An Essay on the Zeroes of an L-function

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We set the following notation.

K a global field

K_ν a local field, completion of K at the place ν of K

A_K the adele ring of K

C_K the idele class group $\mathrm{GL}_1(\mathrm{A}_K)/K^*$

$\hat{C}_K$ the dual group of C_K.

0.

We will summarize the spectral interpretation of critical zeros of $L(\chi, s)$ associated χ of C_K by Alain Connes. Let h be a test function. The Weil explicit formula says

$$\sum_\nu \int'_{K_\nu^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho).$$

Suppose that there exists a representation U of C_K, and that

$$\mathrm{tr}\, U(h) = \sum_\nu \int'_{K_\nu^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu$$

is satisfied. We see that

$$\mathrm{tr}\, U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho)$$

holds. We can say that critical zeros of $L(\chi, s)$ appear as the spectra of the operator U. It is just *the spectral interpretation of critical zeros of* $L(\chi, s)$.

Let

$$X = \mathbb{A}_K / K^*.$$

The left regular representation U of C_K on $L^2{}_\delta(X)$ which is a weighted L^2 space can be used to accomplish our task. Namely, it holds that

$$\mathrm{tr}U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\,\rho = 1/2}} \hat{h}(\chi,\rho) + \infty h(1) .$$

However we will not try to treat the representation $(U, L^2{}_\delta(X))$ directly. Instead of the representation $(U, L^2{}_\delta(X))$, we will think of the operator $Q_\Lambda U$ where U is the left regular representation of C_K on $L^2(X)$. Because, firstly there is a possibility of using some results to compute *Trace* $Q_\Lambda U$, secondly we can eliminate the parameter δ of $L^2{}_\delta(X)$. Now, we can show that

$$\mathit{Trace}\ Q_\Lambda U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\,\rho=1/2}} \hat{h}(\chi,\rho) + \infty h(1) \qquad \Lambda \longrightarrow \infty$$

for the function h which belongs to *Bruhat–Shwartz space* $\mathcal{S}(C_K)$ of functions on C_K.

We try to compute *Trace* $Q_\Lambda U(h)$. This has the relationship to the validity of the Riemann Hypothesis. Suppose that we can compute as follows;

$$\mathit{Trace}\, Q_\Lambda U(h) = 2h(1)\log' \Lambda + \sum_v \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu + o(1) \qquad \Lambda \to \infty$$

where $2\log'\Lambda = \int_{\lambda \in C_K,\, |\lambda| \in [\Lambda^{-1}, \Lambda]} d^*\lambda$. We obtain a trace formula:

$$\hat{h}(0) + \hat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\,\rho=1/2}} \hat{h}(\chi,\rho) + \infty h(1) = 2h(1)\log' \Lambda + \sum_v \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu + o(1) \qquad \Lambda \longrightarrow \infty .$$

The left side is spectral and the right side is geometrical. From the Weil explicit formula, we have seen that

$$\sum_v \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu = \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho) .$$

Therefore, one obtains that

$$\sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho) = \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\,\rho = 1/2}} \hat{h}(\chi,\rho).$$

It means the validity of the Riemann Hypothesis. Conversely, the validity of the Riemann Hypothesis implies that

$$Trace\, \mathrm{Q}_{\Lambda} U(h) = 2h(1)\log' \Lambda + \sum_{v} \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu + o(1) \qquad \Lambda \to \infty.$$

1.

We try to characterize L-functions from the view of the representation theory.

We will begin with the local case. Denote the set of the irreducible representations of K_ν^* by $\mathrm{Irr}(K_\nu^*)$. Let (π_ν, V_{π_ν}) be an irreducible representation of K_ν^*. Put

$$\pi_\nu(f)v = \int_{K_\nu^*} f(g)\pi_\nu(g)v\,d^*g\,, \quad f \in \mathcal{S}(K_\nu).$$

Suppose that $\mathrm{tr}\pi_\nu(f)$ can be defined, namely $\pi_\nu(f)$ is a trace class operator. So we may think that there exists a character $\mathrm{tr}\pi_\nu$ of K_ν^*, and

$$\mathrm{tr}\pi_\nu(f) = \int_{K_\nu^*} f(g)\mathrm{tr}\pi_\nu(g)\,d^*g\,.$$

Define the local zeta function as

$$Z(s, \chi, \Phi) = \int_{K_\nu^*} \Phi(g)\chi(g)|g|^s\,d^*g\,.$$

Here $s \in \mathbb{C}$, χ is a character of K_ν^* and $\Phi \in \mathcal{S}(K_\nu)$. The integral converges at $\mathrm{Re}(s) > 0$. The L-factor $L(s, \chi)$ is defined as $Z(s, \chi, \Phi)/L(s, \chi)$ being entire. We will see that the local zeta function associated with π_ν can be

$$Z(s, \mathrm{tr}\pi_\nu, \Phi) = \int_{K_\nu^*} \Phi(g)\mathrm{tr}\pi_\nu(g)|g|^s\,d^*g\,.$$

The L-factor $L(s, \pi_\nu)$ is defined as $Z(s, \mathrm{tr}\pi_\nu, \Phi)/L(s, \pi_\nu)$ being entire.

Next, we will think of the global case. It is performed on the adele ring of K. Set

$$\pi = \bigotimes_\nu \pi_\nu\,, \qquad V_\pi = \bigotimes_\nu V_{\pi_\nu}\,.$$

We can obtain an irreducible representation (π, V_π) of A_K^*. Denote the set of the irreducible representations of A_K^* by $\mathrm{Irr}(\mathrm{A}_K^*)$. Suppose that $\pi(f)$ where $f \in \mathcal{S}(\mathrm{A}_K)$ is a trace class operator. Then $\mathrm{tr}\pi$ is given as a character of A_K^*. We also obtain the global zeta function

$$Z(s, \mathrm{tr}\pi, \boldsymbol{\Phi}) = \prod_\nu Z(s, \mathrm{tr}\pi_\nu, \Phi)\,.$$

Here $\boldsymbol{\Phi} \in \mathcal{S}(\mathrm{A}_K)$. We define the L-function associated with π as follows;

$$L(s, \pi) = \prod_\nu L(s, \pi_\nu)\,.$$

Each L-factor $L(s, \pi_v)$ gives the Euler factor of $L(s, \pi)$, namely $L(s, \pi)$ has the Euler product. The $L(s, \pi)$ satisfies the functional equation which is given by the functional equation of the global zeta function. Thus, $L(s, \pi)$ is analytically continued to the function which is meromorphic in the whole plain $\mathbb{C}$.

We shall consider an irreducible representation (π, V_π) of C_K. Let $\mathcal{H}_\pi$ be a suitable completion of V_π with a certain inner product. One obtains a unitary representation $(\pi, \mathcal{H}_\pi)$, which is a left regular representation of C_K on $\mathcal{H}_\pi$. We may say that if $\pi \in \mathrm{Irr}(C_K)$ then $\pi \in \hat{C}_K$. Thus,

$$\mathcal{H} = \bigoplus_{\pi \in \hat{C}_K} \mathcal{H}_\pi, \qquad \mathcal{H}_\pi = \left\{ \xi \mid \xi(g^{-1}x) = \pi(g)\xi(x), \forall g \in C_K \right\}.$$

We know that $\mathrm{tr}\pi$ is a character of C_K. We frequently use χ to denote a character of C_K. Then, $\mathrm{tr}\pi = \chi$. Correspondingly, $L(s, \pi) = L(s, \chi)$.

Lastly we will mention trace formulae. The trace formula which is given by a zeta function:

$$\underbrace{\cdots}_{\text{Zero points}} = \underbrace{\cdots}_{\text{Geometrical side}}$$

is a prototype. Selberg's trace formula is that

$$\underbrace{\cdots}_{\text{Eigenvalues of Laplacian}} = \underbrace{\cdots}_{\text{Geometrical side}} .$$

There exists an operator M such that it is commutative with the Laplacian of H. The operator is the integral operator which has $k(z, w)$ as an integral kernel

$$M(f)(z) = \int_{\mathrm{H}} k(z, w) f(w) d\mu(w).$$

The Selberg's trace formula gives the explicit formula of Selberg's zeta function.

The trace formula given by Connes is the same type as Selberg's. It is that

$$\underbrace{\cdots}_{\text{Characters}} = \underbrace{\cdots}_{\text{Geometrical side}} .$$

Here $U(h)$: $C_c^\infty(X) \longrightarrow C_c^\infty(X)$

$$(U(h)\xi)(x) = \int_{C_K} h(g)(U(g)\xi)(x) d^*g .$$

The operator $U(h)$ is the integral operator which has $k_h(x, y)$ as an integral kernel

$$(U(h)\xi)(x) = \int_{C_K} k_h(x, y)\xi(y) d^*y .$$

2.

The space $\mathcal{S}(\mathrm{A}_K)_0$ is given as the codimension 2 subspace of $\mathcal{S}(\mathrm{A}_K)$ such that

$$f(0)=0, \quad \int_X f(x)dx=0.$$

Let $L^2(X)_0$ be the completion of $\mathcal{S}(\mathrm{A}_K)_0$. We obtain an exact sequence:

$$0 \to L^2(X)_0 \to L^2(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0$$

where $\mathbb{C} \oplus \mathbb{C}(1) \cong L^2(X)/L^2(X)_0$.

[Remark] $\mathbb{C}$ is a trivial C_K module:

$$T(g)\lambda = \lambda \qquad g \in C_K, \lambda \in \mathbb{C}.$$

$\mathbb{C}(1)$ is Tate twist:

$$T(g)\lambda = |g|\,\lambda \quad g \in C_K, \lambda \in \mathbb{C}.$$

Here we have to give one's attention to the space X. The space X is *a delicate quotient space*. It must be non-compact. It must be also questionable to think that X contains C_K as a subspace. However, considering the construction of $L^2(C_K)$, if we restrict the function in $L^2(X)$ to C_K then it can be a function on C_K. We can also obtain the following exact sequence:

$$0 \to L^2(X)_0 \xrightarrow{\mathrm{T}} L^2(C_K) \to \mathcal{H} \to 0$$

where $\mathcal{H} \cong L^2(C_K)/\mathrm{Im}(\mathrm{T})$. Let U be a left regular representation of C_K on $L^2(X, dx)$ and V be a left regular representation of C_K on $L^2(C_K, d^*x)$. For $f(x) \in L^2(X, dx)$, let $(\mathrm{T}f)(a)$ be the restriction of $f(x)$ to C_K. Then,

$$(\mathrm{T}f)(a) = |a|^{1/2} f(a) \qquad \forall a \in C_K.$$

Since $dx = |x| d^*x$, we will understand that $(\mathrm{T}f)(a) \in L^2(C_K, d^*x)$. Set

$$(U(g)f)(x) = f(g^{-1}x) \quad \forall g \in C_K, x \in X.$$

It turns out that

$$\begin{aligned} \mathrm{T}(U(g)f)(a) &= \text{the restriction of } f(g^{-1}x) \\ &= |g|^{1/2}(V(g)\mathrm{T}f)(a) \qquad \forall a, g \in C_K. \end{aligned}$$

From this equation, it is that $|g|^{-1/2}\,\mathrm{T}(U(g)f)(a) = V(g)(\mathrm{T}f)(a)$. For $(\mathrm{T}f)(a)$,

$$\begin{aligned} V(g)(\mathrm{T}f)(a) &= \textit{the restriction of } |g|^{-1/2} f(g^{-1}x) \\ &= |a|^{1/2}\,|g|^{-1/2} f(g^{-1}a). \end{aligned}$$

From $f \in \mathcal{S}(\mathrm{A}_K)$, we will see that $|g|^{-1/2} f(x) \in L^2(X)_0$, and that $|g|^{-1/2} f(g^{-1}x) \in L^2(X)_0$. Thus $V(\mathrm{Im}(\mathrm{T})) \subseteq \mathrm{Im}(\mathrm{T})$, namely $\mathrm{Im}(\mathrm{T})$ is an invariant subspace for V. Now, we have to turn one's attention to using $L^2(C_K)$. Because C_K is abelian locally compact, we can't always decompose $L^2(C_K)$ in the direct sum of finite dimensional subspaces. This fact, $L^2(C_K)$ having no finite dimensional subrepresentation, is an obstacle to our attempt computing the trace of U.

"*The second subtle point is that since C_K is abelian and non compact, its regular representation does not contain any finite dimensional subrepresentation so that the Polya-Hilbert space cannot be a subrepresentation (or unitary quotient) of V. There is an easy way out which is to replace $L^2(C_K)$ by $L^2{}_\delta(C_K)$ using the polynominal weight $(\log^2|a|)^{\delta/2}$, i.e. the norm* $\|\xi\|^2_\delta = \int_{C_K} |\xi(a)|^2 (1+\log^2|a|)^{\delta/2} d^*a$." in A. Connes [2].

Because $L^2{}_\delta(C_K)$ is a weighted L^2 space, we can decompose it in the direct sum of finite dimensional subspaces. Let the Hilbert space $L^2{}_\delta(X)$ $(\delta > 1)$ be the space of functions on X with the square norm

$$\|f\|^2_\delta = \int_X |f(x)|^2 (1+(\log|x|)^2)^{\delta/2}\, dx\,.$$

The Hilbert space $L^2{}_\delta(C_K)$ is obtained from the space of functions with the square norm

$$\|f\|^2_\delta = \int_{C_K} |f(g)|^2 (1+(\log|g|)^2)^{\delta/2}\, d^*g$$

where we normalize the Haar measure of the multiplicative group C_K

$$\int_{|g|\in[1,\,\Lambda]} d^*g \sim \log\Lambda \qquad \Lambda \to +\infty\,.$$

We understand that the representation $(V, L^2{}_\delta(C_K))$ isn't unitary because of the suffix $(1+(\log|g|)^2)^{\delta/2}$. However

$$\|V(a)\|_\delta = O((\log|a|)^{\delta/2}) \qquad |a| \longrightarrow \infty.$$

It is also satisfied that

$$\|V(a)\|_\delta = O((\log|a|)^{\delta/2}) \qquad |a| \longrightarrow 0.$$

[Remark] It holds that

$$\|V(a)\|_\delta \le (c\cdot(1+(\log|a|)^2)^{\delta/2})^{1/2}.$$

Here we may say that $\|V(a)\|_\delta \ge 0$. We can compute as follows;

$$\|V(a)\|_\delta^2 \le c\cdot(1+(\log|a|)^2)^{\delta/2}$$

moreover,

$$\|V(a)\|_\delta^{4/\delta} \le c^{4/\delta}\cdot(1+(\log|a|)^2).$$

Thus,

$$\frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} \le c^{4/\delta}\cdot\frac{1+(\log|a|)^2}{(\log|a|)^2}.$$

It turns out that

$$\frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} \le c^{4/\delta} \quad |a|\to\infty \quad \text{and} \quad \frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} \le c^{4/\delta} \quad |a|\to 0.$$

We can show that

$$\frac{\|V(a)\|_\delta^{4/\delta}}{(\log|a|)^2} = \left(\frac{\|V(a)\|_\delta}{\left|(\log|a|)^{\delta/2}\right|}\right)^{4/\delta}.$$

Therefore,

$$\frac{\|V(a)\|_\delta}{\left|(\log|a|)^{\delta/2}\right|} \le c \quad |a|\to\infty \quad \text{and} \quad \frac{\|V(a)\|_\delta}{\left|(\log|a|)^{\delta/2}\right|} \le c \quad |a|\to 0.$$

We have a following decomposition:

$$C_K \cong C_{K,1}\times N.$$

Here $C_{K,1}$ is the maximal compact subgroup: $\{\, g\in C_K \mid |g| = 1 \,\}$ and $N = \mathbb{R}^*_{>0}$. Let χ_0 be a character of $C_{K,1}$. We use $\tilde{\chi}_0$ to denote an extension of χ_0 as a character of C_K. Namely, $\tilde{\chi}_0(g) = \chi_0(g);\ \forall g\in C_{K,1}$. Here $\tilde{\chi}_0$ has the form $\tilde{\chi}_0 = \chi_0|\cdot|^\rho, \rho\in i\mathbb{R}$. Restrict V to $C_{K,1}$, one decompose $L^2{}_\delta(C_K)$ in the direct sum of the finite dimensional subspaces,

$$L^2{}_{\delta,\chi_0} = \left\{\xi\in L^2{}_\delta(C_K)\,\middle|\, \xi(a^{-1}g) = \chi_0(a)\xi(g) \quad \forall g\in C_K\ \forall a\in C_{K,1}\right\}.$$

The dual space $(L^2{}_\delta(C_K))^*$ of $L^2{}_\delta(C_K)$ can be identified with $L^2{}_{-\delta}(C_K)$. It is also decomposed in the direct sum of the subspaces,

$$L^2{}_{-\delta,\chi_0} = \left\{\xi\in L^2{}_{-\delta}(C_K)\,\middle|\, \xi(ag) = \chi_0(a)\xi(g) \quad \forall g\in C_K\ \forall a\in C_{K,1}\right\}.$$

Here, we use the transposed of V

$$(V^{\tau}(a)\eta)(x) = \eta(ax); \quad \eta(x) \in (L^2{}_{\delta}(C_K))^*.$$

The pairing between $L^2{}_{\delta}(C_K)$ and its dual $(L^2{}_{\delta}(C_K))^* = L^2{}_{-\delta}(C_K)$ is given by

$$\langle f, \eta \rangle = \int_{C_K} f(x)\eta(x)d^*x \,.$$

We can obtain the following exact sequences:

$$0 \to L^2_{\delta}(X)_0 \overset{\mathrm{T}}{\to} L^2_{\delta}(C_K) \to \mathcal{H} \to 0\,.$$

Let

$$\mathrm{Im}(\mathrm{T})^0 = \left\{ \eta \in (L^2{}_{\delta}(C_K))^* \;\middle|\; \langle \mathrm{T}f,\ \eta \rangle = 0, \quad \forall f \in \mathcal{S}(\mathrm{A}_K)_0 \right\}.$$

It holds that

$$\eta(x) \in \mathrm{Im}(\mathrm{T})^0 \iff \int_{C_K} \mathrm{T}f(a)\eta(a)d^*a \;=0,\ \ \forall f \in \mathcal{S}(\mathrm{A}_K)_0\,.$$

For $\eta(x) \in L^2{}_{-\delta,\,\chi_0}$, we may think that it has the form:

$$\eta(x) = \tilde{\chi}_0(x)\Psi(|x|).$$

Now

$$\Psi(|x|) = \int_{-\infty}^{\infty} \hat{\Psi}(\mathrm{t})|x|^{it}\,d\mathrm{t}$$

where $\hat{\Psi}(\mathrm{t}) = \int_{C_K} \Psi(a)|a|^{it}\,d^*a$. Thus,

$$\eta(x) = \int_{-\infty}^{\infty} \eta(x;\,\mathrm{t})d\mathrm{t}\,; \quad \eta(x;\,\mathrm{t}) = \tilde{\chi}_0(x)|x|^{it}\,\hat{\Psi}(\mathrm{t})\,.$$

Then,

$$\begin{aligned}\eta(x) \in \mathrm{Im}(\mathrm{T})^0 &\iff \langle\, \mathrm{T}f,\, \eta \,\rangle = 0\\ &\iff \int_{C_K} \mathrm{T}f(a)\int_{-\infty}^{\infty} \tilde{\chi}_0(a)|a|^{it}\,\hat{\Psi}(\mathrm{t})d\mathrm{t}\,d^*a \;= 0\\ &\iff \int_{-\infty}^{\infty}\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}\,\hat{\Psi}(\mathrm{t})d^*a\,d\mathrm{t} \;= 0,\ \ \forall f \in \mathcal{S}(\mathrm{A}_K)_0\,.\end{aligned}$$

As the consequence of Tate's work,

Lemma 2.1. For $\mathrm{Re}(s)>0$, and any character χ_0 of C_K,

$$\int_{C_K} \mathrm{T}f(a)\chi_0(a)|a|^{s-1/2}\, d^*a \;=\; L(\chi_0, s)D'_s(f), \quad \forall f \in \mathcal{S}(\mathrm{A}_K)_0.$$

Here, $D'_s(f)$ is a holomorphic function of s ($\mathrm{Re}(s)>0$).

From this lemma, we can say that

$$\eta(x) \in \mathrm{Im}(\mathrm{T})^0 \iff L(\tilde{\chi}_0, 1/2+\rho) = 0; \;\; \rho \in i\mathbb{R}.$$

Here $\mathcal{H} \cong L^2{}_\delta(C_K)/\mathrm{Im}(\mathrm{T})$. Think of the left regular representation W of C_K on $\mathcal{H}$: $(W, \mathcal{H})$, where one deduces W from V. Restrict W to $C_{K,1}$, one decompose $\mathcal{H}$ in the direct sum of the subspaces,

$$\mathcal{H} = \bigoplus_{\chi_0 \in \hat{C}_{K,1}} d(\chi_0)\mathcal{H}_{\chi_0} \qquad d(\chi_0) < \infty$$

where $\mathcal{H}_{\chi_0} = \left\{ \xi \mid W(a)\xi = \chi_0(a)\xi,\ \forall a \in C_{K,1} \right\}$ and we denote the dimension of $\mathcal{H}_{\chi_0}$ by $d(\chi_0)$. We will also consider its dual. We obtain the transposition W^τ of C_K on $\mathcal{H}^*$: $(W^\tau, \mathcal{H}^*)$, where one deduces W^τ from V^τ. Now, let h be a test function on C_K and set

$$W(h) = \int_{C_K} h(g)W(g)\, d^*g\, .$$

Denote h's Fourier transform by $\hat{h}$:

$$\hat{h}(\chi, z) = \int_{C_K} h(\mu)\chi(\mu)|\mu|^z\, d^*\mu\, .$$

Recall

$$\mathcal{H}^* \cong (L^2{}_\delta(C_K)/\mathrm{Im}(\mathrm{T}))^* \cong \mathrm{Im}(\mathrm{T})^0,$$

moreover

$$\mathrm{tr}W = \mathrm{tr}W^\tau.$$

We can compute

$$\begin{aligned}\int_{C_K} h(g)(V^\tau(g)\eta)(x)\, d^*g \;&=\; \int_{C_K} h(g)(V^\tau(g)\int_{-\infty}^{\infty} \eta(\;;\mathrm{t})d\mathrm{t})(x)\, d^*g \\ &=\; \int_{C_K}\int_{-\infty}^{\infty} h(g)\tilde{\chi}_0(g)|g|^{it}\, \tilde{\chi}_0(x)|x|^{it}\, \hat{\Psi}(\mathrm{t})dt\, d^*g\end{aligned}$$

$$= \int_{-\infty}^{\infty} \hat{h}(\tilde{\chi}_0, it)\tilde{\chi}_0(x)|x|^{it}\hat{\Psi}(t)dt,$$

thus

$$\langle \mathrm{T}f, (V^{\tau}(h)\eta)(x) \rangle = \int_{C_K} \mathrm{T}f(a)\int_{-\infty}^{\infty} \hat{h}(\tilde{\chi}_0, it)\tilde{\chi}_0(a)|a|^{it}\hat{\Psi}(t)dt\, d^*a$$
$$= \int_{-\infty}^{\infty}\int_{C_K} \mathrm{T}f(a)\tilde{\chi}_0(a)|a|^{it}\hat{h}(\tilde{\chi}_0, it)\hat{\Psi}(t)d^*a\, dt.$$

If $\eta(x) \in \mathrm{Im(T)}^0$ then $\langle \mathrm{T}f, (V^{\tau}(h)\eta)(x) \rangle = 0, \forall f \in \mathcal{S}(\mathrm{A}_K)_0$. Therefore, $(V^{\tau}(h)\eta)(x) \in \mathrm{Im(T)}^0$. The above computation shows that

$$\hat{h}(\tilde{\chi}_0, \rho)L(\tilde{\chi}_0, 1/2 + \rho) = 0; \ \rho \in i\mathbb{R}.$$

So, we see that

$$\mathrm{tr}W(h) = \sum_{\substack{L(\tilde{\chi}_0,\, 1/2+\rho) = 0 \\ \rho \in i\mathbb{R}}} \hat{h}(\tilde{\chi}_0, \rho).$$

Let $\chi_0 \in \hat{C}_{K,1}$. Recall $\tilde{\chi}_0 = \chi_0|\cdot|^{\rho}$ $(\rho \in \mathbb{C})$. The action of C_K on $\mathcal{H}_{\chi_0}$ can be $W(g)\xi = \tilde{\chi}_0(g)\xi$, and it turns out that $W(g)\xi = |g|^{\rho}\xi$; $g \in N$. So it is satisfied that

$$|g|^{\mathrm{Re}(\rho)} \leq \|W(g)\|_{\delta}, \quad g \in C_K.$$

Let $W_{\chi_0} = W|_{\mathcal{H}_{\chi_0}}$ and $e^{\mathrm{t}} = g$ $(g \in N)$. We will rewrite the action of N on $\mathcal{H}_{\chi_0}$ as

$$W_{\chi_0}(e^{\mathrm{t}}): \ \mathbb{R} \longrightarrow \ \mathcal{H}_{\chi_0}.$$

The following things

(a) $W_{\chi_0}(e^0) = 1$,

(b) $W_{\chi_0}(e^{\mathrm{t+s}}) = W_{\chi_0}(e^{\mathrm{t}})W_{\chi_0}(e^{\mathrm{s}})$

are satisfied. Thus $W_{\chi_0}(e^{\mathrm{t}})$ is a semi-group. From the theory of semi-group, we can say that

$$W_{\chi_0}(e^{\mathrm{t}}) = e^{\mathrm{t}D_{\chi_0}}$$

where

$$D_{\chi_0}\xi = \lim_{\mathrm{t}\to 0^+} \frac{W_{\chi_0}(e^{\mathrm{t}})\xi - W_{\chi_0}(e^0)\xi}{\mathrm{t}} \qquad \xi \in \mathcal{H}_{\chi_0}.$$

The operator D_{χ_0} has discrete spectra. We may think that the discrete spectrum is given by the element ξ which belongs to $\mathrm{Im(T)}^0$.

Let $\tilde{\chi}_0$ be the unique extension of $\chi_0 \in \hat{C}_{K,1}$ to C_K which is equal to 1 on N. We see that $\chi = \tilde{\chi}_0 |\cdot|^{it_0}$ $(t_0 \in \mathbb{R})$ for $\chi \in \hat{C}_K$. Then $L(\chi, 1/2+it) = L(\tilde{\chi}_0, 1/2+i(t_0+t))$. Thus, as the extension of χ_0, we will use the above unique extension $\tilde{\chi}_0$.

Theorem 2.1. $\chi_0 \in \hat{C}_{K,1}$, $\delta > 1$. Then D_{χ_0} has discrete spectra, $\mathrm{sp}D_{\chi_0} \subset i\mathbb{R}$ is the set of imaginary parts of zeros of the L function with Grossencharacter $\tilde{\chi}_0$ which have real part equal to $1/2$;

$\rho \in \mathrm{sp}D \iff L(\tilde{\chi}_0, 1/2+\rho) = 0$ and $\rho \in i\mathbb{R}$, where $\tilde{\chi}_0$ is the unique extension of χ_0 to C_K which is equal to 1 on N.

Moreover the multiplicity of ρ in $\mathrm{sp}D$ is equal to the largest integer of $n < \frac{1+\delta}{2}$, $n \leq$ *multiplicity of* $1/2+\rho$ as a zero of L.

The action of N is that

$$W_{\chi_0}(e^{t})\xi = |e^{t}|^{\rho}\xi = e^{\rho t}\xi.$$

Then,

$$D_{\chi_0}\xi = \left.\frac{dW_{\chi_0}(e^{t})\xi}{dt}\right|_{t=0} = \rho\xi.$$

Therefore, ρ is the spectrum of D_{χ_0}. Consider

$$\lim_{|g|\to\infty}\frac{|g|^{\alpha}}{\log|g|} = \infty \quad (\alpha > 0) \text{ and } \lim_{|g|\to 0}\frac{|g|^{\alpha}}{\log|g|} = \infty \quad (\alpha < 0).$$

Because $|g|^{\mathrm{Re}(\rho)} \leq \|W(g)\|_{\delta}$; $g \in C_K$, if $\mathrm{Re}(\rho) > 0$ or $\mathrm{Re}(\rho) < 0$ then each of them conflicts with

$$\|V(a)\|_{\delta} = O((\log|a|)^{\delta/2}) \qquad |a| \longrightarrow \infty$$

or

$$\|V(a)\|_{\delta} = O((\log|a|)^{\delta/2}) \qquad |a| \longrightarrow 0$$

Therefore, it is that $\rho = it$ $(t \in \mathbb{R})$. Thus,

$$\tilde{\chi}_0 = \chi_0 |\cdot|^{it} \qquad t \in \mathbb{R}.$$

We see that D_{χ_0} has a purely imaginary spectrum, so we obtain the following corollary.

Corollary 2.2. For any Shwarzt function $h \in \mathcal{S}(C_K)$ the operator $\int_{C_K} h(g)W(g)\, d^*g$ in $\mathcal{H}$ is of trace class, and its trace is given by

$$\operatorname{tr} W(h) = \sum_{\substack{L(\tilde{\chi}_0,\, 1/2+\rho)=0 \\ \rho \in i\mathbb{R}}} \hat{h}(\tilde{\chi}_0, \rho)$$

where the multiplicity is counted as in **Theorem 2.1.** and where Fourier transform $\hat{h}$ of h is defined by $\hat{h}(\chi, z) = \int_{C_K} h(\mu)\chi(\mu)|\mu|^z\, d^*\mu$.

We can obtain the following exact sequences:

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0$$

and

$$0 \to L^2_\delta(X)_0 \xrightarrow{\mathrm{T}} L^2_\delta(C_K) \to \mathcal{H} \to 0\,.$$

We will compute $\operatorname{tr} U(h)$ for $(U, L^2{}_\delta(X))$ from spectral side. From the above first sequence, considering *Lefchetz formula*, we will see that

$$\mathrm{A} = \operatorname{tr} U|_{L^2{}_\delta(X)_0} - \operatorname{tr} U|_{L^2{}_\delta(X)} + \operatorname{tr} U|_{\mathbb{C}\oplus\mathbb{C}(1)}.$$

From the second sequence, we will obtain

$$\mathrm{A}' = \operatorname{tr} U|_{L^2{}_\delta(X)_0} - \operatorname{tr} U|_{L^2{}_\delta(C_K)} + \operatorname{tr} U|_{\mathcal{H}}.$$

Therefore, it is satisfied that

$$\operatorname{tr} U|_{L^2{}_\delta(X)} = \operatorname{tr} U|_{\mathbb{C}\oplus\mathbb{C}(1)} - \operatorname{tr} U|_{\mathcal{H}} + \operatorname{tr} U|_{L^2{}_\delta(C_K)} + \mathrm{A}' - \mathrm{A}.$$

We try to compute $\operatorname{tr} U(h)$ spectrally. Here,

$$U(h) = \int_{C_K} h(g)U(g)\, d^*g\,.$$

The first term $\operatorname{tr} U|_{\mathbb{C}\oplus\mathbb{C}(1)}$ gives

$$\hat{h}(0) + \hat{h}(1)\,.$$

Considering that $\mathrm{T}(U(g)\xi)(a) = |g|^{1/2}(V(g)\mathrm{T}\xi)(a)$,

$$U|_{L^2{}_\delta(C_K)} \text{ is } (|\cdot|^{1/2}V,\ L^2{}_\delta(C_K)) \text{ and } U|_{\mathcal{H}} \text{ is } (|\cdot|^{1/2}V,\ \mathrm{Im(T)}^0).$$

So we will understand that the second term gives

$$\sum_{\substack{L(\tilde{\chi}_0,\rho)=0 \\ \mathrm{Re}\,\rho = 1/2}} \hat{h}(\tilde{\chi}_0,\rho)\,.$$

Finally, the term $\mathrm{tr}U|_{L^2{}_\delta(C_K)} + \mathrm{A}' - \mathrm{A}$ gives $\infty h(1)$. Thus,

$$\mathrm{tr}U(h) = \hat{h}(0) + \hat{h}(1) - \sum_{\substack{L(\tilde{\chi}_0,\rho)=0 \\ \mathrm{Re}\,\rho = 1/2}} \hat{h}(\tilde{\chi}_0,\rho) + \infty h(1)\,.$$

3.

We try to compute $\mathrm{tr}U$ geometrically.

Let S be a finite set of places of K containing all infinite places. Set

$$\mathrm{A_S} = \prod_{v\in \mathrm{S}} K_v \times \prod_{v\notin \mathrm{S}} R_v \quad \text{and} \quad \mathrm{J_S} = \prod_{v\in \mathrm{S}} K_v^* \times \prod_{v\notin \mathrm{S}} R_v^*$$

where R_v is the ring of integers of K_v. The S-units of K is given by

$$\mathcal{O}^*_{\mathrm{S}} = \mathrm{J_S} \cap K^*.$$

The idele class C_K is embedded in $C_{\mathrm{S}} = \mathrm{J_S}/\mathcal{O}^*_{\mathrm{S}}$ and $X_{\mathrm{S}} = \mathrm{A_S}/\mathcal{O}^*_{\mathrm{S}}$ plays the same roll as X. We will think of $L^2(X_{\mathrm{S}})$ which is obtained by a completion of $\mathcal{S}(\mathrm{A_S})$. Let

$$R_\Lambda = \hat{P}_\Lambda P_\Lambda, \qquad \Lambda \in \mathbb{R}_+ .$$

Here P_Λ is the orthogonal projection onto the subspace,

$$P_\Lambda = \left\{ \xi \in L^2(X_{\mathrm{S}}) \,\middle|\, \xi(x) = 0, \ \forall x, \ |x| > \Lambda \right\}$$

while $\hat{P}_\Lambda = F P_\Lambda F^{-1}$ where F is the Fourier transform.

Theorem 3.1. For any $h \in \mathcal{S}_c(C_{\mathrm{S}})$, one has

$$Trace(R_\Lambda U(h)) \ = \ 2\log'(\Lambda) h(1) \ + \ \sum_{v\in \mathrm{S}} \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu \ + \ o(1) \qquad \Lambda \to \infty$$

where $2\log'(\Lambda) = \int_{\lambda\in C_{\mathrm{S}},\, |\lambda| \in [\Lambda^{-1},\, \Lambda]} d^*\lambda$.

Let χ_0 be a character of $C_{\mathrm{S},1}$ which is the subgroup: $\{ g \in C_{\mathrm{S}} \mid |g| = 1 \}$. The Hilbert space $L^2(X_{\mathrm{S}})$ is decomposed in the subspace,

$$L^2_{\chi_0} = \left\{ \xi \in L^2(X_{\mathrm{S}}) \,\middle|\, \xi(a^{-1}x) = \chi_0(a)\xi(x) \quad \forall x \in X_{\mathrm{S}},\, a \in C_{\mathrm{S},1} \right\}.$$

Let $\mathcal{U}_{\mathrm{S}}$ be the image in C_{S} of the open subgroup $\prod R_v^*$. Fix a character χ of $\mathcal{U}_{\mathrm{S}}$, and think of χ_0 whose restriction to $\mathcal{U}_{\mathrm{S}}$ is equal to χ. Set

$$L^2(X_S)_\chi = \left\{ \xi \in L^2(X_S) \,\middle|\, \xi(a^{-1}x) = \chi(a)\xi(x) \quad \forall x \in X_S,\ a \in \mathcal{U}_S \right\}.$$

We can find $h_\chi \in \mathcal{S}(C_S)$ such that

$$\mathrm{Supp}(h_\chi) = \mathcal{U}_S \qquad h_\chi(x) = \lambda\,\overline{\chi}(x) \quad \forall x \in \mathcal{U}_S$$

where the constant λ is determined by corresponding normalization of the Haar measure on C_S.

Let $B_\Lambda = \mathrm{Im}(P_\Lambda) \cap \mathrm{Im}(\hat{P}_\Lambda)$ be the intersection of the ranges of the projection P_Λ and $\hat{P}_\Lambda$. We will think of $B_\Lambda{}^\chi$ which is the intersection of B_Λ with $L^2(X_S)_\chi$. For each character χ of $\mathcal{U}_S$, we can find a vector $\eta_\chi \in L^2(X_S)_\chi$ such that

$$U(g)(\eta_\chi) \in B_\Lambda \qquad g \in C_S,\ \Lambda^{-1} \le |g| \le \Lambda.$$

Then $B_\Lambda{}^\chi$ is given as the linear span of $U(g)(\eta_\chi)$:

$$B_\Lambda{}^\chi = \sum_{g \in D_S\ |g| \in [\Lambda^{-1},\ \Lambda]} \lambda_g U(g)(\eta_\chi) \qquad D_S = C_S/\mathcal{U}_S.$$

Set

$$(B_\Lambda{}^\chi)^0 = \textit{The whole of} \sum_{g \in D_S\ |g| \in [\Lambda^{-1},\ \Lambda]:\ \text{finite sum}} \lambda_g U(g)(\eta_\chi).$$

It turns out that $(B_\Lambda{}^\chi)^0 \subseteq B_\Lambda{}^\chi \subseteq L^2(X_S)_\chi$. We may say that $(B_\Lambda{}^\chi)^0$ is dense in $B_\Lambda{}^\chi$. So, from the compactness of $\left\{ g \in C_S \,\middle|\, \Lambda^{-1} \le |g| \le \Lambda \right\}$, we can consider that $B_\Lambda{}^\chi$ is a vector space which has a countable basis at most. It must be hard to show that $B_\Lambda{}^\chi = L^2(X_S)_\chi$ for sufficient large Λ. We will replace R_Λ by the orthogonal projection Q_Λ on $\mathrm{Im}(P_\Lambda) \cap \mathrm{Im}(\hat{P}_\Lambda)$. Suppose that $B_\Lambda{}^\chi = L^2(X_S)_\chi$ for sufficient large Λ. Then we can identify $\mathrm{tr} R_\Lambda U$ with $\mathrm{tr}\mathrm{Q}_\Lambda U$ of $(U,\ L^2(X_S))$. From the **Theorem 3.1.**, we can show the following.

Corollary. Let Q_Λ be the orthogonal projection on the subspace of $L^2(X_S)$ spanned by the $f \in \mathcal{S}(\mathrm{A}_S)$, which vanish as well as Fourier transform for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_S)$ have compact support. Then when $\Lambda \longrightarrow \infty$, one has

$$\mathit{Trace}(\mathrm{Q}_\Lambda U(h)) = 2h(1)\log'(\Lambda) + \sum_{v \in S} \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu + o(1)$$

where $2\log'(\Lambda) = \int_{\lambda \in C_S,\ |\lambda| \in [\Lambda^{-1},\ \Lambda]} d^*\lambda$.

We can get from the above corollary an $\mathbb{S}$-independent global formulation:

$$Trace\,\mathrm{Q}_{\Lambda}U(h) = 2h(1)\log'\Lambda + \sum_{v}\int'_{K_v^*}\frac{h(\mu^{-1})}{|1-\mu|}d^*\mu + o(1) \qquad \Lambda\to\infty$$

where $\mathrm{Q}_{\Lambda}U$ is a trace class operator for $(U, L^2(X))$.

In order to obtain the identification of $\mathrm{tr}\mathrm{Q}_{\Lambda}U$ with $\mathrm{tr}R_{\Lambda}U$, we have to show that $B_{\Lambda}{}^{\chi} = L^2(X_{\mathrm{S}})_{\chi}$ for sufficient large Λ. If C_{S} is compact then the compactness must be sufficient for us to show the equation. If C_K were compact, we could show the validity of the Riemann hypothesis.

So it must be interesting to think of the compactification of C_K. With this interest, we will examine the space $Y = \mathrm{A}_K/K$. As the same way in the case of X, we can obtain $L^2(Y)$ and $L^2(Y)_0$. We will think of the case $K = \mathbb{Q}$. It holds that

$$\mathrm{A}_{\mathbb{Q}} = \prod_{p<\infty}\mathbb{Z}_p\times[0,\ 1) + \mathbb{Q} \text{ and } \mathrm{A}_{\mathbb{Q}}{}^* = (\prod_{p<\infty}\mathbb{Z}_p{}^*\times\mathbb{R}^*{}_{>0})\cdot\mathbb{Q}^*.$$

Thus, it turns out that

$$Y = \mathrm{A}_{\mathbb{Q}}/\mathbb{Q} \cong \prod_{p<\infty}\mathbb{Z}_p\times[0,\ 1] \text{ and } C_{\mathbb{Q}} = \mathrm{A}_{\mathbb{Q}}{}^*/\mathbb{Q}^* \cong \prod_{p<\infty}\mathbb{Z}_p{}^*\times\mathbb{R}^*{}_{>0}.$$

Think of $r \mapsto 2/\pi\tan(r)^{-1}$; $r\in\mathbb{R}^*{}_{>0}$, it must be allowed to say that

$$\mathbb{R}^*{}_{>0} \textit{ is embedded in } [0,\ 1].$$

Thus,

$$C_{\mathbb{Q}} \textit{ is embedded in } \prod_{p<\infty}\mathbb{Z}_p{}^*\times[0,\ 1].$$

It may be allowed to say that

$$Y = \left\{x\in Y \,\middle|\, |x|<1\right\} \cup \left\{x\in Y \,\middle|\, |x|=1\right\}$$

and that $\left\{x\in Y \,\middle|\, |x|=1\right\}$ consists of the boundary of Y. Denote it by ∂Y. It must correspond to $\prod_{p<\infty}\mathbb{Z}_p{}^*\times\{1\}$. Let

$$\mathcal{C}_{\mathbb{Q}} = \prod_{p<\infty}\mathbb{Z}_p{}^*\times(0,\ 1].$$

We will think that $\mathcal{C}_{\mathbb{Q}}$ is the compactification of $C_{\mathbb{Q}}$. We expect that $\mathcal{C}_{\mathbb{Q}}$ fills the same role of $C_{\mathbb{Q}}$.

We can obtain an exact sequence:

$$0 \to L^2(Y)_0 \xrightarrow{\mathrm{T}} L^2(\mathcal{C}_{\mathrm{Q}}) \to \mathcal{H} \to 0$$

where $\mathcal{H} \cong L^2(\mathcal{C}_{\mathrm{Q}})/\mathrm{Im(T)}$. Let U be a left regular representation of $\mathcal{C}_{\mathrm{Q}}$ on $L^2(Y, dx)$ and V be a left regular representation of $\mathcal{C}_{\mathrm{Q}}$ on $L^2(\mathcal{C}_{\mathrm{Q}}, d^*x)$. One deduces the left regular representation W of $\mathcal{C}_{\mathrm{Q}}$ on $\mathcal{H}$ from V. One may be allowed to say that $\mathcal{C}_{\mathrm{Q}}$ is compact because it must be complete and totally bounded. So one can decompose $L^2(\mathcal{C}_{\mathrm{Q}})$ in the direct sum of 1–dimensional subspaces,

$$L^2{}_{\chi_0} = \left\{ \xi \in L^2(\mathcal{C}_{\mathrm{Q}}) \,\middle|\, \xi(a^{-1}g) = \chi_0(a)\xi(g) \quad \forall g, a \in \mathcal{C}_{\mathrm{Q}} \right\}.$$

The dual space $(L^2(\mathcal{C}_{\mathrm{Q}}))^*$ of $L^2(\mathcal{C}_{\mathrm{Q}})$ can be identified with $L^2(\mathcal{C}_{\mathrm{Q}})$.

[Remark] The left regular representation U of $\mathcal{C}_{\mathrm{Q}}$ on $L^2(Y, dx)$ isn't unitary. But the left regular representation T of Y on $L^2(Y, dx)$:

$$(T(g)\xi)(x) = \xi(-g+x) \qquad g, x \in Y$$

is unitary. Because Y is abelian and compact, we obtain the following decomposition:

$$L^2(Y) = \bigoplus_{\chi \in \hat{Y}} L^2_\chi(Y) \qquad\qquad T_\chi = T|_{L^2_\chi(Y)}$$

where T_χ is 1-dimensional representation.

Here Y is compact. Thus the following formula:

$$\mathrm{tr}U|_{L^2(Y)_0} = \mathrm{tr}U|_{L^2(\mathcal{C}_{\mathrm{Q}})} - \mathrm{tr}U|_{\mathcal{H}} + \mathrm{A}$$

becomes meaningful.

Now, our problem is to compute $\mathrm{tr}U|_{L^2(Y)_0}$. Basically, we may think that this problem is how to construct $L^2(Y)_0$. Set

$$\Delta = |x|^2 \frac{d^2}{dx^2}$$

which is a differential operator on Y. We shall think of the eigenvalue problems:

$$\Delta\xi - \lambda\xi = 0, \quad \xi(x) = 0 \text{ on } \partial Y$$

on the analogy of Sturm-Liouville problem. Recall that the action of $\mathcal{C}_{\mathrm{Q}}$ on the functions on Y is

$$(U(g)\phi)(x) = \phi(g^{-1}x) \quad \forall g \in \mathcal{C}_{\mathbb{Q}},\, x \in Y.$$

It turns out that $U(g)$ and Δ are commutative. Hence they shares the same eigen-space. We try to construct the $L^2(Y)_0$ space as the space of eigenfunctions of Δ.

[Remark] One computes

$$(U(g)|x|^2 \frac{d^2}{dx^2}\phi)(x) = (U(g)\,|\cdot|^2\,\phi''\,)(x) = \left|g^{-1}x\right|^2 \phi''\,(g^{-1}x).$$

It holds that $dgx = |g|dx$, so

$$|x|^2 \frac{d^2}{dx^2}(U(g)\phi)(x) = |x|^2 \frac{d^2}{dx^2}\phi(g^{-1}x) = |x|^2 \frac{d^2 g^{-1}x}{dx^2}\,\frac{d^2}{d\left(g^{-1}x\right)^2}\,\phi(g^{-1}x)$$
$$= \left|g^{-1}x\right|^2 \phi''\,(g^{-1}x).$$

[Remark] The Δ becomes a differential operator on X. Since $dg^{-1}x = |g^{-1}|dx$, if one restricts U to $C_{K,1}$ then $\left|d(g^{-1}x)\right| = |dx|$. Namely, $|dx|$ is invariant under the action $U(g);\ \forall g \in C_{K,1}$. Thus, the Laplacian $\frac{d^2}{dx^2}$ is $C_{K,1}-$ invariant.

If we can show that the Laplacian $\frac{d^2}{dx^2}$ is C_K- invariant then we can say that $U(g);\ \forall g \in C_K$ is isometry, namely unitary. On the other hand it does not always mean that $U(g);\forall g \in C_K$ is unitary.

Considering $\xi'' = \lambda \frac{\xi}{|x|^2}$, it turns out that

$$\left(\xi'\,\overline{\xi}\right)' = \xi''\,\overline{\xi} + \xi'\,\overline{\xi}' = \lambda\frac{\xi\,\overline{\xi}}{|x|^2} + \xi'\,\overline{\xi}'.$$

One compute

$$\int_Y \xi'\overline{\xi}'\,dx \;+\; \lambda\int_Y \frac{\xi\,\overline{\xi}}{|x|^2}dx \;=\; \int_Y \left(\xi'\overline{\xi}\right)' dx.$$

We can write that $\int_Y \left(\xi'\overline{\xi}\right)' dx = \int_{\partial Y} \xi'\overline{\xi}\,dx$. From the boundary condition, it holds that

$$\int_{\partial Y} \xi'\overline{\xi}\,dx = 0.$$

Therefore, we obtain

$$\int_Y \xi'\overline{\xi}'\,dx = -\lambda\int_Y \frac{\xi\,\overline{\xi}}{|x|^2}dx.$$

Here $\int_Y \xi' \overline{\xi}' dx$, $\int_Y \frac{\xi \overline{\xi}}{|x|^2} dx \geq 0$. Thus, $\lambda \leq 0$. Write a function $\xi(x)$ on Y

$$\xi(x) = \xi(u\mathrm{t}) \qquad u \in \prod_{p<\infty} \mathbb{Z}_p \text{ , t} \in [0,\ 1].$$

We will compute as follows.

(a)
$$\frac{\partial}{\partial u} x = \mathrm{t}.$$

(b)
$$\frac{\partial}{\partial u} \xi(x) = \frac{\partial x}{\partial u} \frac{\partial}{\partial x} \xi(x) = \mathrm{t}\, \xi'(x).$$

[Remark] From the definition, the following things are satisfied.

(a)
$$dx = du\, d\mathrm{t}, \text{ so } \frac{dx}{du} = d\mathrm{t}.$$

(b)
$$\frac{d}{du} \xi(x) = \frac{dx}{du} \frac{d}{dx} \xi(x) = \xi'(x) d\mathrm{t}.$$

It holds that $\frac{\partial^2}{\partial u^2} = \mathrm{t}^2 \frac{d^2}{dx^2}$, so we see that $|u|^2 \frac{\partial^2}{\partial u^2} = |u|^2 \mathrm{t}^2 \frac{d^2}{dx^2} = |x|^2 \frac{d^2}{dx^2}$. Thus, we can identify Δ with $|u|^2 \frac{\partial^2}{\partial u^2}$. We will think of the eigenvalue problems:

$$|u|^2 \frac{\partial^2}{\partial u^2} \xi(x) - \lambda \xi(x) = 0.$$

Let

$$\eta(u) = \begin{cases} \xi(u0) & \cdots \quad u \in \prod_{p<\infty} \mathbb{Z}_p^{\ *} \\ \xi(u1) & \cdots \quad \text{otherweise} \end{cases} .$$

Then we can interpret the eigenvalue problems as the following problem;

$$|u|^2 \frac{\partial^2}{\partial u^2} \eta(u) - \lambda \eta(u) = 0, \quad \eta(u) = 0 \text{ on } \prod_{p<\infty} \mathbb{Z}_p^{\ *}.$$

Here we will identify $\prod_{p<\infty} \mathbb{Z}_p^{\ *}$ with ∂Y.

[Remark] Here,

$$|u|^2\frac{\partial^2}{\partial u^2}\xi(x) = |u|^2|\mathrm{t}|^2\xi''(u\mathrm{t}).$$

From the above definition of $\eta(u)$, let

$$\eta''(u) = |0|^2\xi''(u0) \quad \forall u\in \prod_{p<\infty}\mathbb{Z}_p^* \quad \text{and} \quad \eta''(u) = |1|^2\xi''(u1) \quad \forall u\notin \prod_{p<\infty}\mathbb{Z}_p^* .$$

Then we can say that $|u|^2\frac{\partial^2}{\partial u^2}\xi(x)$ gives $|u|^2\frac{\partial^2}{\partial u^2}\eta(u)$.

We can show that $\lambda \leq 0$. Here, think of *the heat equation*:

$$\begin{cases} \dfrac{\partial}{\partial \mathrm{t}}\xi(u\mathrm{t}) = |u|^2\dfrac{\partial^2}{\partial u^2}\xi(u\mathrm{t}) \\ \eta(u) = \xi(u0) = 0 \quad \text{on } \prod_{p<\infty} Z_p^* \end{cases} .$$

Let $\eta_\lambda(u)$ be an eigenfunction of $|u|^2\frac{\partial^2}{\partial u^2}$ with eigenvalue λ. Then, $e^{-\lambda \mathrm{t}}\eta_\lambda(u)$ is a particular solution. We obtain general solutions

$$\xi(x) = \sum_\lambda c_\lambda \cdot e^{-\lambda \mathrm{t}}\eta_\lambda(u).$$

Here, c_λ is the constant. There exists some function, called *heat kernel*, $p(\mathrm{t}, \mu, \nu)$ on Y^2 and we can say that

$$\xi(u\mathrm{t}) = \int_Y p(\mathrm{t},u,v)\eta(v)dv .$$

From the theory of semi-group, it holds that

$$\sum_\lambda e^{-\mathrm{t}\lambda} = \int_Y p(\mathrm{t},u,u)du .$$

We can say that such a function $\eta(u)$ associated with $|x|^s$

$$|\cdot|^s(u) = \begin{cases} |u0|^s \ \dots \ u\in \prod_{p<\infty}\mathbb{Z}_p^* \\ |u1|^s \ \dots \ \text{oherweise} \end{cases}$$

is an eigenfunction of Δ with eigenvalue $\lambda = s(s-1) \leq 0$. It must be allowed that

$L^2(Y)$ is decomposed in the subspace $\{\mathrm{c}|\cdot|^s \mid \mathrm{c}\in\mathbb{C}\}$.

Here $|0|^s = 0$, so we can say that $|\cdot|^s \in L^2(Y)_0$. Moreover, since $L^2(Y)$ is a Hilbert space, $\{|\cdot|^s\}$ is discrete. Now

$$(U(g)|\cdot|^s)(x) = |g^{-1}x|^s = |g^{-1}|^s\,|x|^s \qquad \forall g \in \mathcal{C}_{\mathrm{Q}},\ x \in Y.$$

We shall think that $|g^{-1}|^s = |g|^{-s}$ is extended as a quasi-character of $\mathcal{C}_{\mathrm{Q}}$. We may be allowed to think that the quasi-character $|g|^{-s}$ is equivalent to a quasi-character $|g|^s$. We may say that

$$\mathrm{tr}U|_{L^2(Y)_0} \text{ extends over } \{\,|a|^s \mid s(s-1) = \lambda\}\,.$$

Moreover,

$$\mathrm{tr}U|_{L^2(\mathcal{C}_{\mathrm{Q}})} \text{ extends over } \{\chi(a)|a|^{1/2} \mid \chi \text{ is a character of } \mathcal{C}_{\mathrm{Q}}\}$$

and

$\mathrm{tr}U|_{\mathcal{H}}$ extends over

$\{\pi(a)|a|^{1/2} \mid \pi$ is the character of $\mathcal{C}_{\mathrm{Q}}$ which is given by $\eta(x) \in \mathrm{Im}(\mathrm{T})^0\}$.

Recall $\mathrm{T}(U(g)\xi)(a) = |g|^{1/2}(V(g)\mathrm{T}\xi)(a)$,

$$\begin{aligned}\mathrm{T}(U(g)|\cdot|^s)(a) &= |g|^{1/2}(V(g)\mathrm{T}|\cdot|^s)(a)\\ &= |g|^{1/2}(V(g)|\cdot|^{1/2+s})(a)\\ &= |g|^{-s}|a|^{1/2+s}\end{aligned}$$

$|g|^{-s}$ being equivalent to $|g|^s$

$$= |g|^s|a|^{1/2+s}.$$

Thus we see that $\mathrm{tr}U|_{L^2(\mathcal{C}_{\mathrm{Q}})}$ contains $\mathrm{tr}U|_{L^2(Y)_0}$ and it also expands over such a quasi-character as $|a|^s$.

The compactness of Y guarantees to compute

$$\mathrm{tr}\,U(h) = \sum_p \int'_{\mathrm{Q}_p^*} \frac{h(\mu^{-1})}{|1-\mu|}\, d^*\mu\,.$$

Thus we can say that

$$\sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho) = \sum_{\substack{L(\chi,\rho)=0\\ \mathrm{Re}\,\rho = 1/2}} \hat{h}(\chi,\rho)\,.$$

[Remark] In order to obtain an expected formula like **Theorem 3.1.**, we need the evaluation of a certain *error term*. We shall compare $(U, L^2(Y))$ with $(U, L^2{}_\delta(X))$. For the latter, U is also trace-class so that we formally get $\mathrm{tr}U(h) = \sum_v \int'_{K_v^*} \frac{h(\mu^{-1})}{|1-\mu|} d^*\mu$. However, since $L^2{}_\delta(X)$ is a weighted space L^2, we can't always obtain the expected formula. On the other hand, in the case of $L^2(Y)$, we can expect to obtain the desired formula.

On the other hand, the compactness must also guarantee $\lambda \leq -1/4$. So, we will see a certain relationship between the validity of the Riemann hypothesis and the fact that $\lambda \leq -1/4$. We shall suppose that the compactification of $C_{\mathbb{Q}}$ is equivalent to the fact that $\lambda \leq -1/4$ for the eigenvalue λ of Δ on X. Then, we may say that the validity of the Riemann hypothesis is equivalent to showing $\lambda \leq -1/4$.

4.

We will think of the case $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathrm{A}_\mathbb{Q})$.

Let (π, V) be an irreducible admissible infinite dimensional representation of $\mathrm{GL}_2(\mathrm{A}_\mathbb{Q})$ with central character ω. Here, ω is a quasi-character of $\mathrm{GL}_2(\mathrm{A}_\mathbb{Q})$ defined by

$$\pi\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \omega(a)\,\mathrm{id}V \qquad a\in\mathrm{A}_\mathbb{Q}^*.$$

Suppose that $W(\pi, \psi)$ is the ψ-Whittaker model. Let χ be a character of $\mathrm{A}_\mathbb{Q}^*/\mathbb{Q}^*$. The *Jacquet—Langlands zeta integrals* are defined by

$$Z(s, W, \chi; g) = \int_{\mathrm{A}_\mathbb{Q}^*} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\chi(a)|a|^{s-1/2}\, d^*a$$

and

$$Z^\vee(s, W, \chi; g) = \int_{\mathrm{A}_\mathbb{Q}^*} W\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\chi(a)|a|^{s-1/2}\,\omega^{-1}(a)\, d^*a\,.$$

There exists $s_0\in\mathbb{R}$ such that $Z(s, W, \chi; g)$ and $Z^\vee(s, W, \chi; g)$ absolutely converge whenever $\mathrm{Re}(s)>s_0$ for all $g\in\mathrm{GL}_2(\mathrm{A}_\mathbb{Q})$ and $W\in W(\pi, \psi)$. There exists a unique L-function $L(s, \pi\otimes\chi)$ ($(\pi\otimes\chi)(g) = \pi(g)\chi(\det g)$) such that

$$\phi(s, W, \chi; g) = Z(s, W, \chi; g)/L(s, \pi\otimes\chi)$$

is entire in s for all $g\in\mathrm{GL}_2(\mathrm{A}_\mathbb{Q})$ and $W\in W(\pi, \psi)$. Therefore, we may say

$$Z(s, W, \chi; g) = 0 \Longleftrightarrow L(s, \pi\otimes\chi) = 0.$$

Moreover, we will see that

$$Z(s, W, \chi; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = 0 \Longleftrightarrow L(s, \pi\otimes\chi) = 0.$$

It must be instructive to compare the *Jacquet—Langlands zeta integral* with the *Tate integral*. The *Tate integral* is defined by

$$Z(s, \chi, \Phi) = \int_{\mathrm{A}_\mathbb{Q}^*} \Phi(a)\chi(a)|a|^s\, d^*a$$

where χ is a character of $\mathrm{A}_\mathbb{Q}^*$ and $\Phi\in\mathcal{S}(\mathrm{A}_\mathbb{Q})$. We will see that $\Phi(x)$ corresponds to $|x|^{-1/2}W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g)$.

[Remark] We may say that $W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \in L^2(\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*, d^*x)$. Thus we see that

$$|x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g) \in L^2(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*, dx).$$

The *Jacquet-Langlands zeta integral* is defined by

$$Z(s, \chi, |x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g)) = \int_{\mathbb{A}_{\mathbb{Q}}^*} |a|^{-1/2} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \chi(a) |a|^s d^*a .$$

Considering $\phi(s, W; g) = Z(s, W; g)/L(s, \pi)$, of which χ is trivial, we will understand that the L-function $L(s, \pi)$ is determined associated with $|x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g)$. In GL_1 case, we may similarly think that there exists a unique L-function $L(s, \pi)$ which is determined associated with a certain $\Phi \in \mathcal{S}(\mathbb{A}_{\mathbb{Q}})$, and that an L-function $L(s, \pi \otimes \chi)$ is given by $Z(s, \chi, \Phi)$.

Set

$$\phi^{\vee}(s, W, \chi^{-1}; g) = Z^{\vee}(s, W, \chi^{-1}; g)/L(s, \pi^{\vee} \otimes \chi^{-1}).$$

Here $\pi^{\vee}$ is the contragredient representation of π and $\pi^{\vee} = \omega(\det)^{-1}\pi$. Then there exists a unique exponential function $\varepsilon(s, \pi, \chi, \psi)$ such that

$$\phi^{\vee}(1-s, W, \chi^{-1}; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g) = \varepsilon(s, \pi, \chi, \psi)\, \phi(s, W, \chi; g).$$

We shall think of the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. We may think that a right regular representation $(V, L^2(\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}), d^*g))$ is given.

Theorem 4.1. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. One obtains $\pi \simeq \otimes_p \pi_p$. We will think of the case where χ is trivial.

(1) The L-function $L(s, \pi)$ has the Euler product:

$$L(s, \pi) = \Pi_p L(s, \pi_p).$$

(2) There exists a exponential function $\varepsilon(s, \pi, \psi)$, and the functional equation:

$$L(s, \pi) = \varepsilon(s, \pi, \psi) L(1-s, \pi^{\vee})$$

is satisfied.

Proposition 4.2. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathrm{A}_{\mathbb{Q}})$. It has its Whittaker model.

Recall the sequence

$$0 \to L^2(X)_0 \xrightarrow{\mathrm{T}} L^2(C_{\mathbb{Q}}) \to \mathcal{H} \to 0 .$$

We may say that

$$|x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g) \in L^2(X, dx)_0.$$

Think of the pairing $\langle \mathrm{T}f, \eta \rangle$ for $f \in L^2(X, dx)_0$ and $\eta \in (L^2(C_{\mathbb{Q}}))^*$. Then,

$$\langle \mathrm{T}f, \eta \rangle = \langle W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g), \eta \rangle = \int_{-\infty}^{\infty} \int_{\mathrm{A}_{\mathbb{Q}}^*} W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \chi(a) |a|^{it} \hat{\Psi}(\mathrm{t}) d^*a \, d\mathrm{t} .$$

This computation makes us to say that

$$\eta \in \mathrm{Im}(\mathrm{T})^0 \Longleftrightarrow Z(1/2+i\mathrm{t}, W, \chi; g) = 0 \Longleftrightarrow L(1/2+i\mathrm{t}, \pi) = 0 \quad \mathrm{t} \in \mathbb{R}.$$

Therefore, also in GL_2 case, we can give the same spectral interpretation of critical zeros of $L(s, \pi)$.

Expect that $\{ |x|^{-1/2} W(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} g) \mid W \in W(\pi, \psi) \}$ are dense in $L^2(X, dx)_0$. Then we may say that $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathrm{A}_{\mathbb{Q}})$ has no complementary series representation, so if $Z(s, W, \chi; g) = 0$ then $s = 1/2+i\mathrm{t}$. This must accomplish the spectral interpretation of critical zeros of $L(\chi, s)$, and we can confirm the Riemann hypothesis.

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