Calculation of the Sum for Sequence of Square Natural Numbers

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ABSTRACT. This sum for natural values is, of course, already calculated by Bernoulli himself – at least modern or relatively recent authors that deal with it usually refer to take into account Bernoulli numbers. But, apparently, this method is rather cumbersome. Therefore, there can be suggested another, easier way to do this, but without claiming of its superfluous rigidity.

This sum $\sum_{k=0}^{n} k^2$ for natural values $k, n \in \mathbb{N}$ is, of course, already calculated by Bernoulli himself – at least modern or relatively recent authors that deal with it usually refer to take into account Bernoulli numbers. But, apparently, this method is rather cumbersome. Therefore, we can suggest another, easier way to do this, however, without claiming of its superfluous rigidity.

To estimate approximate form of sought expression, one can note the following. Given the known the fact that the set of natural numbers is the subset of the set of positive real numbers $\mathbb{N} \subset \mathbb{R}_+$, one can presumably assume that some of their properties are similar to each other (while omitting them any differences – to assess this, apparently, it would be enough). Instead of the initial sum, writing easily taking integral $\int_0^x \theta^2\, d\theta = x^3/3$, we can conclude that the desired sum may be represented by a product of three natural numbers as multiplicands, which is divisible by 3, since their value is natural in any case.

Further, it can be noted that powering by any natural degree and squaring, particularly, doesn't change a parity of a natural number. In addition, the desired sum can take both even or odd values – depending on the parity of the number n, and the formula is a fractional expression with 3 in denominator and even number in the numerator. Taking into account an optional parity of the sought expression, we can conclude that denominator must be 6.

The simplest form of an even number, which may be represented as the product of two natural numbers, is the product of numbers having distinctly different parity – i.e. $n(n+1)$ in order to get the result of division as odd values too, the third factor must be necessarily odd – that is $2n+1$. In addition, it's easy to show that one of these three factors is necessarily divisible by 3. Under the assumption that for the factors $n = 3l$ or $n+1 = 3m$ such a conclusion is obvious. Therefore, suppose that e.g. $n = 3l+1$, then $n+1 = 3l+1$, i.e. none of these numbers divisible by 3. But then the number $2n+1 = 2(3l+1)+1 = 6l+3$, i.e. it is divisible by 3.

Summarizing what has been said, we can finally write down

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

For certainty we can make a test for small values of n. For $n = 0$ the result is obvious. For $n = 1$ we have the same one. For $n = 2$ we get

$$1^2 + 2^2 = \frac{2 \cdot 3 \cdot 5}{6} = 5$$

For $n = 3$ we get

$$1^2 + 2^2 + 3^2 = 14 = \frac{3 \cdot 4 \cdot 7}{6} = \frac{84}{6} = 14$$

… etc.

However, the general result may be verified inductively.