THE KAKEYA TUBE CONJECTURE IMPLIES THE KAKEYA CONJECTURE

J.ASPEGREN

ABSTRACT. In this article we will give a proof that the Kakeya tube conjecture implies the Kakeya conjecture.

1. INTRODUCTION

We define the δ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T^\delta_\omega(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot\omega| \leq \frac{1}{2}, |proj_{\omega^\perp}(x-a)| \leq \delta\}.$$

In this paper any constant can depend on dimension n. We define the (spherical) Hausdorff content $H^s(K)$ of a subset of $K \subset \mathbb{R}^n$ as follows. Let $r > 0$ and let $0 < r_j < r$ then

$$H^s_r(K) = \inf\{\sum_{j=i}^{\infty} r_j^s | K \subset \bigcup_{j=1}^{\infty} B(x_j, r_j/2)\},$$

where each $B(x_j, r/2)$ is a ball with a diameter strictly less than r. The (spherical) $s-$ dimensional Hausdorff content of K is defined as $\lim_{r\to 0} H^s_r(K)$. We define the Hausdorff dimension as

$$Dim_H(K) = \inf\{s \geq 0 | H^s(K) = 0\}.$$

We will give a proof that the result

$$\bigcup_{\omega\in\Omega} T_\omega \approx 1$$

for maximal set of δ - tubes implies the Kakeya conjecture:

Theorem 1. *Any Kakeya set has full Hausdorf dimension.*

2. THE PROOF

Let K be a Kakeya set, that is, a set that contains an unit line in every direction. let $\bigcup_{j=1}^{\infty} B_j(x, \frac{r_j}{2})$ be a cover of K with balls of diameters less than $1 > r > r_j > 0$. Let $n > n - \alpha > 0$ be such that

$$\sum_{j=1}^{\infty} r_j^{n-\alpha} < 1. \tag{1}$$

1991 *Mathematics Subject Classification.* 42B37,28A75.
Key words and phrases. Kakeya conjectures.

If the Hausdorff content is zero that kind of cover exists. By compactness of the Kakeya set we can take a subcover with diameters such that $1 > r > r_j \geq \delta > 0$, where at least one $r_j \sim \delta$. Now, assume

$$\sum_{j=1}^{M} r_j^n \gtrsim |\bigcup_{j=1}^{M} B_j| \gtrsim |\bigcup_{i=1}^{N} T_i| \gtrapprox 1. \tag{2}$$

The second inequality above follows because the balls cover the middle lines of the tubes, so there exists a constant such that the second inequality above is valid. Using inequality (1) and (2) we obtain

$$C_{\alpha/k}\delta^{-\alpha/k} \sum_{j=1}^{M} r_j^n > \sum_{j=1}^{M} r_j^{n-\alpha}. \tag{3}$$

Thus,

$$\sum_{j=1}^{M} r_j^n (C_{\alpha/k}\delta^{-\alpha/k} - r_j^{-\alpha}) > 0. \tag{4}$$

It follows that for the average value of a power of diameters it holds that

$$C_{\alpha/k}\delta^{-\alpha/k} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha} \geq \frac{1}{M^{-\alpha}}(\sum_{j=1}^{M} r_j)^{-\alpha}, \tag{5}$$

where we used Jensen‘s inequality. Thus,

$$c_\alpha \frac{1}{M}\sum_{j=1}^{M} r_j > \delta^{1/k}. \tag{6}$$

From above it follows that

$$\frac{(c_\alpha)^n}{M}(\sum_{j=1}^{M} r_j^n) \geq (\frac{c_\alpha}{M})^n(\sum_{j=1}^{M} r_j)^n > \delta^{n/k},$$

where we used Jensen‘s inequality again. Thus, from above and inequality (1)

$$C_\alpha > M\delta^{n/k}.$$

It follows from above that

$$\delta^{-n/k}C_\alpha > M \tag{7}$$

We can do the steps (3), (4) and (5) again for $\epsilon = \alpha/2$ and obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha}. \tag{8}$$

Let k and a small δ be such that

$$\delta^{-\alpha/3} > C_\alpha \delta^{-n/k}.$$

From above and inequalities (7) and (8) we obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \delta^{\alpha/3}\sum_{j=1}^{M} r_j^{-\alpha} > \delta^{\alpha/3}\delta^{-\alpha} = \delta^{-2/3\alpha}, \tag{9}$$

which is a contradiction when δ is small.

References

[1] J. Bourgain, *Besicovitch Type Maximal Operators and Applications to Fourier Anal- ysis, Geometric and Functional Analysis 1* (1991), 147-187.

[2] J. Bourgain, *Harmonic analysis and combinatorics: How much may they contribute to each other?*,IMU/Amer. Math. Soc. (2000), 13âĂŞ32

[3] A. Còrdoba, *The Kakeya Maximal Function and the Spherical Summation Multipli- ers*, American Journal of Mathematics 99 (1977), 1-22.

[4] R.O. Davies, *Some Remarks on the Kakeya Problem*, Proc. Camb. Phil. Soc. 69 (1971), 417-421.

[5] Z. Dvir, *On the Size of Kakeya Sets in Finite Fields*, J. Amer. Math. Soc. 22 (2009), 1093-1097.

[6] K.J. Falconer *The Geometry of Fractal Sets*, Cambridge University Press 1985.

[7] E.Kroc, *The Kakeya problem*, available at http://ekroc.weebly.com/uploads/2/1/6/3/21633182/mscessay-final.pdf

[8] N.H. Katz, I. Laba and T. Tao, *An improved bound on the Minkowski dimension of Besicovitch sets in R3* , Annals of Math. (2) 152, 2 (2000), 383âĂŞ446.

[9] N.H. Katz, T. Tao, *New bounds for Kakeya problems*, J. Anal. Math. 87 (2002), 231-263.

[10] I. Laba, T. Tao, *An improved bound for the Minkowski dimension of Besicovitch sets in medium dimension*, Geometric and Functional Analysis 11 (2001), 773âĂŞ806.

[11] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press (1993)

[12] T. Tao, *Lecture Notes*, available at math.ucla.edu/ tao/254b.1.99s/ (1999)

[13] T. Tao, *The Bochner-Riesz Conjecture Implies the Restriction Conjecture*, Duke Math. J. 96 (1999), 363-375.

[14] T.Tao, *From rotating needles to stability of waves: emerging connections between com- binatorics, analysis, and pde*,Notices Amer. Math. Soc., 48(3),(2001),294âĂŞ303.

[15] T. Tao *https://terrytao.wordpress.com/2009/05/15/the-two-ends-reduction-for-the-kakeya-maximal-conjecture/*

[16] T. Wolff, *An Improved Bound for Kakeya Type Maximal Functions*, Rev. Mat. Iberoamericana 11 (1995), 651-674.

[17] T. Wolff, *Recent work connected with the Kakeya problem* Prospects in mathematics (Princeton, NJ, 1996), (1999),129âĂŞ162.

E-mail address: jaspegren@outlook.com