a single particle and gives a value for its mass in dependence of all mass around in the entire universe. Most surprising is that the calculated values are of the right order even though the numbers that are going into the equations are extremely large and extremely small.

8. References.

[1] A. Antognini, et. al. Science 339 (2013) 417-420.
[2] Bardeen, Cooper, Schrieffer. Theory of Superconductivity: Physical Review 108, (1957) 1175.
[3] W. D. Heacox, The Expanding Universe, Cambridge University Press 2015.
[4] W. Heitler. Elementary Wave Mechanics, Oxford Clarendon Press 1945.
[5] A. Messiah. Quantum Mechanics Vol. 1, North Holland Publishing Company, Amsterdam 1969.
[6] A. Messiah. Quanten-Mechanik, Band 2, Walter de Gruyter, Berlin 1985.
[7] E. P. Ney. Electromagnetism and Relativity: Harper &Row, New York 1965.
[8] W. Panowsky, M. Philips. Classical Electricity and Magnetism, Addison Wesly Publishing Company 1972.
[9] H.J. Veringa, H.J. New Basic Theory on Gravity, Journal of Modern Physics **7** (2016) 1818-1828.
[10] H.J. Veringa. H.J. Part II of New Basic Theory on Gravity, Journal of Modern Physics **7** (2016) 2266-2280.
[11] Wikipedia.org/proton.