A harmonic oscillator in a potential energy proportional to the square of The second derivative of the coordinate with respect to time

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Abstract:
In search to construct a Lagrangian functional of a damped harmonic oscillator I thought to study higher derivatives of coordinates with respect to time in the Lagrangian of a simple harmonic oscillator by adding a term proportional to the square of the second derivative of the coordinate with respect to time in its Lagrangian. In Newtonian mechanics a damping term is added directly to the equation of motion of a simple harmonic oscillator, whereas in Lagrangian and Hamiltonian mechanics (Analytical Mechanics as opposed to Vectroial Mechanics of Newton) adding a term to the Lagrangian of the simple harmonic oscillator wouldn't reveal whether the term is a damping driving or a forced driving agent until one study the solutions of the equation of motion.

Here, The Euler-Lagrange and equation of motion of a harmonic oscillator in a potential energy proportional to the square of the second derivative of the coordinate with respect to time have been formulated and discussed. The equation of motion is derived from Euler-Lagrange equation by performing the partial derivatives on the Lagrangian functional of the second variation of the calculus of variations.

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Introduction:

The harmonic oscillator model is very important in physics (Classical physics and Quantum models of natural phenomena); because any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations. Harmonic oscillators occur widely in nature and are exploited in many manmade devices, such as clocks and radio circuits. They are the source of virtually all sinusoidal vibrations and waves.

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Discussion

A Lagrangian functional of simple harmonic oscillator

A Lagrangian functional of simple harmonic oscillator in one dimension may be written as:

$$L = -\frac{1}{2}k\ x^2 + \frac{1}{2}m\ \dot{x}^2$$

The first term is the potential energy and the second term is kinetic energy of the simple harmonic oscillator.

The equation of motion of the simple harmonic oscillator is derived from the Euler-Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

To give

$$-kx - m\ddot{x} = 0$$

This is the same as the equation of motion of the simple harmonic oscillator resulted from application of Newton's second law to a mass attached to spring of spring constant k and displaced to a position x from equilibrium position.

Solving this differential equation, we find that the motion is described by the function

$$x(t) = x_0 \cos(\omega_0\ t - \varphi),$$

where $x_0 = x(t = t_0)$ and $\omega_0 = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$.

Damped harmonic oscillator in the Newtonian Mechanics

In real oscillators, friction, or damping, slows the motion of the system. Due to frictional force, the velocity decreases in proportion to the acting frictional force. While simple harmonic motion oscillates with only the restoring force acting on the system, damped harmonic motion experiences friction. In many vibrating systems the frictional force F_f can be modeled as being proportional to the velocity v of the object: $F_f = -b\mathrm{v}$, where b is called the *viscous damping coefficient.*

Balance of forces (Newton's second law) for damped harmonic oscillators is then

$$F = F_{ext} = F_f - kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

When no external forces are present (i.e. when $F_{ext} = 0$), this can be rewritten into the form

$$\frac{d^2x}{dt^2} + 2\gamma\omega_0\frac{dx}{dt} + {\omega_0}^2 x = 0$$

where

$\omega_0 = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$ is called the" un-damped angular frequency of the oscillator" and $\gamma = \frac{b}{2\sqrt{mk}}$ is called the "damping ratio".

The value of the damping ratio γ critically determines the behavior of the system. A damped harmonic oscillator can be:

- *Over-damped* ($\gamma > 1$): The system returns (exponentially decays) to steady state without oscillating. Larger values of the damping ratio γ return to equilibrium slower.
- *Critically damped* ($\gamma = 1$): The system returns to steady state as quickly as possible without oscillating (although overshoot can occur). This is often desired for the damping of systems such as doors.
- *Under-damped* ($\gamma < 1$): The system oscillates (with a slightly different frequency than the un-damped case) with the amplitude gradually decreasing to zero. The angular frequency of the under-damped harmonic oscillator is given by $\omega_1 = \omega_0\sqrt{1-\gamma^2}$; the exponential decay of the under-damped harmonic oscillator is given by $\lambda = \omega_0\gamma$.

The Q factor of a damped oscillator is defined as

$$Q = 2\pi \frac{energy\ stored}{energy\ lost\ per\ cycle}$$

Q is related to the damping ratio by the equation $Q = \frac{1}{2\gamma}$.

Second Variations of the Calculus of Variations of scalar functions

It is known that the Euler-Lagrange equation resulting from applying the second variations of the Calculus of Variations of a Lagrangian functional $L(t, q(t), \dot{q}(t), \ddot{q}(t))$ of a single independent variable $q(t)$, its first and second derivatives $\dot{q}(t)$, $\ddot{q}(t)$ of following action

$$I[q(t)] = \int L(t, q(t), \dot{q}(t), \ddot{q}(t))\ dt$$

When varied with respect to the arguments of integrand and the variation are set to zero, i.e.

$$0=\delta\ I[q(t)]=\delta\int L(t,q(t),\dot{q}(t),\ddot{q}(t))\ dt$$
$$=\int \delta[L(t,q(t),\dot{q}(t),\ddot{q}(t))]\ dt$$
$$=\int\left\{\frac{\partial L}{\partial t}\delta t+\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)+\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right]\delta q+\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)-\left(\delta q\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right)+\left(\frac{\partial L}{\partial \ddot{q}}\delta\dot{q}\right)\right]\right\}dt$$

is given by

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}+\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}=0$$

Provided that the variations δq and $\delta\dot{q}$ vanish at the end points of the integration.

The Model:

(1) A negative term proportional to square of second derivative of coordinate with respect to time

A Lagrangian functional of a simple harmonic oscillator in one dimension in a potential energy proportional to square of second derivative of coordinate with respect to time may be written as:

$$L=-\frac{1}{2}k\ x^2+\frac{1}{2}m\ \dot{x}^2-\frac{1}{2}n\ddot{x}^2$$

where n is a positive parameter.

The first two terms are the potential energy and kinetic energy of a simple harmonic oscillator. The third term has been added to study effects on motion of the harmonic oscillator of higher derivatives in the Lagrangian functional via varying the parameter n.

The equation of motion is derived from Euler-Lagrange equation by performing the partial derivatives on the Lagrangian functional:

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}+\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}=0$$

To give the equation of motion

$$-kx-\frac{d}{dt}(m\dot{x})-\frac{d^2}{dt^2}(n\ddot{x})=0$$
$$-kx-m\ddot{x}-n\ddot{\ddot{x}}=0$$

Or,

$$kx+m\ddot{x}+n\ddot{\ddot{x}}=0$$

where k,m,n are spring constant, mass of particle and the yet unknown factor n, respectively. They are positive and have finite real valued constants, and none have zero value (i.e. $k\neq 0, m\neq 0, n\neq 0; k\neq\infty, m\neq\infty, n\neq\infty$).

where the units of the constants in the SI are $[k]=\frac{N}{m}$, $[m]=kg$, $[n]=kgs^2$

Assume a solution of the form

$$\boxed{x = x_0 \exp[\pm i(\omega\, t - \varphi)]}$$

Where x_0 the amplitude in [m], $i = \sqrt{-1}$ is the imaginary unit, ω is the angular frequency in [Hz], t is the time in [s] and φ is a phase factor in [dimensionless].

$x = x_0 \exp[+i(\omega t - \varphi)]$	$x = x_0 \exp[-i(\omega t - \varphi)]$
$\dot{x} = i\omega x$	$\dot{x} = -i\omega x$
$\ddot{x} = -\omega^2 x$	$\ddot{x} = -\omega^2 x$
$\dddot{x} = -i\omega^3 x$	$\dddot{x} = +i\omega^3 x$
$\ddddot{x} = +\omega^4 x$	$\ddddot{x} = +\omega^4 x$

Substituting in the equation of motion, we get

$$kx + m(-\omega^2)x + n(\omega^4)x = 0$$

$$[k + m(-\omega^2) + n(\omega^4)]x = 0$$

Assume $x(t) \neq 0$ at all times, then,

$$[k + m(-\omega^2) + n(\omega^4)] = 0$$

$$\boxed{k - m\omega^2 + n\omega^4 = 0}$$

This has four solutions, using the general method for determining roots of a quadratic equation ($az^2 + bz + c = 0, a \neq 0.$):

$$z_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with,

$$z = \omega^2; a = n, b = -m, c = k$$

The solutions are:

$$\omega = \pm\sqrt{\frac{m \pm \sqrt{m^2 - 4(n)(k)}}{2(n)}}$$

$$= \pm\sqrt{\frac{m}{2n} \pm \frac{m}{2n}\sqrt{1 - \frac{4nk}{m^2}}}$$

Define; $\omega_0{}^2 \equiv \dfrac{k}{m}$ where ω_0 is the natural frequency of the oscillator.

Then, ω may be written as

$$\omega = \pm\sqrt{\frac{m}{2n} \pm \frac{m}{2n}\sqrt{1 - \frac{4\omega_0{}^2}{m}n}}$$

Then, the fours roots are

$$\omega_1 = +\sqrt{\frac{m}{2n} + \frac{m}{2n}\sqrt{1 - \frac{4\omega_0{}^2}{m}n}}, \qquad \omega_2 = +\sqrt{\frac{m}{2n} - \frac{m}{2n}\sqrt{1 - \frac{4\omega_0{}^2}{m}n}}$$

$$\omega_3 = -\omega_1, \qquad \omega_4 = -\omega_2$$

Types of motion for different values of the positive parameter (n)

<u>Case (1):</u> $n \cong 0$

The harmonic oscillator should oscillate with its natural frequency when the external potential energy term vanish or has very small value close to zero for fixed values of the simple harmonic oscillator parameters k, m and / or ω_0.

Setting $n \cong 0$ in the four solutions of ω above, and approximating the inner radical:

$$\sqrt{1 - \frac{4\omega_0{}^2}{m}n} = [1 - \frac{1}{2}(\frac{4\omega_0{}^2}{m}n)] = [1 - \frac{2\omega_0{}^2}{m}n]$$

We, get

$$\omega_1 = +\sqrt{\frac{m}{2n} + \frac{m}{2n}(1 - \frac{2\omega_0{}^2}{m}n)} = +\sqrt{\frac{m}{2n} + \frac{m}{2n} - \frac{m}{2n}\frac{2\omega_0{}^2}{m}n} = +\sqrt{\frac{m}{n} - \omega_0{}^2},$$

$$\omega_2 = +\sqrt{\frac{m}{2n} - \frac{m}{2n}(1 - \frac{2\omega_0{}^2}{m}n)} = +\sqrt{\frac{m}{2n} - \frac{m}{2n} + \frac{m}{2n}\frac{2\omega_0{}^2}{m}n} = +\sqrt{\omega_0{}^2} = +\omega_0,$$

$$\omega_3 = -\sqrt{\frac{m}{2n} + \frac{m}{2n}(1 - \frac{2\omega_0{}^2}{m}n)} = -\sqrt{\frac{m}{2n} + \frac{m}{2n} - \frac{m}{2n}\frac{2\omega_0{}^2}{m}n} = -\sqrt{\frac{m}{n} - \omega_0{}^2},$$

$$\omega_4 = -\sqrt{\frac{m}{2n} - \frac{m}{2n}(1 - \frac{2\omega_0{}^2}{m}n)} = -\sqrt{\frac{m}{2n} - \frac{m}{2n} + \frac{m}{2n}\frac{2\omega_0{}^2}{m}n} = -\sqrt{\omega_0{}^2} = -\omega_0,$$

So, we choose $\omega = \omega_2$ and $\omega = \omega_4$ to be the solutions in case $n \cong 0$. The corresponding equations of motion are that of a simple harmonic oscillator as it should.

<u>Case (2):</u> $n = \dfrac{m}{4\omega_0{}^2}$ (i.e. $1 - \dfrac{4\omega_0{}^2}{m}n = 0$)

Substituting $n = \dfrac{m}{4\omega_0{}^2}$ in ω_2, we get

$$\omega = \pm\sqrt{\frac{m}{2n} - \frac{m}{2n}\sqrt{1 - \frac{4\omega_0{}^2}{m}n}} = \pm\sqrt{\frac{m}{2(\frac{m}{4\omega_0{}^2})}} = \pm\sqrt{2\omega_0{}^2} = \pm\sqrt{2}\omega_0$$

The corresponding equations of motion are that of a simple harmonic oscillator oscillating at a frequency ω of a value larger by ($\cong 41\%$) of the value of natural frequency ω_0 of free oscillation.

Case (3): $0<1-\frac{4\omega_0^{\ 2}}{m}n<1$ (e.g. $1-\frac{4\omega_0^{\ 2}}{m}n=0.64$)

Substituting $1-\frac{4\omega_0^{\ 2}}{m}n=0.64$ or,

$(1-\frac{4\omega_0^{\ 2}}{m}n=0.64 \rightarrow n=\frac{m}{4\omega_0^{\ 2}}(1-0.64)=0.36\frac{m}{4\omega_0^{\ 2}}=0.9\frac{m}{\omega_0^{\ 2}} \rightarrow \frac{m}{2n}=\frac{\omega_0^{\ 2}}{1.8})$ as an example in the equation of ω, we get the following solutions:

$$\omega=\pm\sqrt{\frac{m}{2n}-\frac{m}{2n}(0.64)}=\pm\sqrt{\frac{m}{2n}[1-(0.64)]}=\pm\sqrt{\frac{\omega_0^{\ 2}}{1.8}[1-(0.64)]}=\pm\approx 0.45\omega_0$$

The corresponding equations of motion are that of a simple harmonic oscillator oscillating at a frequency ω of almost the half the value of the value of natural frequency ω_0 of free oscillation.

Case (4): $n>\frac{m}{4\omega_0^{\ 2}}$ (i.e. $1-\frac{4\omega_0^{\ 2}}{m}n<0$)

Substituting $1-\frac{4\omega_0^{\ 2}}{m}n<0$ in the equation of ω, we get

$$\omega=\pm\sqrt{\frac{m}{2n}-\frac{m}{2n}\sqrt{1-\frac{4\omega_0^{\ 2}}{m}n}}=\pm\sqrt{\frac{m}{2n}-\frac{m}{2n}\sqrt{-(\frac{4\omega_0^{\ 2}}{m}n-1)}}$$
$$=\pm\sqrt{\frac{m}{2n}-\frac{m}{2n}i\sqrt{\frac{4\omega_0^{\ 2}}{m}n-1}}$$

It is a square root of a complex number (i.e. it is a complex number), when a complex number appears in the power of a complex number written in its polar form a decay behavior appears.

Changing variables,

$$\omega^2\equiv\xi=\frac{m}{2n}-\frac{m}{2n}i\sqrt{\frac{4\omega_0^{\ 2}}{m}n-1}$$

Its modulus is

$$|\xi|=\sqrt{\left(\frac{m}{2n}\right)^2+\left(\frac{m}{2n}\sqrt{\frac{4\omega_0^{\ 2}}{m}n-1}\right)^2}=\sqrt{\left(\frac{m}{2n}\right)^2+\left(\frac{m}{2n}\right)^2\frac{4\omega_0^{\ 2}}{m}n-\left(\frac{m}{2n}\right)^2}=\sqrt{\left(\frac{m^2}{4n^2}\right)\frac{4\omega_0^{\ 2}}{m}n}$$
$$=\sqrt{\frac{m^2}{4n^2}\frac{4\omega_0^{\ 2}}{m}n}=\sqrt{\frac{m}{n}\frac{\omega_0^{\ 2}}{1}}=\sqrt{\frac{m}{n}}\omega_0$$

Its arguments are

$$\theta = \arctan(\frac{y}{x}) = \arctan\left(\frac{-\frac{m}{2n}\sqrt{\frac{4\omega_0{}^2}{m}n-1}}{\frac{m}{2n}}\right) = \arctan\left(-\sqrt{\frac{4\omega_0{}^2}{m}n-1}\right)$$

It is in the fourth quarter in the complex plane.
In Cartesian form;

$$\xi = |\xi|[\cos(\theta) + i\sin(\theta)]$$

Now, taking the square root of both sides of $\omega^2 = \xi$, we get

$$\omega = \sqrt{\xi} = \sqrt{|\xi|\cos(\theta) + i\sin(\theta)}$$
$$= |\xi|^{\frac{1}{2}}\sqrt{\cos(\theta) + i\sin(\theta)}$$
$$= \pm\left(\sqrt{\frac{m}{n}}\omega_0\right)^{\frac{1}{2}}\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})$$

and, the other root;

$$\omega = \pm\left(\sqrt{\frac{m}{n}}\omega_0\right)^{\frac{1}{2}}\cos(\frac{\theta+2\pi}{2}) + i\sin(\frac{\theta+2\pi}{2})$$

Now, inserting the value in the equation of motion $x = x_0\ \exp[\pm i(\omega t-\varphi)]$, yields

$$x = x_0\exp\pm i\{[\sqrt{\sqrt{\frac{m}{n}}\omega_0}\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})]t-\varphi\}$$
$$= x_0\exp\pm\{i[(\sqrt{\sqrt{\frac{m}{n}}\omega_0}\cos(\frac{\theta}{2}))t-\varphi]-\sin(\frac{\theta}{2})t\}$$

and,

$$x = x_0\exp\pm i\{[\sqrt{\sqrt{\frac{m}{n}}\omega_0}\cos(\frac{\theta+2\pi}{2}) + i\sin(\frac{\theta+2\pi}{2})]t-\varphi\}$$
$$= x_0\exp\pm\{i[(\sqrt{\sqrt{\frac{m}{n}}\omega_0}\cos(\frac{\theta+2\pi}{2}))t-\varphi]-\sin(\frac{\theta+2\pi}{2})t\}$$

Taking the negative sign in the exponential we get a smoothly damping solution with a damping constant $\sin(\frac{\theta}{2})$ and $\sin(\frac{\theta+2\pi}{2})$, otherwise we get a growing solution if the positive sign is taken and with growing constant $\sin(\frac{\theta}{2})$ and $\sin(\frac{\theta+2\pi}{2})$.

(2) A positive term proportional to square of second derivative of coordinate with respect to time

The Lagrangian functional may be given by

$$L = -\frac{1}{2}k\ x^2 + \frac{1}{2}m\ \dot{x}^2 + \frac{1}{2}n\ddot{x}^2$$

The equation of motion is derived from Euler-Lagrange equation by performing the partial derivatives on the Lagrangian functional:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}} = 0$$

To give the equation of motion this may be written as

$$-kx - \frac{d}{dt}(m\dot{x}) + \frac{d^2}{dt^2}(n\ddot{x}) = 0$$

where k, m, n are spring constant, mass of particle and the yet unknown factor n , respectively. They are positive and have finite real valued constants, and none have zero value (i.e. $k \neq 0, m \neq 0, n \neq 0; k \neq \infty, m \neq \infty, n \neq \infty$).

Then, the equation of motion may further be written as

$$-kx - m\ddot{x} + n\ddddot{x} = 0$$

Assume a solution of a form

$$x = x_0\ \exp[\pm i(\omega t - \varphi)]$$

$x = x_0 \exp[+i(\omega t - \varphi)]$	$x = x_0 \exp[-i(\omega t - \varphi)]$
$\dot{x} = i\omega x$	$\dot{x} = -i\omega x$
$\ddot{x} = -\omega^2 x$	$\ddot{x} = -\omega^2 x$
$\dddot{x} = -i\omega^3 x$	$\dddot{x} = +i\omega^3 x$
$\ddddot{x} = +\omega^4 x$	$\ddddot{x} = +\omega^4 x$

Substituting in the equation of motion, we get

$$-kx - m(-\omega^2)x + n(\omega^4)x = 0$$

$$[-k - m(-\omega^2) + n(\omega^4)]x = 0$$

Assume $x(t) \neq 0$ at all times, then,

$$[-k - m(-\omega^2) + n(\omega^4)] = 0$$

$$\boxed{-k + m\omega^2 + n\omega^4 = 0}$$

This has four solutions, using the general method for determining roots of a quadratic equation ($az^2 + bz + c = 0, a \neq 0.$):

$$z_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with,

$$z = \omega^2; a = n, b = m, c = -k$$

The solutions are:

$$\omega = \pm\sqrt{\frac{-m \pm \sqrt{m^2 - 4(n)(-k)}}{2(n)}}$$

$$= \pm\sqrt{\frac{-m}{2n} \pm \frac{m}{2n}\sqrt{1+\frac{4nk}{m^2}}}$$

Then, ω may be written as

$$\omega = \pm\sqrt{\frac{-m}{2n} \pm \frac{m}{2n}\sqrt{1+\frac{4\omega_0^{\ 2}}{m}n}}$$

where $\omega_0^{\ 2} \equiv \dfrac{k}{m}$ is the natural frequency of the oscillator.

Then, the fours roots are

$$\omega_1 = +\sqrt{\frac{-m}{2n} + \frac{m}{2n}\sqrt{1+\frac{4\omega_0^{\ 2}}{m}n}}, \qquad \omega_2 = +\sqrt{\frac{-m}{2n} - \frac{m}{2n}\sqrt{1+\frac{4\omega_0^{\ 2}}{m}n}}$$

$$\omega_3 = -\omega_1, \qquad \omega_4 = -\omega_2$$

Types of motion for different values of the positive parameter (n)

<u>Case (1):</u> $n \cong 0$

The harmonic oscillator should oscillate with its natural frequency when the external potential energy term vanish or has very small value close to zero for fixed values of the simple harmonic oscillator parameters k, m and / or ω_0.

Setting $n \cong 0$ in the four solutions of ω above, and approximating the inner radical:

$$\sqrt{1+\frac{4\omega_0^{\ 2}}{m}n} = [1+\frac{1}{2}(\frac{4\omega_0^{\ 2}}{m}n)] = [1+\frac{2\omega_0^{\ 2}}{m}n]$$

We, get

$$\omega_1 = +\sqrt{\frac{-m}{2n} + \frac{m}{2n}(1+\frac{2\omega_0^{\ 2}}{m}n)} = +\sqrt{\frac{-m}{2n} + \frac{m}{2n} + \frac{m}{2n}\frac{2\omega_0^{\ 2}}{m}n} = +\sqrt{\omega_0^{\ 2}} = +\omega_0,$$

$$\omega_2 = +\sqrt{\frac{-m}{2n} - \frac{m}{2n}(1+\frac{2\omega_0^{\ 2}}{m}n)} = +\sqrt{\frac{-m}{2n} - \frac{m}{2n} - \frac{m}{2n}\frac{2\omega_0^{\ 2}}{m}n} = +\sqrt{\frac{-m}{n} - \omega_0^{\ 2}},$$

$$\omega_3 = -\sqrt{\frac{-m}{2n} + \frac{m}{2n}(1+\frac{2\omega_0^{\ 2}}{m}n)} = -\sqrt{\frac{-m}{2n} + \frac{m}{2n} + \frac{m}{2n}\frac{2\omega_0^{\ 2}}{m}n} = -\sqrt{\omega_0^{\ 2}} = -\omega_0,$$

$$\omega_4 = -\sqrt{\frac{-m}{2n} - \frac{m}{2n}(1+\frac{2\omega_0^{\ 2}}{m}n)} = -\sqrt{\frac{-m}{2n} - \frac{m}{2n} - \frac{m}{2n}\frac{2\omega_0^{\ 2}}{m}n} = -\sqrt{\frac{-m}{n} - \omega_0^{\ 2}},$$

So, we choose $\omega=\omega_1$ and $\omega=\omega_3$ to be the solutions in case $n\cong 0$. The corresponding equations of motion are that of a simple harmonic oscillator as it should.

<u>Case (2):</u> Other values of the positive parameter (n)

For all other values of the positive parameter n the value of $\omega=\omega_1$ and $\omega=\omega_3$ are real and the motion is oscillatory:

$$\omega_1=+\sqrt{\frac{-m}{2n}+\frac{m}{2n}\sqrt{1+\frac{4\omega_0^{2}}{m}n}},$$
$$\omega_3=-\omega_1$$

As an example: I take

$$\frac{4\omega_0^{2}}{m}n=0.44\rightarrow n=\frac{0.44m}{4\omega_0^{2}}\rightarrow\frac{m}{2n}=\frac{2\omega_0^{2}}{0.44}=\frac{\omega_0^{2}}{0.22}\ ,$$

Then, ω_1 take the value

$$\omega_1=+\sqrt{\frac{-\omega_0^{2}}{0.22}+\frac{\omega_0^{2}}{0.22}\sqrt{1+0.44}}=+\sqrt{\frac{-\omega_0^{2}}{0.22}+\frac{\omega_0^{2}}{0.22}(1.2)}=+\sqrt{(0.2)\frac{\omega_0^{2}}{0.22}}=+\approx 0.95\omega_0,$$

And,

$$\frac{4\omega_0^{2}}{m}n=9999\rightarrow n=\frac{9999m}{4\omega_0^{2}}\rightarrow\frac{m}{2n}=\frac{2\omega_0^{2}}{9999}\ ,$$

Then, ω_1 take the value;

$$\omega_1=+\sqrt{\frac{-2\omega_0^{2}}{9999}+\frac{2\omega_0^{2}}{9999}\sqrt{1+9999}}=+\sqrt{\frac{-2\omega_0^{2}}{9999}+\frac{2\omega_0^{2}}{9999}\sqrt{10000}}=+\sqrt{(99)\frac{2\omega_0^{2}}{9999}}=+\approx 0.14\omega_0,$$

The frequency of oscillation decreases with increasing the value of the positive parameter n.

Conclusion:

Studying systems with additional nonlinear term in their Lagrangian functional may lead to interesting results. In our simple example of a simple harmonic oscillator with a third -positive and negative term proportional to the square of the second derivative of coordinates with respect to time- it is observed that oscillatory, damping and growing behaviors emerged from the solution of the equation of motion. Although it doesn't substitute the direct treatment of the Newtonian Mechanics of the damping harmonic oscillator it could be an additional method for study it. This method could be applied to other systems e.g. non-linear classical fields and may be to nonlinear quantum fields.

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