Confronting the Galilean Transformation with the Field Shapes of a Constant-Velocity Point Charge

Steven Kenneth Kauffmann*

Abstract

The space-time Galilean transformation is predicated on a salient theme of Galilean/Newtonian physics: relative motion at constant velocity has no physical consequences beyond the minimum which is required by that motion's existence. Therefore, since the electric field produced by a point charge at rest is spherically symmetric around the charge's location, and since a point charge at rest produces zero magnetic field, Galilean physics implies that a point charge moving at constant velocity produces an electric field which is spherically symmetric around that charge's instantaneous location and that it produces zero magnetic field. But the Biot-Savart-Maxwell Law has it that a point charge moving at nonzero constant velocity produces nonzero magnetic field, and Faraday's Law has it that this time-varying magnetic field, which has zero component along the line of the charge's motion, produces an electric field which isn't spherically symmetric around the charge's instantaneous location. Thus the space-time Galilean transformation is violated by electromagnetic phenomena in a definite way, and must be modified. The needed modification produces the space-time Lorentz transformation, which can straightforwardly be shown to never change the speed of electromagnetic radiation. The fate of the Galilean/Newtonian constant-velocity relative-motion paradigm was actually already sealed when it was observed that the presence of direct current in a wire deflects an adjacent compass needle.

Introduction

The theoretical physics idea which underlies the space-time Galilean transformation of constant velocity $\mathbf{v}$, namely,

$$\mathbf{r}' = (\mathbf{r} - \mathbf{v}t), \ \ t' = t, \tag{1}$$

is that relative motion at constant velocity has no physical consequences beyond the minimum which is required by that motion's existence.

Therefore since the electric field of a point charge at rest, namely $e\mathbf{r}/|\mathbf{r}|^3$, manifests spherical symmetry around that charge's location, Galilean logic implies that the electric field of a point charge traveling at constant velocity $\mathbf{v}$ ought to manifest precisely the same spherical symmetry around that charge's instantaneous moving location $\mathbf{v}t$; indeed its electric field ought to be precisely $e(\mathbf{r} - \mathbf{v}t)/|(\mathbf{r} - \mathbf{v}t)|^3$. By the same token, the Galilean prediction for the constant-velocity point charge's electric potential would, in the appropriately chosen gauge, be $e/|(\mathbf{r} - \mathbf{v}t)|$. Since a point charge *at rest* produces *precisely zero magnetic field*, and also, in the appropriately-chosen gauge, *zero vector potential*, the Galilean prediction would be that a constant-velocity point charge produces *zero magnetic field* and also, in the appropriately chosen gauge, *zero vector potential*.

However an outright *zero* value for the magnetic field of a constant-velocity point charge flies utterly and completely in the face of the Biot-Savart-Maxwell Law of electromagnetism, which is,

$$\nabla \times \mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c. \tag{2a}$$

Direct current which is impelled by a battery in a length of copper wire can, to reasonable approximation, be thought of as consisting of well-shielded electric charges moving along at constant velocity, and there is no doubt whatsoever that they produce a magnetic field, one in accord with the Biot-Savart Law (the fact that the electric field of the charges in the wire is well-shielded *eliminates the effect of the Maxwell source term* $\dot{\mathbf{E}}/c$). There is thus no doubt that electromagnetic physics is at loggerheads with the theoretical physics idea which underlies the space-time Galilean transformation, and that that transformation *requires modification*.

The *way* in which the electric field of a constant-velocity point charge *differs* from $e(\mathbf{r}-\mathbf{v}t)/|(\mathbf{r}-\mathbf{v}t)|^3$, or the *way* in which its electric potential *differs* from $e/|(\mathbf{r} - \mathbf{v}t)|$ can reasonably be expected *to speak volumes* concerning *how* the space-time Galilean transformation of Eq. (1) *needs to be modified*.

The calculation of the electric and magnetic fields of a constant-velocity point charge was first carried out by Oliver Heaviside in 1888, and Heaviside's results were famously cited by George FitzGerald in 1889 and by Hendrik A. Lorentz in 1892 in the endeavors of those two physicists to attain theoretical understanding of the null readings for the hypothetical "aether wind" in the 1887 Michelson-Morley experiment [1].

* Retired, American Physical Society Senior Life Member, E-mail: SKKauffmann@gmail.com

While the fact that a constant-velocity point charge *produces* a nonzero *magnetic* field is *a first-order consequence of the Biot-Savart-Maxwell Law of electromagnetism*, the *alteration* of a constant-velocity point charge's *electric* field and *electric* potential *from what would be expected from the space-time Galilean transformation* is *a second-order consequence, via Faraday's Law*, namely via,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c, \tag{2b}$$

of the *existence* of that first-order *magnetic* field. Furthermore, because that first-order magnetic field departs altogether from spherical symmetry—calculation shows that it has *zero* component along the line of the point charge's motion—its electric-field consequence that is produced by Faraday's Law *isn't spherically symmetric around the charge's instantaneous moving location*.

A sensible way to deal with such "fed-through" physical phenomena is to *combine* the *first-order* electromagnetic field equations, such as the one of Biot-Savart-Maxwell and the one of Faraday, into *second-order* "driven-wave" type equations. The treatment below of the electromagnetic potentials and fields of the constant-velocity point charge is carried out *entirely* using such *second-order* driven-wave type electromagnetic field equations.

We turn now to the *details* of calculating the electromagnetic potentials and fields of a constant-velocity point charge. A nonzero magnetic field is obtained, one which has zero component along the point charge's line of motion, and therefore *no semblance whatsoever of spherical symmetry*. Obtained along with that magnetic field is *the distortion induced by it*, via Faraday's Law, of the electric-field and electric-potential shapes $e(\mathbf{r}-\mathbf{v}t)/|(\mathbf{r}-\mathbf{v}t)|^3$ and $e/|(\mathbf{r}-\mathbf{v}t)|$ whose spherically-symmetric forms around the constant-velocity point charge's instantaneous location $\mathbf{v}t$ follow from the space-time Galilean transformation. Taking into account *the distortion which is obtained of these shapes that follow from the space-time Galilean transformation* enables *repair* of the space-time Galilean transformation to be effected.

The electromagnetic potentials and fields of a constant-velocity point charge

The two electromagnetic laws given by Eqs. (2a) and (2b) are *completed* by the *addition* of *both* Coulomb's Law,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2c}$$

and the Gauss Law,

$$\nabla \cdot \mathbf{B} = 0. \tag{2d}$$

The technical development of the second-order driven-wave type electromagnetic field equations begins by applying the curl operator to both sides of both the Biot-Savart-Maxwell and Faraday Laws. The resulting field divergence terms are then eliminated by insertions of the Coulomb and Gauss Laws. The electric and magnetic fields can then be fully decoupled from each other by eliminating the remaining field curl terms in the second-order equations by insertions of the original Biot-Savart-Maxwell and Faraday Laws. The two second-order driven-wave type electric and magnetic field equations which result are,

$$(1/c^2)\ddot{\mathbf{E}} - \nabla^2\mathbf{E} = -4\pi\left(\nabla\rho + (1/c^2)\partial\mathbf{j}/\partial t\right), \quad (1/c^2)\ddot{\mathbf{B}} - \nabla^2\mathbf{B} = 4\pi(\nabla\times\mathbf{j})/c. \tag{3}$$

Since these electric-field and magnetic-field driven-wave type equations *have complicated source terms*, it is convenient to introduce the electromagnetic scalar potential ϕ and the electromagnetic vector potential $\mathbf{A}$ *in Lorentz gauge*, which *also* satisfy driven-wave type equations, *albeit ones with much simpler source terms*,

$$(1/c^2)\ddot{\phi} - \nabla^2\phi = 4\pi\rho, \quad (1/c^2)\ddot{\mathbf{A}} - \nabla^2\mathbf{A} = 4\pi\mathbf{j}/c. \tag{4a}$$

In *addition* to satisfying these driven-wave type equations, ϕ and $\mathbf{A}$ *are related to each other by the Lorentz condition*,

$$(1/c)\dot{\phi} + \nabla\cdot\mathbf{A} = 0. \tag{4b}$$

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ *are obtained from* ϕ *and* $\mathbf{A}$ *via the two relations*,

$$\mathbf{E} = -\nabla\phi - (1/c)\dot{\mathbf{A}}, \quad \mathbf{B} = \nabla\times\mathbf{A}. \tag{4c}$$

Combining the two relations of Eq. (4c) with the simple driven-wave type equations for ϕ and $\mathbf{A}$ of Eq. (4a) is readily seen to yield the more complicated driven-wave type equations for $\mathbf{E}$ and $\mathbf{B}$ of Eq. (3). Also the second relation of Eq. (4c) implies the Gauss Law for $\mathbf{B}$ as given by Eq. (2d), while the first relation of

Eq. (4c) combined with *both* the Lorentz condition of Eq. (4b) *and* the simple driven-wave type equation for ϕ of Eq. (4a) yields Coulomb's Law for $\mathbf{E}$ as given by Eq. (2c).

Now a point charge of strength e traveling with constant vector velocity $\mathbf{v}$ has the charge density,

$$\rho(\mathbf{r},t) = e\delta^{(3)}(\mathbf{r}-\mathbf{v}t), \tag{5a}$$

and it has the current density,

$$\mathbf{j}(\mathbf{r},t) = e\mathbf{v}\delta^{(3)}(\mathbf{r}-\mathbf{v}t) = \mathbf{v}\rho(\mathbf{r},t). \tag{5b}$$

It can be readily shown that the charge density of Eq. (5a) and the current density of Eq. (5b) together satisfy the required equation of continuity, namely $\dot{\rho}(\mathbf{r},t)+\nabla\cdot\mathbf{j}(\mathbf{r},t)=0$. With these source terms, which are appropriate to a point charge of strength e traveling at constant vector velocity $\mathbf{v}$, the two simple driven-wave type equations of Eq. (4a) read,

$$(1/c^2)\ddot{\phi} - \nabla^2\phi = 4\pi e\delta^{(3)}(\mathbf{r}-\mathbf{v}t), \quad (1/c^2)\ddot{\mathbf{A}} - \nabla^2\mathbf{A} = (\mathbf{v}/c)4\pi e\delta^{(3)}(\mathbf{r}-\mathbf{v}t), \tag{5c}$$

which makes it apparent that,

$$\mathbf{A}(\mathbf{r},t) = (\mathbf{v}/c)\phi(\mathbf{r},t), \tag{5d}$$

so we only need to solve,

$$(1/c^2)\ddot{\phi}(\mathbf{r},t) - \nabla^2\phi(\mathbf{r},t) = 4\pi e\delta^{(3)}(\mathbf{r}-\mathbf{v}t), \tag{5e}$$

a task which we undertake *by applying Fourier transforms and their inverses.*

Insertion of the Fourier *ansatz*,

$$\phi(\mathbf{r},t) = \int d^3\mathbf{k}\, d\omega\, e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\bar{\phi}(\mathbf{k},\omega), \tag{6a}$$

into Eq. (5e) yields,

$$\begin{gathered}\left(|\mathbf{k}|^2-(\omega/c)^2\right)\bar{\phi}(\mathbf{k},\omega) = 4\pi e(2\pi)^{-4}\int d^3\mathbf{r}\, dt\, e^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\delta^{(3)}(\mathbf{r}-\mathbf{v}t) = \\ 4\pi e(2\pi)^{-4}\int dt\, e^{-i((\mathbf{k}\cdot\mathbf{v})t+\omega t)} = 4\pi e(2\pi)^{-3}\delta(\omega+(\mathbf{k}\cdot\mathbf{v})),\end{gathered} \tag{6b}$$

so the Fourier transform $\bar{\phi}(\mathbf{k},\omega)$ of $\phi(\mathbf{r},t)$ is evaluated to be,

$$\bar{\phi}(\mathbf{k},\omega) = \frac{e}{2\pi^2}\,\frac{\delta(\omega+(\mathbf{k}\cdot\mathbf{v}))}{|\mathbf{k}|^2-(\omega/c)^2}. \tag{6c}$$

Insertion of Eq. (6c) into Eq. (6a) produces,

$$\phi(\mathbf{r},t) = \frac{e}{2\pi^2}\int d^3\mathbf{k}\, d\omega\, e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}\frac{\delta(\omega+(\mathbf{k}\cdot\mathbf{v}))}{|\mathbf{k}|^2-(\omega/c)^2}. \tag{7a}$$

Using the delta function to carry out the integration over ω then yields,

$$\phi(\mathbf{r},t) = \frac{e}{2\pi^2}\int d^3\mathbf{k}\,\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}}{|\mathbf{k}|^2-(\mathbf{k}\cdot(\mathbf{v}/c))^2}. \tag{7b}$$

In order to evaluate the integral in Eq. (7b) we choose a Cartesian coordinate system whose x-axis points in the direction of $\mathbf{v}$. In that Cartesian coordinate system, $\mathbf{v}=(v,0,0)$ and $\mathbf{k}\cdot\mathbf{v}=k^1 v$, so Eq. (7b) can be written,

$$\phi(x,y,z,t) = \frac{e}{2\pi^2}\int dk^1\, dk^2\, dk^3\,\frac{e^{i[k^1(x-vt)+k^2y+k^3z]}}{(k^1)^2(1-(v/c)^2)+(k^2)^2+(k^3)^2}. \tag{7c}$$

We now change the three integration variables from (k^1,k^2,k^3) to $(q^1,q^2,q^3) \stackrel{\text{def}}{=} (k^1(1-(v/c)^2)^{\frac{1}{2}},k^2,k^3)$, which implies that $(k^1,k^2,k^3)=(\gamma q^1,q^2,q^3)$, where $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}$. With that change of the integration variables, Eq. (7c) becomes,

$$\phi(x,y,z,t) = \frac{e\gamma}{2\pi^2}\int dq^1\, dq^2\, dq^3\,\frac{e^{i[q^1\gamma(x-vt)+q^2y+q^3z]}}{(q^1)^2+(q^2)^2+(q^3)^2}, \tag{7d}$$

which can be written more compactly as,

$$\phi(x,y,z,t) = \frac{e\gamma}{2\pi^2}\int d^3\mathbf{q}\,\frac{e^{i\mathbf{q}\cdot\mathbf{R}(x,y,z,t;v,\gamma)}}{|\mathbf{q}|^2}, \tag{7e}$$

where the time-dependent radius-like vector field $\mathbf{R}(x,y,z,t;v,\gamma)$ is of course seen from Eq. (7d) to be,

$$\mathbf{R}(x,y,z,t;v,\gamma) \stackrel{\text{def}}{=} (\gamma(x-vt),y,z), \tag{8a}$$

in which γ is defined as,

$$\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}. \tag{8b}$$

The integral over $d^3\mathbf{q}$ which appears in Eq. (7e) is a standard one which is familiar from the case of the Coulomb potential for a point charge at rest; as a matter of fact, in the *special case* that the point charge e has its *speed* $|v|$ put equal to *zero*, $\phi(x,y,z,t)$ *must* reduce to the straightforward static Coulomb result $e/|\mathbf{r}|$, where $\mathbf{r}=(x,y,z)$. That fact implies that the *integral* which occurs in Eq. (7e) has the value $2\pi^2/|\mathbf{R}(x,y,z,t;v,\gamma)|$, namely that,

$$\int d^3\mathbf{q}\,\frac{e^{i\mathbf{q}\cdot\mathbf{R}(x,y,z,t;v,\gamma)}}{|\mathbf{q}|^2} = 2\pi^2/|\mathbf{R}(x,y,z,t;v,\gamma)|,$$

a result which alternatively can be obtained by simply carrying out the relatively straightforward integration over $d^3\mathbf{q}$. From this result and Eq. (7e) it follows that,

$$\phi(x,y,z,t) = \frac{e\gamma}{|\mathbf{R}(x,y,z,t;v,\gamma)|} = \frac{e\gamma}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{1}{2}}}, \tag{9a}$$

where we have used Eq. (8a) to obtain the second equality.

Since $\gamma=(1-(v/c)^2)^{-\frac{1}{2}}>1$, the electric potential ϕ of Eq. (9a) *deviates from the form* $e/|(\mathbf{r}-\mathbf{v}t)| = e/((x-vt)^2+y^2+z^2)^{\frac{1}{2}}$ *which is predicted by the space-time Galilean transformation for* $\mathbf{r}=(x,y,z)$ *and* $\mathbf{v}=(v,0,0)$. Specifically, Eq. (9a) shows that the details of electromagnetic theory *have distorted the spherical symmetry of the electric potential* ϕ *around the constant-velocity point charge's instantaneous location* $\mathbf{v}t=(vt,0,0)$ *which is predicted by the Galilean transformation*, with that distortion occurring along the point charge's line of motion.

We next obtain the constant-velocity point charge's vector potential and magnetic field. Since from Eq. (5d), $\mathbf{A}(x,y,z,t)=(\mathbf{v}/c)\phi(x,y,z,t)$, and from the discussion below Eq. (7b), $\mathbf{v}=(v,0,0)$, we obtain from Eq. (9a) that,

$$\mathbf{A}(x,y,z,t) = \frac{e\gamma((v/c),0,0)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{1}{2}}}. \tag{9b}$$

From Eq. (9b) it is apparent that the three components of $\mathbf{A}(x,y,z,t)$ are,

$$A^1(x,y,z,t) = \frac{e\gamma(v/c)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{1}{2}}},\;\; A^2(x,y,z,t)=0,\;\; A^3(x,y,z,t)=0,$$

and therefore the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ produced by the moving point charge is given by,

$$\mathbf{B}=\nabla\times\mathbf{A}=(0,\partial A^1/\partial z,-\partial A^1/\partial y) = \frac{e\gamma(v/c)(0,-z,y)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}}. \tag{9c}$$

This magnetic field clearly *vanishes entirely* when the point charge is *at rest*, namely when $v=0$, and *it has no semblance whatsoever of spherical symmetry around the point charge's instantaneous location since it has vanishing component along the point charge's line of motion*. But as we have strongly emphasized in the Introduction, the fact that this magnetic field *is nonzero at all* contradicts the theoretical physics idea which underlies the space-time Galilean transformation, namely that constant relative velocity has no physical consequences beyond the minimum required by its existence. As we have *also* emphasized, this nonzero *time-varying* magnetic field $\mathbf{B}$, *which departs so strongly from spherical symmetry around the point charge's instantaneous location* $\mathbf{v}t=(vt,0,0)$, modifies, via Faraday's Law $\nabla\times\mathbf{E}=-\dot{\mathbf{B}}/c$, the constant-velocity point charge's electric field $\mathbf{E}$ and electric potential ϕ (shown in Eq. (9a)) in a way *that distorts the electric-field shape* $e(\mathbf{r}-\mathbf{v}t)/|(\mathbf{r}-\mathbf{v}t)|^3$ *and the electric-potential shape* $e/|(\mathbf{r}-\mathbf{v}t)|$ *which follow from the space-time Galilean transformation* (and are consequently spherically-symmetric around the constant-velocity point charge's instantaneous location $\mathbf{v}t$). That distortion is clearly apparent in the Eq. (9a) expression for the electric potential ϕ of the constant-velocity point charge; the distortion occurs because of the presence of the factor $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}} > 1$. The result for the electric *field* $\mathbf{E}$ of the constant-velocity point charge will be similarly distorted by the presence of that factor $\gamma > 1$.

We next work out $\mathbf{E}$, this constant-velocity point charge's electric field. Since $\mathbf{E} = -\nabla\phi - (1/c)\dot{\mathbf{A}}$, we need to use Eq. (9a) to calculate,

$$-\nabla\phi(x,y,z,t) = \frac{e\gamma(\gamma^2(x-vt),y,z)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}},$$

and *in addition* we need to use Eq. (9b) to calculate,

$$-(1/c)\dot{\mathbf{A}}(x,y,z,t) = \frac{e\gamma(-(v/c)^2\gamma^2(x-vt),0,0)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}}.$$

We can now assemble the above two intermediate results to obtain $\mathbf{E}$, the constant-velocity point charge's electric field,

$$\mathbf{E} = -\nabla\phi - (1/c)\dot{\mathbf{A}} = \frac{e\gamma((x-vt),y,z)}{(\gamma^2(x-vt)^2+y^2+z^2)^{\frac{3}{2}}} = \frac{e\gamma((x-vt),y,z)}{|\mathbf{R}(x,y,z,t;v,\gamma)|^3}. \tag{9d}$$

Comparison of Eq. (9d) with Eq. (9c) shows that $(\mathbf{E}\cdot\mathbf{B})$ *always vanishes identically*. This interesting result is related to exact *non-Galilean* space-time transformation properties of the electromagnetic fields; we will return to this topic at greater length further on.

We note that $\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}$ is very close to unity when $(v/c)^2 \ll 1$, so under those circumstances Eq. (9d) shows that the electric field $\mathbf{E}$ is nearly spherically symmetric around the instantaneous location $\mathbf{v}t = (vt,0,0)$ of the constant-velocity point charge. However, as $(v/c)^2 \to 1$, γ can become *arbitrarily large*, which can *markedly distort the electric field* $\mathbf{E}$ *away from spherical symmetry*, weakening it along the line of motion of the point charge relative to its strength *perpendicular* to that line of motion.

We note that *the key entity which underlies the character of both the electric field* $\mathbf{E}$ *of* Eq. (9d) *and the electric potential* ϕ *of* Eq. (9a) *is the constant-rate translating, radius-like vector field* $\mathbf{R}(x,y,z,t;v,\gamma) = (\gamma(x-vt),y,z)$ *of* Eq. (8a), *which is distorted away from spherical symmetry (around the time-varying location* $(\gamma vt,0,0)$*) along its line of motion*. A crucial property of $(\gamma(x-vt),y,z)$ is that when $(v/c)^2 \ll 1$, so that γ is very close to unity, *it reduces to the three spatial components* $\mathbf{r}' = (x',y',z') = ((x-vt),y,z) = (\mathbf{r}-\mathbf{v}t)$ *of the space-time Galilean transformation which is given by* Eq. (1) (bearing in mind that we have meantime *specialized* to a Cartesian coordinate system where $\mathbf{v} = (v,0,0)$, so that $(\mathbf{r}-\mathbf{v}t) = (x,y,z) - (vt,0,0) = ((x-vt),y,z)$—see the discussion below Eq. (7b)).

However, although $(\gamma(x-vt),y,z)$ *reduces* to the three spatial components $(x',y',z') = ((x-vt),y,z)$ of the space-time Galilean transformation when $(v/c)^2 \ll 1$, $(\gamma(x-vt),y,z)$ becomes *vastly different* from $((x-vt),y,z)$ as $(v/c)^2 \to 1$. Thus the Galilean transformation becomes *less and less consistent* with electromagnetic theory as the transformation speed $|v|$ approaches the universal constant c of electromagnetic theory.

It is therefore clear that the Galilean transformation must be *replaced* by a transformation which has the property that,

$$(x',y',z') = (\gamma(x-vt),\ y,\ z), \tag{10a}$$

where, of course,

$$\gamma \stackrel{\text{def}}{=} (1-(v/c)^2)^{-\frac{1}{2}}. \tag{10b}$$

The *three* requirements of Eq. (10a) for the replacement transformation for the Galilean transformation *aren't sufficient to determine* t' *in terms of* t *and* x. However, the Galilean transformation, as given by Eq. (1) with $\mathbf{v} = (v,0,0)$, namely,

$$(x',y',z',t') = ((x-vt),\ y,\ z,\ t),$$

is readily verified to have the *inverse*,

$$(x,y,z,t) = ((x'+vt'),\ y',\ z',\ t').$$

Thus the *inverse* of the Galilean transformation is the *same* as the Galilean transformation itself *except that* v *is replaced by* $-v$. That fact is completely sensible from a physical standpoint; indeed, from a physical standpoint it is well-nigh *inconceivable* that replacing v by $-v$ *wouldn't* invert the transformation. There is no reason which is apparent to doubt *that the correct replacement transformation for the Galilean transformation is as well inverted by replacing v by $-v$.* Therefore in *addition* to Eqs. (10a) and (10b), we *also* expect the correct replacement transformation to satisfy,

$$(x, y, z) = (\gamma(x' + vt'),\ y',\ z'), \tag{10c}$$

which takes into account the fact that γ *doesn't change* when v is replaced by $-v$, as is seen from Eq. (10b).

In consequence of Eq. (10c) we obtain that $x' = -vt' + x/\gamma$, while, of course, Eq. (10a) states that $x' = \gamma(x - vt)$. We can therefore *deduce* that $-vt' + x/\gamma = \gamma(-vt + x)$, and from *that* we uniquely work out t' in terms of t and x, which is exactly what we *need* to fully specify the definition of the *replacement transformation* for the *electromagnetically invalid* Galilean transformation. The *result* of thus working out t' is,

$$t' = \gamma\left(t - (x/v)\left(1 - \left(1/\gamma^2\right)\right)\right) = \gamma\left(t - \left(vx/c^2\right)\right),$$

where the last equality follows from Eq. (10b).

We combine the above result for t' in terms of t and x with Eq. (10a) to obtain the full expression for the replacement transformation,

$$(x', y', z', t') = \left(\gamma(x - vt),\ y,\ z,\ \gamma\left(t - \left(vx/c^2\right)\right)\right), \tag{11a}$$

where,

$$\gamma \stackrel{\text{def}}{=} (1 - (v/c)^2)^{-\frac{1}{2}}. \tag{11b}$$

The replacement transformation given by Eqs. (11a) and (11b) above is, of course, *the Lorentz transformation which corresponds to the constant velocity* $\mathbf{v} = (v, 0, 0)$ [2]. Note that the Lorentz transformations of constant velocity $\mathbf{v} = (v, 0, 0)$ defined by Eqs. (11a) and (11b) are physically viable *only* for $|v| < c$, *so we always assume, even when we don't explicitly mention it, that $|v| < c$ when we discuss the Lorentz transformations of constant velocity* $\mathbf{v} = (v, 0, 0)$ *defined by* Eqs. (11a) *and* (11b).

It is readily seen that in the limit $(v/c) \to 0$ the Lorentz transformations of constant velocity $\mathbf{v} = (v, 0, 0)$ of Eqs. (11a) and (11b) go over into the Galilean transformations of Eq. (1), namely they go over into,

$$(\mathbf{r}', t') = (x', y', z', t') = ((x - vt),\ y,\ z,\ t) = ((\mathbf{r} - \mathbf{v}t),\ t),$$

where $\mathbf{r}' = (x', y', z')$, $\mathbf{r} = (x, y, z)$ and $\mathbf{v} = (v, 0, 0)$.

Because of the Eq. (10c) condition which was imposed on them, the Lorentz transformations of constant velocity $\mathbf{v} = (v, 0, 0)$ of Eqs. (11a) and (11b) are *inverted* by replacing v by $-v$, so *every* Lorentz transformation of constant velocity $\mathbf{v} = (v, 0, 0)$ defined by Eqs. (11a) and (11b) has the *inverse*,

$$(x, y, z, t) = \left(\gamma(x' + vt'),\ y',\ z',\ \gamma\left(t' + \left(vx'/c^2\right)\right)\right), \tag{11c}$$

which of course *is identical in form to the Lorentz transformation of constant velocity* $\mathbf{v} = (-v, 0, 0)$.

Also of interest is $(d\mathbf{r}'/dt')$, *the Lorentz transformation of velocity* $(d\mathbf{r}/dt)$ *that arises from the* Eq. (11a) *space-time Lorentz transformation, namely*,

$$(d\mathbf{r}'/dt') = (d\mathbf{r}'/dt)\,/(dt'/dt) = ((dx'/dt),\ (dy'/dt),\ (dz'/dt))\,/(dt'/dt). \tag{11d}$$

Eq. (11d) shows that obtaining the *velocity* Lorentz transformation $(d\mathbf{r}'/dt')$ that arises from the Eq. (11a) *space-time* Lorentz transformation inherently involves *two steps*: the *first step* is the calculation of *the derivative with respect to t of the space-time Lorentz transformation* $(\mathbf{r}',\ t') = (x',\ y',\ z',\ t')$, namely,

$$\begin{aligned}(d\,(\mathbf{r}',\ t')\,/dt) &= ((d\mathbf{r}'/dt)\,,\ (dt'/dt)) = (d(x',\ y',\ z',\ t')/dt) = ((dx'/dt),\ (dy'/dt),\ (dz'/dt),\ (dt'/dt)) \\ &= \left(\gamma((dx/dt) - v),\ (dy/dt),\ (dz/dt),\ \gamma\left(1 - \left(v/c^2\right)(dx/dt)\right)\right),\end{aligned} \tag{11e}$$

and *the second step* is to *divide* the three-vector "space-rate" part of that derivative by its "time-rate" part,

$$\begin{aligned}(d\mathbf{r}'/dt') &= (d\mathbf{r}'/dt)\,/(dt'/dt) = ((dx'/dt),\ (dy'/dt),\ (dz'/dt))\,/(dt'/dt) \\ &= \frac{(\gamma((dx/dt) - v),\ (dy/dt),\ (dz/dt))}{\gamma\left(1 - \left(v/c^2\right)(dx/dt)\right)} = \frac{((\gamma(dx/dt) + (-v/c)\gamma(c)),\ (dy/dt),\ (dz/dt))}{(1/c)(\gamma(c) + (-v/c)\gamma(dx/dt))},\end{aligned} \tag{11f}$$

where *the final rendition* in Eq. (11f) of the $(d\mathbf{r}'/dt')$ that arises from the Eq. (11a) space-time Lorentz transformation *has been reexpressed to accord with a notation convention that is useful further on*.

It is to be noted that because $|dx/dt| \le |d\mathbf{r}/dt|$ and $|v| < c$, *the denominators in the* Eq. (11f) *renditions of the* $(d\mathbf{r}'/dt')$ *that arises from* Eq. (11a) *don't vanish if* $|d\mathbf{r}/dt| \le c$, which is *necessary* if $(d\mathbf{r}'/dt')$ *is to make physical sense*.

In the *limit* $(v/c) \to 0$, the Eq. (11f) *velocity* Lorentz transformation $(d\mathbf{r}'/dt')$ that arises from Eq. (11a) *goes over into the velocity Galilean transformation*, namely into,

$$(d\mathbf{r}'/dt') = (((dx/dt) - v),\ (dy/dt),\ (dz/dt)) = ((d\mathbf{r}/dt) - \mathbf{v}),$$

where $(d\mathbf{r}/dt) = ((dx/dt),\ (dy/dt),\ (dz/dt))$, and of course $\mathbf{v} = (v, 0, 0)$.

A *key property* of the *velocity* Lorentz transformation $(d\mathbf{r}'/dt')$ that arises from Eq. (11a) is that *if the velocity magnitude* $|d\mathbf{r}/dt|$ *is equal to c, then the transformed velocity magnitude* $|d\mathbf{r}'/dt'|$ *is also equal to c for any value whatsoever of v which satisfies the fundamental restriction* $|v| < c$. We show this by using Eq. (11f) together with the fact that $\gamma^2(1 - (v/c)^2) = 1$ to calculate $(c^2 - |d\mathbf{r}'/dt'|^2)$,

$$\left(c^2 - |d\mathbf{r}'/dt'|^2\right) = \frac{(\gamma(c) + (-v/c)\gamma(dx/dt))^2 - (\gamma(dx/dt) + (-v/c)\gamma(c))^2 - (dy/dt)^2 - (dz/dt)^2}{((1/c)(\gamma(c) + (-v/c)\gamma(dx/dt)))^2} =$$
$$\frac{c^2 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2}{((1/c)(\gamma(c) + (-v/c)\gamma(dx/dt)))^2} = \frac{\left(c^2 - |d\mathbf{r}/dt|^2\right)}{((1/c)(\gamma(c) + (-v/c)\gamma(dx/dt)))^2}, \tag{11g}$$

which makes it apparent that $|d\mathbf{r}/dt| = c$ implies that $|d\mathbf{r}'/dt'| = c$ for all v which satisfy $|v| < c$.

Therefore it is true in particular that photons, the quanta of electromagnetic radiation, *always have speed c irrespective of what the velocity parameter v of their source happens to be* (provided, of course, that $|v| < c$). This *universal speed c of the quanta of electromagnetic radiation* is the *reason* that a Michelson-Morley type of experiment *necessarily produces null readings for the hypothetical "aether wind"* [3].

Eq. (11a) and the fact that $\gamma^2(1 - (v/c)^2) = 1$ *furthermore imply that so long as* $|v| < c$, $(ct')^2 - (x')^2 - (y')^2 - (z')^2 = (ct)^2 - x^2 - y^2 - z^2$, which is shown in detail as follows,

$$(ct')^2 - (x')^2 - (y')^2 - (z')^2 = \gamma^2(ct - (v/c)x)^2 - \gamma^2(x - vt)^2 - (y)^2 - (z)^2 = (ct)^2 - x^2 - y^2 - z^2. \tag{12a}$$

The *entire set of homogeneous linear transformations of* (x, y, z, t) *which satisfy*,

$$(ct')^2 - (x')^2 - (y')^2 - (z')^2 = (ct)^2 - x^2 - y^2 - z^2. \tag{12b}$$

comprises what is known as the homogeneous Lorentz-transformation group [4], of which the constant-velocity $\mathbf{v} = (v, 0, 0)$ space-time Lorentz transformations with $|v| < c$ that are defined by Eqs. (11a) and (11b) *is of course a subset*.

The set of homogeneous linear transformations which preserve the quadratic form $(ct)^2 - x^2 - y^2 - z^2$, as is set out by Eq. (12b), obviously includes those which leave time t invariant and preserve the quadratic form $x^2 + y^2 + z^2$. Thus the homogeneous Lorentz transformations include *ordinary three-dimensional spatial rotations* as well as the constant-velocity $\mathbf{v} = (v, 0, 0)$ Lorentz-transformation "boosts" described by Eqs. (11a) and (11b). The two simple *discrete* linear homogeneous transformations of *time reversal*, namely $t \to -t$, and *parity inversion*, namely $(x, y, z) \to (-x, -y, -z)$, *also clearly preserve the quadratic form* $(ct)^2 - x^2 - y^2 - z^2$, so the homogeneous Lorentz transformations include these as well. Being a *group*, the homogeneous Lorentz transformations include *any product* of constant-velocity Lorentz-transformation "boosts", three-dimensional spatial rotations, time reversals and parity inversions.

In order *to describe in detail* the characteristics *of the matrix elements* of a *general* homogeneous linear transformation which preserves the quadratic form $(ct)^2 - x^2 - y^2 - z^2$, it is very convenient to swap the notation (x, y, z, t) for the standard four-vector notation x^α, where $x^0 \stackrel{\text{def}}{=} ct$, $x^1 \stackrel{\text{def}}{=} x$, $x^2 \stackrel{\text{def}}{=} y$ and $x^3 \stackrel{\text{def}}{=} z$, and to as well swap the notation (x', y', z', t') for the four-vector notation $(x')^\mu$, where $(x')^0 \stackrel{\text{def}}{=} ct'$, $(x')^1 \stackrel{\text{def}}{=} x'$, $(x')^2 \stackrel{\text{def}}{=} y'$, and $(x')^3 \stackrel{\text{def}}{=} z'$. If we now introduce the sixteen (arbitrary) dimensionless *matrix elements* Λ^σ_γ, where $\sigma, \gamma = 0, 1, 2, 3$, the fact that the space-time point (x', y', z', t') is an (arbitrary) homogeneous linear transformation of the space-time point (x, y, z, t) is conveyed by the compact expression,

$$(x')^\mu = \Lambda^\mu_\alpha x^\alpha, \tag{13a}$$

where $\mu = 0, 1, 2, 3$, and we have adopted the Einstein convention *that repeated indices are automatically assumed to be summed over* (namely "contracted"), *provided that one is a subscript index* (namely a "covariant" index) *and the other is a superscript index* (namely a "contravariant" index).

The sixteen dimensionless matrix elements Λ^σ_γ, where $\sigma, \gamma = 0, 1, 2, 3$, *can't*, in fact, *be entirely arbitrary* because *they must ensure* that a *transformed* space-time point $(x')^\mu$, where $\mu = 0, 1, 2, 3$, which is given by Eq. (13a) *accords with the quadratic-form preservation requirement that is set out in* Eq. (12b), *namely that*,

$$\left((x')^0\right)^2 - \left((x')^1\right)^2 - \left((x')^2\right)^2 - \left((x')^3\right)^2 = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2,$$

for any arbitrary space-time point x^α, where $\alpha = 0, 1, 2, 3$.

To obtain *the consequences, within the context of the Einstein index convention which we have adopted, for the sixteen dimensionless matrix elements* Λ^σ_γ, *where* $\sigma, \gamma = 0, 1, 2, 3$, *of the above quadratic-form preservation requirement that is imposed on* Eq. (13a), we introduce the fixed-value 4×4 diagonal matrix (i.e., the fixed-value *second-rank "covariant" symmetric tensor*) $\eta_{\kappa\lambda}$, where $\kappa, \lambda = 0, 1, 2, 3$, which is defined as follows,

$$\eta_{00} \stackrel{\text{def}}{=} 1,\ \eta_{11} \stackrel{\text{def}}{=} -1,\ \eta_{22} \stackrel{\text{def}}{=} -1,\ \eta_{33} \stackrel{\text{def}}{=} -1 \text{ and } \eta_{\kappa\lambda} \stackrel{\text{def}}{=} 0 \text{ when } \kappa \neq \lambda. \tag{13b}$$

This fixed-value second-rank symmetric (in fact diagonal) "covariant" (namely subscript-indexed) tensor $\eta_{\kappa\lambda}$, where $\kappa, \lambda = 0, 1, 2, 3$, *enables us to express, within the context of the Einstein index convention, the quadratic-form preservation requirement stated above as the following equality*,

$$(x')^\mu \eta_{\mu\nu} (x')^\nu = x^\alpha \eta_{\alpha\beta} x^\beta. \tag{13c}$$

Insertion of Eq. (13a) into Eq. (13c) produces,

$$x^\alpha \Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta x^\beta = x^\alpha \eta_{\alpha\beta} x^\beta,$$

which, because the x^α, *where* $\alpha = 0, 1, 2, 3$, *are arbitrary, imposes the sixteen equations,*

$$\Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta = \eta_{\alpha\beta}, \tag{13d}$$

where $\alpha, \beta = 0, 1, 2, 3$, *on the sixteen matrix elements* Λ^σ_γ, where $\sigma, \gamma = 0, 1, 2, 3$. Eq. (13d), in conjunction with the Eq. (13b) definition of $\eta_{\alpha\beta}$, where $\alpha, \beta = 0, 1, 2, 3$, is the necessary and sufficient condition for a 4×4 dimensionless matrix Λ^σ_γ, where $\sigma, \gamma = 0, 1, 2, 3$, to produce a *general* homogeneous Lorentz transformation via Eq. (13a), namely via $(x')^\mu = \Lambda^\mu_\alpha x^\alpha$, where $\mu = 0, 1, 2, 3$.

The *form* of Eq. (13d) tells us that a *general* homogeneous Lorentz-transformation dimensionless matrix Λ^σ_γ, where $\sigma, \gamma = 0, 1, 2, 3$, *preserves the fixed-value second-rank symmetric (in fact diagonal) covariant tensor* $\eta_{\kappa\lambda}$, where $\kappa, \lambda = 0, 1, 2, 3$, which is defined above by Eq. (13b).

However, because the definition of $\eta_{\kappa\lambda}$ given by Eq. (13b) implies that $\eta_{\kappa\lambda} = \eta_{\lambda\kappa}$, where $\kappa, \lambda = 0, 1, 2, 3$, it is readily seen that *six* of the sixteen equations which are given by Eq. (13d) *are in fact redundant*. Indeed, the six Eq. (13d) equations for which $\alpha > \beta$ *merely repeat the six* Eq. (13d) *equations for which* $\beta > \alpha$.

Therefore the sixteen dimensionless matrix elements Λ^σ_γ, where $\sigma, \gamma = 0, 1, 2, 3$, which comprise a general homogeneous Lorentz-transformation 4×4 matrix *necessarily* can be expressed *in terms of six continuously-variable dimensionless parameters*, just as the nine dimensionless matrix elements which comprise a three-dimensional rotation matrix *necessarily* can be expressed *in terms of three continuously-variable angle parameters, such as the three Euler angles*.

To work out the *consequences* of Eq. (13d), it is extremely useful to *as well* express it in *non-indexed* 4×4 matrix notation, namely as,

$$\Lambda^{\mathrm{T}} \eta \Lambda = \eta. \tag{13e}$$

To begin with, the Eq. (13e) form of the necessary and sufficient condition for Λ to be a general homogeneous Lorentz transformation 4×4 matrix considerably facilitates the demonstration that these 4×4 matrices *do in fact comprise a group*. It is, for example, apparent from Eq. (13e) that the 4×4 *identity matrix* I is indeed a general homogeneous Lorentz transformation. Given any two general homogeneous Lorentz transformation 4×4 matrices Λ_1 and Λ_2, which of course satisfy,

$$\Lambda_1^{\mathrm{T}} \eta \Lambda_1 = \eta \ \text{ and } \ \Lambda_2^{\mathrm{T}} \eta \Lambda_2 = \eta,$$

it is readily seen that,

$$\Lambda_2^{\mathrm{T}} \left(\Lambda_1^{\mathrm{T}} \eta \Lambda_1\right) \Lambda_2 = \Lambda_2^{\mathrm{T}} \eta \Lambda_2 = \eta,$$

and therefore, since,

$$\Lambda_2^{\mathrm{T}}\Lambda_1^{\mathrm{T}} = (\Lambda_1\Lambda_2)^{\mathrm{T}}\,,$$

it is true that,

$$(\Lambda_1\Lambda_2)^{\mathrm{T}}\,\eta\,(\Lambda_1\Lambda_2) = \eta,$$

and therefore the *product* $(\Lambda_1\Lambda_2)$ of any two general homogeneous Lorentz transformation 4×4 matrices Λ_1 and Λ_2 is itself a general homogeneous Lorentz transformation 4×4 matrix.

To show that the *inverse* of a general homogeneous Lorentz transformation 4×4 matrix Λ is itself a general homogeneous Lorentz transformation 4×4 matrix, *we begin by showing that* Λ^{-1} *always exists.* Multiplying Eq. (13e) from the left by η produces,

$$\left(\eta\Lambda^{\mathrm{T}}\eta\right)\Lambda = \eta^2 = \mathrm{I},$$

where the second equality follows immediately from the Eq. (13b) *definition* of η. This result implies that,

$$\Lambda^{-1} = \eta\Lambda^{\mathrm{T}}\eta,$$

and since $\Lambda\Lambda^{-1} = \mathrm{I}$,

$$\Lambda\eta\Lambda^{\mathrm{T}}\eta = \mathrm{I}.$$

Multiplying this equality from the right by η and using the fact that $\eta^2 = \mathrm{I}$ yields the useful *lemma*,

$$\Lambda\eta\Lambda^{\mathrm{T}} = \eta, \tag{13f}$$

namely that *if* Λ *is a general homogeneous Lorentz transformation* 4×4 *matrix, then so is its transpose* Λ^{T}.

We now multiply Eq. (13f) from both the left and from from the right by η. Combining the result of that multiplication with the fact that $\eta^3 = \eta$ (which follows from $\eta^2 = \mathrm{I}$) permits us to write down the relation,

$$(\eta\Lambda\eta)\eta\left(\eta\Lambda^{\mathrm{T}}\eta\right) = \eta.$$

Since from the Eq. (13b) definition of η we see that η is *symmetric*, namely that $\eta^{\mathrm{T}} = \eta$, we can rewrite the above relationship as,

$$\left(\eta\Lambda^{\mathrm{T}}\eta\right)^{\mathrm{T}}\eta\left(\eta\Lambda^{\mathrm{T}}\eta\right) = \eta,$$

which, since we have shown that $\left(\eta\Lambda^{\mathrm{T}}\eta\right) = \Lambda^{-1}$, can in turn be rewritten,

$$\left(\Lambda^{-1}\right)^{\mathrm{T}}\eta\Lambda^{-1} = \eta.$$

This, together with Eq. (13e), shows that if Λ is a general homogeneous Lorentz transformation 4×4 matrix, then so is Λ^{-1}, *completing the demonstration that the general homogeneous Lorentz transformation* 4×4 *matrices described by* Eq. (13e) *comprise a group*.

We next study the *velocity* Lorentz transformations $(d\mathbf{r}'/dt')$ that arise from *general* space-time Lorentz transformations *of the homogeneous Lorentz group*. This will result in a massive *extension* of the Eq. (11f) *velocity* Lorentz transformations because *those particular* $(d\mathbf{r}'/dt')$ arise *exclusively* from Eq. (11a) constant-velocity $\mathbf{v} = (v, 0, 0)$ pure "boost" space-time Lorentz transformations.

Eq. (11d) shows that a *velocity* Lorentz transformation $(d\mathbf{r}'/dt')$ is by its nature expected to have a *denominator* factor. The *vanishing* of the denominator factor of a *velocity* Lorentz transformation *wouldn't* be physically acceptable; fortunately we have seen that the *particular* denominator factor of the Eq. (11f) $(d\mathbf{r}'/dt')$ *doesn't vanish* if $|d\mathbf{r}/dt| \le c$. This turns out to be the case *as well* for the denominator factors of the $(d\mathbf{r}'/dt')$ that arise from *general* space-time Lorentz transformations of the homogeneous Lorentz group, but *demonstrating* that fact requires two little-known lemmas regarding the properties of the general homogeneous Lorentz-transformation Λ matrices. The first lemma required is Eq. (13f), namely that the *transpose* Λ^{T} of such a Λ matrix is *itself* such a general homogeneous Lorentz-transformation matrix. The *second* lemma required is *the transpose modification* of *a particular one* of the sixteen equations of Eq. (13d) that are satisfied by *all* general homogeneous Lorentz-transformation Λ matrices.

We now derive this second lemma. Eq. (13d) is of course, $\Lambda^{\mu}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}_{\beta} = \eta_{\alpha\beta}$, where $\alpha, \beta = 0, 1, 2, 3$, and $\eta_{\alpha\beta}$ is defined by Eq. (13b). Expanding out the $\mu\nu$ summation implicit in Eq. (13d) produces,

$$\left(\Lambda^0_\alpha\Lambda^0_\beta - \Lambda^1_\alpha\Lambda^1_\beta - \Lambda^2_\alpha\Lambda^2_\beta - \Lambda^3_\alpha\Lambda^3_\beta\right) = \eta_{\alpha\beta}. \tag{13g}$$

The *particular* equation we need of the sixteen of Eq. (13g) is the one with $\alpha = \beta = 0$, i.e.,

$$\left(\Lambda^0_0\right)^2 - \left(\Lambda^1_0\right)^2 - \left(\Lambda^2_0\right)^2 - \left(\Lambda^3_0\right)^2 = 1,$$

in which, as allowed by the Eq. (13f) *transpose* lemma, we replace all occurrences of Λ by Λ^{T}, namely,

$$\left((\Lambda^{\mathrm{T}})^0_0\right)^2 - \left((\Lambda^{\mathrm{T}})^1_0\right)^2 - \left((\Lambda^{\mathrm{T}})^2_0\right)^2 - \left((\Lambda^{\mathrm{T}})^3_0\right)^2 = 1,$$

with the result that,

$$\left(\Lambda^0_0\right)^2 - \left(\Lambda^0_1\right)^2 - \left(\Lambda^0_2\right)^2 - \left(\Lambda^0_3\right)^2 = 1. \tag{13h}$$

Eq. (13h) will be key to showing that *the denominator factors of general* $(d\mathbf{r}'/dt')$ *don't vanish if* $|d\mathbf{r}/dt| \le c$.

We now work out the $(d\mathbf{r}'/dt')$ that arises from a *general homogeneous space-time Lorentz transformation* $(x')^\mu = \Lambda^\mu_\alpha x^\alpha$, where Λ^μ_α conforms to the necessary and sufficient condition given by Eq. (13d) (or *equivalently* to the 4×4 matrix condition given by Eq. (13e)), and where the *fixed* 4×4 matrix η which occurs in the Eq. (13d) and Eq. (13e) conditions *is defined by* Eq. (13b). Emulating *the two steps that are set out before and after* Eq. (11e), we *first* differentiate both sides of $(x')^\mu = \Lambda^\mu_\alpha x^\alpha$ with respect to t, which produces,

$$(d(x')^\mu/dt) = \Lambda^\mu_\alpha\,(dx^\alpha/dt)\,, \tag{14a}$$

where $\mu = 0, 1, 2, 3$. Since by definition $(x')^0 = ct'$, the equality of the $\mu = 0$ component of the left-hand side of Eq. (14a) to the $\mu = 0$ component of its right-hand side yields,

$$(dt'/dt) = (1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)\,. \tag{14b}$$

The *second step* is *dividing* the left-hand side of Eq. (14a) by the left-hand side of Eq. (14b), and *correspondingly dividing* the right-hand side of Eq. (14a) by the right-hand side of Eq. (14b). Since,

$$\left(d\,(x')^0\,/dt\right) = c\,(dt'/dt)\,,\quad \left(d\left((x')^1\,,\ (x')^2\,,\ (x')^3\right)/dt\right) = \left(d\,(x',\ y',\ z')\,/dt\right) = (d\mathbf{r}'/dt)\,,$$

and $(d\mathbf{r}'/dt)\,/\,(dt'/dt) = (d\mathbf{r}'/dt')$, the result of dividing each side of Eq. (14a) by the corresponding side of Eq. (14b) can, in an explicitly four-component notation (which isn't Lorentz-covariant), be expressed as,

$$(c,\ (d\mathbf{r}'/dt')) = \left(c,\ \frac{\boldsymbol{\Lambda}_\alpha\,(dx^\alpha/dt)}{(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)}\right), \tag{14c}$$

where the indexed three-vector Lorentz-transformation entity $\boldsymbol{\Lambda}_\alpha$ is defined as,

$$\boldsymbol{\Lambda}_\alpha \overset{\text{def}}{=} \left(\Lambda^1_\alpha,\ \Lambda^2_\alpha,\ \Lambda^3_\alpha\right), \tag{14d}$$

with the domain of the index α being the four integer values 0, 1, 2 and 3.

The equality of the rightmost three components of the left-hand side of Eq. (14c) *to the corresponding three components* of the right-hand side of Eq. (14c), namely,

$$(d\mathbf{r}'/dt') = \frac{\boldsymbol{\Lambda}_\alpha\,(dx^\alpha/dt)}{(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)}, \tag{14e}$$

comprises the general *velocity* Lorentz transformation *that arises from the general homogeneous space-time Lorentz transformation* $(x')^\mu = \Lambda^\mu_\alpha x^\alpha$, where $\boldsymbol{\Lambda}_\alpha$ is defined in terms of Λ^μ_α by Eq. (14d) and where,

$$t \overset{\text{def}}{=} x^0/c,\quad \mathbf{r} = (x,\ y,\ z) \overset{\text{def}}{=} (x^1,\ x^2,\ x^3),\quad t' \overset{\text{def}}{=} (x')^0/c \quad\text{and}\quad \mathbf{r}' = (x',\ y',\ z') \overset{\text{def}}{=} ((x')^1,\ (x')^2,\ (x')^3).$$

Eq. (11f) is *the special case* of Eq. (14e) which occurs for the Eq. (11a) constant-velocity $\mathbf{v} = (v, 0, 0)$ pure "boost" space-time Lorentz transformation. For that *special case*, $\Lambda^0_0 = \Lambda^1_1 = \gamma$, $\Lambda^0_1 = \Lambda^1_0 = (-v/c)\gamma$, $\Lambda^2_2 = \Lambda^3_3 = 1$, and the remaining ten Λ^μ_α are equal to zero, as comparing Eq. (11f) to Eq. (14e) confirms.

Vanishing of the denominator factor $(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)$ of the Eq. (14e) general velocity Lorentz transformation $(d\mathbf{r}'/dt')$ *wouldn't be physically acceptable*. Fortunately, if $|d\mathbf{r}/dt| \le c$, we can show with the aid of Eq. (13h) that $(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)$ *doesn't vanish*.

We begin the demonstration that $(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)$ doesn't vanish if $|d\mathbf{r}/dt| \le c$ by expanding out its implicit four-term summation over the index α,

$$(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt) = \Lambda^0_0 + (1/c)\left[(dx/dt)\Lambda^0_1 + (dy/dt)\Lambda^0_2 + (dz/dt)\Lambda^0_3\right]. \tag{15a}$$

We can put *an upper bound* on the expression $\left|(1/c)\left[(dx/dt)\Lambda^0_1 + (dy/dt)\Lambda^0_2 + (dz/dt)\Lambda^0_3\right]\right|$ by applying the Schwarz inequality to the three-term inner product within its square brackets,

$$\begin{gathered}\left|(1/c)\left[(dx/dt)\Lambda^0_1 + (dy/dt)\Lambda^0_2 + (dz/dt)\Lambda^0_3\right]\right| \le \\ (1/c)\left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\right)^{\frac{1}{2}}\left(\left(\Lambda^0_1\right)^2 + \left(\Lambda^0_2\right)^2 + \left(\Lambda^0_3\right)^2\right)^{\frac{1}{2}}.\end{gathered} \tag{15b}$$

Since $\left((dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2\right)^{\frac{1}{2}} = |d\mathbf{r}/dt|$, Eq. (15b) implies that,

$$\left|(1/c)\left[(dx/dt)\Lambda^0_1 + (dy/dt)\Lambda^0_2 + (dz/dt)\Lambda^0_3\right]\right| \le \left(\left(\Lambda^0_1\right)^2 + \left(\Lambda^0_2\right)^2 + \left(\Lambda^0_3\right)^2\right)^{\frac{1}{2}} \text{ if } |d\mathbf{r}/dt| \le c. \tag{15c}$$

From Eq. (13h) we *furthermore* see that,

$$\left|\Lambda^0_0\right| = \left(1 + \left(\Lambda^0_1\right)^2 + \left(\Lambda^0_2\right)^2 + \left(\Lambda^0_3\right)^2\right)^{\frac{1}{2}} > \left(\left(\Lambda^0_1\right)^2 + \left(\Lambda^0_2\right)^2 + \left(\Lambda^0_3\right)^2\right)^{\frac{1}{2}}. \tag{15d}$$

Therefore from Eqs. (15d) and (15c) it follows that,

$$\left|\Lambda^0_0\right| > \left|(1/c)\left[(dx/dt)\Lambda^0_1 + (dy/dt)\Lambda^0_2 + (dz/dt)\Lambda^0_3\right]\right| \text{ if } |d\mathbf{r}/dt| \le c, \tag{15e}$$

which together with Eq. (15a) implies that $(1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)$ *doesn't vanish if* $|d\mathbf{r}/dt| \le c$.

We now turn to the *demonstration* of the *key theorem* that $|d\mathbf{r}/dt| = c$ implies that $|d\mathbf{r}'/dt'| = c$. In Eq. (11g) we demonstrated that theorem for the Eq. (11f) special case of Eq. (14e). As was done in Eq. (11g), we calculate $(c^2 - |d\mathbf{r}'/dt'|^2)$. It is straightforwardly seen from Eqs. (14e) and (14d) that,

$$\left(c^2 - |d\mathbf{r}'/dt'|^2\right) = \frac{(dx^\alpha/dt)\left(\Lambda^0_\alpha\Lambda^0_\beta - \Lambda^1_\alpha\Lambda^1_\beta - \Lambda^2_\alpha\Lambda^2_\beta - \Lambda^3_\alpha\Lambda^3_\beta\right)(dx^\beta/dt)}{\left((1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)\right)^2}. \tag{16a}$$

Eq. (13g) of course makes it immediately apparent that,

$$\left(\Lambda^0_\alpha\Lambda^0_\beta - \Lambda^1_\alpha\Lambda^1_\beta - \Lambda^2_\alpha\Lambda^2_\beta - \Lambda^3_\alpha\Lambda^3_\beta\right) = \eta_{\alpha\beta}, \tag{16b}$$

where $\alpha, \beta = 0, 1, 2, 3$ and $\eta_{\alpha\beta}$ is defined by Eq. (13b). Inserting this simplification into Eq. (16a) produces,

$$\left(c^2 - |d\mathbf{r}'/dt'|^2\right) = \frac{(dx^\alpha/dt)\,\eta_{\alpha\beta}\,(dx^\beta/dt)}{\left((1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)\right)^2} = \frac{\left(c^2 - |d\mathbf{r}/dt|^2\right)}{\left((1/c)\Lambda^0_\alpha\,(dx^\alpha/dt)\right)^2}, \tag{16c}$$

where the *second* equality of Eq. (16c) follows from the Eq. (13b) definition of $\eta_{\alpha\beta}$ together with the properties of (dx^α/dt). In particular, since $\eta_{\alpha\beta} = 0$ when $\alpha \ne \beta$, $\left((dx^\alpha/dt)\,\eta_{\alpha\beta}\,(dx^\beta/dt)\right)$ reduces to,

$$\left(\eta_{00}\left(dx^0/dt\right)^2 + \sum_{i=1}^{3}\eta_{ii}\left(dx^i/dt\right)^2\right) = \left((d(ct)/dt)^2 - (dx/dt)^2 - (dy/dt)^2 - (dz/dt)^2\right) = \left(c^2 - |d\mathbf{r}/dt|^2\right).$$

Eq. (16c) *manifestly* implies the *theorem* that if $|d\mathbf{r}/dt| = c$, then $|d\mathbf{r}'/dt'| = c$.

We next turn to the matter of obtaining a formal solution for the 4×4 matrices of the homogeneous Lorentz group in terms of six continuously-variable dimensionless parameters, i.e., of "solving" the sixfold indeterminate Eq. (13d) equation system. To *orient our thinking* with regard to such a seemingly ambiguous and daunting task, we *first* explore *the simpler case of the orthogonal group in three dimensions*, which is called O(3). The analog of Eq. (13e) for O(3) is the 3×3 dimensionless matrix equation,

$$\Xi^{\mathrm{T}}\Xi = \mathrm{I},$$

which *in fully indexed form analogous to* Eq. (13d) is,

$$\sum_{j=1}^{3} \Xi_i^j \, \Xi_k^j = \delta_{ik}, \text{ where } i, k = 1, 2, 3,$$

whose $i \leftrightarrow k$ *index symmetry* makes these nine equations for the nine matrix elements of Ξ *threefold indeterminate*, so the general "solution" for Ξ has three continuously-variable parameters.

If we now try to express Ξ as an *exponential* of a 3×3 matrix J, the equation $\Xi^{\mathrm{T}}\Xi = \mathrm{I}$ becomes,

$$\exp(J^{\mathrm{T}}) \exp(J) = \mathrm{I},$$

which is clearly *satisfied* if the dimensionless matrix J is *antisymmetric*, namely if,

$$J^{\mathrm{T}} = -J.$$

A *general dimensionless* 3×3 *antisymmetric matrix such as* J *can be expressed in terms of three dimensionless continuously-variable angle parameters* ϕ_1, ϕ_2 and ϕ_3 as follows,

$$J(\phi_1, \phi_2, \phi_3) = \begin{vmatrix} 0 & \phi_3 & \phi_2 \\ -\phi_3 & 0 & \phi_1 \\ -\phi_2 & -\phi_1 & 0 \end{vmatrix} = \phi_1 R_1 + \phi_2 R_2 + \phi_3 R_3,$$

where the three antisymmetric (3×3)-matrix *rotation generators* R_1, R_2 and R_3 manifestly are,

$$R_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}, \quad R_2 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix} \text{ and } R_3 = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

The dimensionless 3×3 matrix $\exp(\phi_1 R_1)$, which in detail is,

$$\exp(\phi_1 R_1) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & \sin\phi_1 \\ 0 & -\sin\phi_1 & \cos\phi_1 \end{vmatrix},$$

rotates a three-vector $\mathbf{r} = (x, y, z)$ by the angle ϕ_1 about the x-axis (namely in the y-z plane), while the dimensionless 3×3 matrices $\exp(\phi_2 R_2)$ and $\exp(\phi_3 R_3)$ *respectively* effect analogous rotations of (x, y, z) by the angle ϕ_2 about the y-axis, and by the angle ϕ_3 about the z-axis.

A straightforward formal representation of the *general* O(3) dimensionless 3×3 matrix Ξ in terms of the three continuously-variable angle parameters ϕ_1, ϕ_2 and ϕ_3 is,

$$\Xi(\phi_1, \phi_2, \phi_2, n_P) = (P)^{n_P} \exp(\phi_1 R_1 + \phi_2 R_2 + \phi_3 R_3),$$

where P denotes *the parity-inversion matrix*, namely,

$$P = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix},$$

which is readily seen to *also* be one of the *general* O(3) transformation matrices Ξ which satisfy $\Xi^{\mathrm{T}}\Xi = \mathrm{I}$. The *discrete* variable n_P in the *general* form of Ξ which is written down above ranges over *the two integer values* 1 *and* 2.

Finally, it is straightforward to verify that the closed commutator (or Lie) algebra for the three (3×3)-matrix rotation generators R_1, R_2 and R_3 which are written down above has a cyclic pattern,

$$[R_1, R_2] = R_3, \ [R_2, R_3] = R_1, \ [R_3, R_1] = R_2,$$

which evokes the cyclic pattern of the cross products of the unit vectors in the x, y and z directions.

Use of this closed commutator algebra in conjunction with the Campbell-Baker-Hausdorff decomposition theorem for the exponential of a linear combination of matrices permits the compact but opaque expression

$\exp(\phi_1 R_1 + \phi_2 R_2 + \phi_3 R_3)$ in the general form of Ξ which is written down above to be recast as a more readily comprehensible *succession* of rotations about the x, y and z axes.

The *key* to the foregoing formal solution of O(3) was *clearly* insertion of *the exponential ansatz* $\Xi = \exp(J)$ into *the non-indexed matrix equation* $\Xi^{\mathrm{T}}\Xi = \mathrm{I}$. Therefore we seek a formal solution of the homogeneous Lorentz group by inserting *the analogous exponential ansatz* $\Lambda = \exp(K)$ into *the analogous non-indexed matrix equation* $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$, which produces,

$$\exp(K^{\mathrm{T}})\eta\exp(K) = \eta. \tag{17a}$$

Since $\eta^2 = \mathrm{I}$ it is straightforward to verify that Eq. (17a) is satisfied if,

$$K^{\mathrm{T}} = -\eta K\eta. \tag{17b}$$

Indeed, given that $\eta^2 = \mathrm{I}$, Eq. (17b) is immediately obtained when Eq. (17a) *is solved through first order in* K. It is straightforward to verify, however, that the Eq. (17b) result *is in fact exactly equivalent to* Eq. (17a).

Eq. (17b) turns out *to actually hold* for *any* dimensionless 4×4 matrix K *which is antisymmetric in its space-space matrix elements, symmetric in its space-time matrix elements and has vanishing time-time matrix element*, i.e., Eq. (17b) holds for *any* dimensionless 4×4 matrix K of the form,

$$K(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3) = \begin{vmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & 0 & \phi_3 & \phi_2 \\ u_2 & -\phi_3 & 0 & \phi_1 \\ u_3 & -\phi_2 & -\phi_1 & 0 \end{vmatrix}, \tag{17c}$$

where u_1, u_2, u_3, ϕ_1, ϕ_2 and ϕ_3 are *six continuously-variable dimensionless parameters*. Together with our *ansatz* $\Lambda = \exp(K)$, Eq. (17c) formally solves Eq. (13e)—which is equivalent to the sixteen equations given by Eq. (13d)—*in complete consonance with the six-fold indeterminacy of the* Eq. (13d) *equation system that is explicitly pointed out below* Eq. (13d).

In *addition* to these solutions of the form $\Lambda = \exp(K)$, Eq. (13e) is *as well* clearly satisfied by the dimensionless 4×4 diagonal *parity-inversion matrix* P,

$$P \stackrel{\mathrm{def}}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \tag{17d}$$

whose square is the identity matrix I, and Eq. (13e) is *also* satisfied by the *time-reversal matrix* T,

$$T \stackrel{\mathrm{def}}{=} \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \tag{17e}$$

whose general properties are strongly analogous to the general properties of P. Therefore *in addition* to the six continuously-variable parameters u_1, u_2, u_3, ϕ_1, ϕ_2 and ϕ_3, the general 4×4 dimensionless matrix solution Λ of Eq. (13e) (or of Eq. (13d)) *also features two discrete parameters* n_P *and* n_T,

$$\Lambda(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3,n_P,n_T) = (T)^{n_T}(P)^{n_P}\exp(K(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3)), \tag{17f}$$

where both n_P and n_T range over the two integer values 1 and 2.

Just as was done for the dimensionless 3×3 matrix $J(\phi_1,\phi_2,\phi_3)$ which pertains to O(3), it is very useful indeed to express the dimensionless 4×4 matrix $K(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3)$ of Eq. (17c) as the parametric linear combination of its six (4×4)-matrix general homogeneous Lorentz-transformation *generators*,

$$K(u_1,u_2,u_3,\phi_1,\phi_2,\phi_3) = u_1B_1 + u_2B_2 + u_3B_3 + \phi_1R_1 + \phi_2R_2 + \phi_3R_3, \tag{17g}$$

where, from Eq. (17c), we read off that,

$$B_1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},\quad B_2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \text{ and } B_3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \tag{17h}$$

are the symmetric (4×4)-matrix constant-velocity Lorentz "boost" generators in, respectively, the x, y and z directions. We also read off from Eq. (17c) that,

$$R_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad R_2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \text{ and } R_3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \tag{17i}$$

are the antisymmetric (4×4)-matrix spatial-rotation generators about, respectively, the x, y and z axes, which very closely correspond to their (3×3)-matrix counterpart rotation generators that are familiar from our discussion above of O(3).

The relevance of the six dimensionless (4×4)-matrix generators B_1, B_2, B_3, R_1, R_2 and R_3 given by Eqs. (17h) and (17i) above is made explicit by using Eq. (17g) to reexpress Eq. (17f) as,

$$\Lambda(u_1, u_2, u_3, \phi_1, \phi_2, \phi_3, n_P, n_T) = (T)^{n_T}(P)^{n_P} \exp(u_1 B_1 + u_2 B_2 + u_3 B_3 + \phi_1 R_1 + \phi_2 R_2 + \phi_3 R_3)\,. \tag{17j}$$

From Eq. (17j) we see, inter alia, that a constant-velocity Lorentz "boost" in the x-direction *by itself* of the four-vector $((ct), x, y, z)$ entails multiplying that four-vector by the 4×4 matrix $\exp(u_1 B_1)$, which is symmetric and of course involves *only* the x-direction "boost" generator B_1 in tandem with the dimensionless "rapidity" parameter u_1, *whose value we require to correctly correspond to the* Eq. (11a) *constant-velocity parameter v of that x-direction Lorentz "boost"*. In detail,

$$\begin{vmatrix} (ct') \\ x' \\ y' \\ z' \end{vmatrix} = \exp(u_1 B_1) \begin{vmatrix} (ct) \\ x \\ y \\ z \end{vmatrix} = \begin{vmatrix} \cosh u_1 & \sinh u_1 & 0 & 0 \\ \sinh u_1 & \cosh u_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} (ct) \\ x \\ y \\ z \end{vmatrix} = \begin{vmatrix} (\cosh u_1)(ct) + (\sinh u_1)x \\ (\cosh u_1)x + (\sinh u_1)(ct) \\ y \\ z \end{vmatrix}, \tag{17k}$$

which will reproduce the x-direction Lorentz "boost" with constant-velocity parameter v of Eqs. (11a) and (11b) *if the dimensionless "rapidity" parameter u_1 can be chosen such that*,

$$\textit{both}\;\; \cosh u_1 = \left(1 - (v/c)^2\right)^{-\frac{1}{2}} \;\; \textit{and} \;\; \sinh u_1 = \left(1 - (v/c)^2\right)^{-\frac{1}{2}}(-(v/c)) \;\; \textit{are satisfied}. \tag{17l}$$

The left-hand sides of the *two* requirements laid down by Eq. (17l) are linked by the mathematical identity $(\cosh u_1)^2 - (\sinh u_1)^2 = 1$, but since the difference of the squares of the corresponding right-hand sides of those two requirements is *also* equal to unity, there is *no* incompatibility. Therefore it is *only* necessary to require the dimensionless "rapidity" parameter u_1 to satisfy $\tanh u_1 = -(v/c)$, which yields,

$$u_1 = \tanh^{-1}(-(v/c)) = \tfrac{1}{2}\ln[(1 - (v/c))/(1 + (v/c)]. \tag{17m}$$

We see that Eq. (17m) doesn't produce real-valued finite dimensionless "rapidity" u_1 *unless* $|v| < c$, which condition is, of course, *necessary for* Eqs. (11a) *and* (11b) *to make physical sense*.

We use Eqs. (17h) and (17i) to work out the closed commutator algebra (Lie algebra) of the six generators B_1, B_2, B_3, R_1, R_2 and R_3 of the general homogeneous Lorentz-transformation group. From our study of O(3) above, we are already familiar with the simple cyclic closed commutator subalgebra of the three rotation generators R_1, R_2 and R_3 among themselves, namely,

$$[R_1, R_2] = R_3, \;\; [R_2, R_3] = R_1, \;\; [R_3, R_1] = R_2. \tag{17n}$$

Commutators among themselves of the "boost" generators B_1, B_2 and B_3, however, necessarily produce *rotation* generators, along with subtle breaking of the cyclic pattern,

$$[B_1, B_2] = R_3, \;\; [B_2, B_3] = R_1, \;\; [B_3, B_1] = -R_2. \tag{17o}$$

Mixed rotation/"boost" commutators necessarily produce "boost" generators, with no discernible semblance of the cyclic pattern,

$$\begin{gathered} [R_1, B_2] = -B_3, \; [B_1, R_2] = B_3, \; [R_2, B_3] = B_1, \; [B_2, R_3] = -B_1, \; [R_3, B_1] = -B_2, \; [B_3, R_1] = -B_2, \\ [R_1, B_1] = [R_2, B_2] = [R_3, B_3] = 0. \end{gathered} \tag{17p}$$

Use of this closed commutator algebra in conjunction with the Campbell-Baker-Hausdorff decomposition theorem for the exponential of a linear combination of matrices permits the rather opaque expression $\exp(u_1B_1 + u_2B_2 + u_3B_3 + \phi_1R_1 + \phi_2R_2 + \phi_3R_3)$ which occurs in the Eq. (17j) general form of Λ to be recast as a more readily comprehensible *succession* of straightforward "boosts" and rotations.

The four electromagnetic field equations given by Eqs. (2) *are actually covariant under Lorentz transformations*, and can be reexpressed *in the tensor notation whose indices run over the four values* 0*,* 1*,* 2 *and* 3 (so as to *explicitly reflect* that Lorentz-transformation covariance) which we found to be so helpful in dealing with calculations involving the *general* homogeneous Lorentz transformations. In particular the **E** and the **B** fields are incorporated into a single 4×4 second-rank antisymmetric Lorentz-contravariant tensor $F^{\mu\nu} = -F^{\nu\mu}$ as follows,

$$F^{00} = F^{11} = F^{22} = F^{33} = 0, \quad F^{01} = -F^{10} = -E^1, \quad F^{02} = -F^{20} = -E^2, \quad F^{03} = -F^{30} = -E^3,$$

$$F^{23} = -F^{32} = -B^1, \quad F^{31} = -F^{13} = -B^2, \quad F^{12} = -F^{21} = -B^3.$$

Furthermore, the charge-density scalar ρ and the current-density vector $\mathbf{j}$ are incorporated into a single current-density Lorentz-contravariant four-vector j^ν as follows: $j^0 = c\rho$ and j^1, j^2 and j^3 *are the three precisely corresponding components of* $\mathbf{j}$. The explicitly Lorentz-invariant presentation of the equation of continuity, which of course is $\dot\rho + \nabla\cdot\mathbf{j} = 0$, has the simple form,

$$\partial_\nu j^\nu = 0,$$

where the Lorentz-covariant four-vector partial-derivative operator ∂_ν is defined as follows,

$$\partial_0 \stackrel{\text{def}}{=} (1/c)(\partial/\partial t), \quad \partial_1 \stackrel{\text{def}}{=} (\partial/\partial x), \quad \partial_2 \stackrel{\text{def}}{=} (\partial/\partial y), \quad \partial_3 \stackrel{\text{def}}{=} (\partial/\partial z).$$

The use of ∂_μ, $F^{\mu\nu}$ and j^ν permits the explicitly Lorentz-contravariant four-vector *joint presentation* of the Biot-Savart-Maxwell Law of Eq. (2a) and Coulomb's Law of Eq. (2c) in the simple form,

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu/c.$$

Because $F^{\mu\nu} = -F^{\nu\mu}$, this dynamical equation also implies the equation of continuity, i.e., the antisymmetry of $F^{\mu\nu}$ permits the implication that $\partial_\nu j^\nu = 0$ to be drawn from the above dynamical equation.

The use of $\varepsilon_{\mu\nu\alpha\beta}$, which is the Lorentz-covariant fourth-rank *totally-antisymmetric tensor symbol*, furthermore permits the explicitly Lorentz-covariant four-vector *joint presentation* of Faraday's Law of Eq. (2b) and the Gauss Law of Eq. (2d) in the simple form,

$$\varepsilon_{\mu\nu\alpha\beta}\partial^\nu F^{\alpha\beta} = 0,$$

where ∂^ν *is the Lorentz-contravariant version of the above-defined Lorentz-covariant four-vector partial-derivative operator* ∂_ν, namely $\partial^\nu = \eta^{\nu\sigma}\partial_\sigma$, where the Lorentz-contravariant $\eta^{\alpha\beta}$ is technically *the inverse matrix* of the Lorentz-covariant $\eta_{\mu\nu}$, but that *inverse* relationship to $\eta_{\mu\nu}$ implies *that* $\eta^{\alpha\beta}$ *is also purely diagonal, with exactly the same matrix elements* as $\eta_{\mu\nu}$, so,

$$\partial^0 \stackrel{\text{def}}{=} (1/c)(\partial/\partial t), \quad \partial^1 \stackrel{\text{def}}{=} -(\partial/\partial x), \quad \partial^2 \stackrel{\text{def}}{=} -(\partial/\partial y), \quad \partial^3 \stackrel{\text{def}}{=} -(\partial/\partial z).$$

Lorentz-transformation *invariants* that in particular are bilinear combinations of the electromagnetic field tensor $F^{\mu\nu}$ in which all tensor indices have been contracted are sometimes of special physical interest. We earlier noted from Eqs. (9d) and (9c) that the bilinear electromagnetic field entity $(\mathbf{E}\cdot\mathbf{B})$ *vanishes identically* for a point charge moving at constant velocity. We can now understand this fact in the context of the concept of Lorentz-transformation invariance by taking note that $(\mathbf{E}\cdot\mathbf{B})$ *is indeed a Lorentz-transformation invariant* because,

$$(\mathbf{E}\cdot\mathbf{B}) = \tfrac{1}{8}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}.$$

Since $(\mathbf{E}\cdot\mathbf{B})$ *certainly vanishes in the rest frame of the point charge where we know that* $\mathbf{B} = \mathbf{0}$*, and* $(\mathbf{E}\cdot\mathbf{B})$ *is as well a Lorentz-transformation invariant,* $(\mathbf{E}\cdot\mathbf{B})$ *must also vanish for a point charge moving at any constant velocity* (whose magnitude of course is less than c).

Finally we would like to understand exactly *which* aspect of electromagnetic physics is incompatible with Galilean principles. Taking the Galilean limit $c\to\infty$ of the four electromagnetic laws, which are,

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\times\mathbf{E} = -\dot{\mathbf{B}}/c, \qquad \nabla\cdot\mathbf{B} = \mathbf{0}, \qquad \nabla\times\mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c,$$

produces,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla \cdot \mathbf{B} = \mathbf{0}, \qquad \nabla \times \mathbf{B} = \mathbf{0},$$

so we see that in the $c \to \infty$ Galilean limit *all of the sources and all of the effects of the magnetic field* $\mathbf{B}$ *fall away*! Thus it definitely is *magnetism* which is incompatible with Galilean principles. We can understand the *details* of this incontrovertible fact by scrutinizing the right-hand side of the Biot-Savart-Maxwell Law,

$$\nabla \times \mathbf{B} = (4\pi\mathbf{j} + \dot{\mathbf{E}})/c,$$

which tells us that it is *motion* which produces magnetism, specifically the *motion* of electric charge and the *motion* of the electric field; *without motion* there is patently *no source* for the magnetic field. At the *same time*, the magnetic field *isn't picky in the slightest about the character of the motion of its sources*; the Biot-Savart-Maxwell Law attests that *a previously static* assortment of electric charge and electric field is, *once set into uniform motion at nonzero constant velocity*, a perfectly *effective source of magnetic field*—our magnetic-field result $\mathbf{B}$ of Eq. (9c) *is an exact example of that fact*. But as we have emphasized in the discussion below Eq. (9c), it is a fact which flies utterly in the face of Galilean principles and the space-time Galilean transformation, which have it that relative motion at constant velocity has no physical consequences beyond the minimum which is required by the existence of that motion: the *production* of completely detectable and physically consequential magnetic field *merely* by dint of *the magnitude of constant velocity of relative motion* is *just as unequivocal* and *startling* a contradiction of Galilean/Newtonian principles as are the much more *celebrated* slowing of clocks and contractions of lengths which *also* occur *merely* by dint of *the magnitude of constant velocity of relative motion*. Indeed the production of magnetic field by dint of the magnitude of constant velocity of relative motion is a *first-order* effect of that velocity magnitude, while the distortion of electric fields, the slowing of clocks and the contraction of lengths are vastly more subtle *second-order* effects of that velocity magnitude. To bluntly put this matter in perspective, the discovery that the presence of direct current in a wire deflects an adjacent compass needle *already sealed the fate of the Galilean/Newtonian paradigm*, notwithstanding that scientists understandably didn't have the wit to *realize* that for a very, very long time.

References

[1] Length contraction–Wikipedia,
https://en.wikipedia.org/wiki/Length_contraction.

[2] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961), Section 2b, Eqs. (10)–(11), p. 38.

[3] Michelson-Morley experiment–Wikipedia,
https://en.wikipedia.org/wiki/Michelson%E2%80%93Morley_experiment.

[4] S. S. Schweber, op. cit., Section 2b, Eqs. (12)–(13), p. 38.