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The $3n \pm p$ Conjecture: A Generalization of Collatz Conjecture

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ABSTRACT. The Collatz conjecture is an open conjecture in mathematics named so after Lothar Collatz who proposed it in 1937. It is also known as $3n + 1$ conjecture, the Ulam conjecture (after Stanislaw Ulam), Kakutanis problem (after Shizuo Kakutani) and so on. Several various generalization of the Collatz conjecture has been carried. In this paper a new generalization of the Collatz conjecture called as the $3n \pm p$ conjecture; where p is a prime is proposed. It functions on $3n + p$ and $3n - p$, and for any starting number n, its sequence eventually enters a finite cycle and there are finitely many such cycles. The $3n \pm 1$ conjecture, is a special case of the $3n \pm p$ conjecture when p is 1.

1 INTRODUCTION

The Collatz conjecture is long standing open conjecture in number theory. Paul Erdos had commented about the Collatz conjecture that *"Mathematics may not be ready for such problems"*. The Collatz conjecture has been extensively studied by several researchers [1, 2, 3, 4, 5]. A novel theoretical framework was formulated for information discovery using the Collatz conjecture data by Idowu [6]. Generalizing the odd part of the Collatz conjecture was studied by [7]. Several various generalization of the Collatz conjecture was studied by [8]. Various generalization are listed and given in number theory website of Keith Matthews [9].
This paper proposes a new conjecture which is a generalization of the Collatz conjecture. This new conjecture is called as the $3n \pm p$ conjecture, where p is a prime. This paper is organised into four sections. First section is introductory in nature. Section two recalls the Collatz conjecture and its various generalizations so that the paper is self contained.

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Section three proposes the new $3n \pm p$ conjecture and illustrates it by examples. The conclusions and future study are given in the last section.

2 COLLAT'Z CONJECTURE AND ITS VARIOUS GENERALIZATIONS

2.2. Collatz Conjecture

The $3n + 1$ conjecture or the Collatz conjecture is summarized as follows: Take any positive integer n. If n is even divide it by 2 to get $n/2$. If n is odd multiply it by 3 and add 1 to obtain $3n + 1$. Repeat the process (which has been called "Half Or Triple Plus One" or HOTPO) indefinitely. The conjecture states that no matter what number you start with you will always eventually reach 1.

Consider the following operation on an arbitrary positive integer: If the number is even divide it by two, if the number is odd, triple it and add one. This is illustrated by example of taking numbers from 4 to 10 and the related sequence is obtained:

- n = 4; related sequence is 4, 2, 1.
- n = 5, related sequence is 5, 16, 8, 4, 2, 1.
- n = 6, related sequence is 6, 3, 10, 5, 16, 8, 4, 2, 1.
- n = 7, related sequence is 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
- n = 8, related sequence is 8, 4, 2, 1.
- n = 9, related sequence is 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
- n = 10, related sequence is 10, 5, 16, 8, 4, 2, 1.

In simple modular arithmetic notation the Collatz conjecture can be represented as

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 2) \\ \frac{3n+1}{2} & \text{if } n \equiv 1 (\text{mod } 2) \end{cases} \tag{1}$$

Note: Only powers of two converge to one quickly.

2.2. Various generalization of the Collatz Conjecture

Several researchers have studied and generalized the Collatz conjecture. Some generalize by taking different values for 2 as 3, 5, etc [9]. Keith Matthew [9] has studied for $3n + 371$ and so on. Some natural generalizations of the Collatz Problem was done by Carnielli [8]. Lu Pei has given a generalization of $3x - 1$ mapping in [9].
The generalization of the $3n - 1$ mapping due to Lu Pei is given verbatim from [9].
Consider the mapping $T_d : Z \leftarrow Z$. Let $d \geq 2$. Then

$$T_d(n) = \begin{cases} \frac{n}{d} & \text{if } n \equiv 0(\text{mod } 2) \\ \frac{(d+1)n-i}{2} & \text{if } n \equiv \text{i}(\text{mod } 2) \end{cases} \tag{2}$$

where $-d/2 \leq i \leq d/2; i \neq 0$.
In case, $d = 2$ it gives the $3n - 1$ mapping:

$$T_2(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0(\text{mod } 2) \\ \frac{3n-1}{2} & \text{if } n \equiv 1(\text{mod } 2) \end{cases} \tag{3}$$

This is a special case of a version of a mapping studied by Herbert Moller [10] and is also an example of a relatively prime mapping, in the language of Matthews and Watts, where $m_0 = 1$ and m_i = d+1 for 1$\leq i$ ¡ d and where we have the inequality

$$m_0 m_1 \ldots m_d = (d+1)^{d-1} < d^d$$

So it seems certain that the sequence of iterates

$$n, T_d(n), T_d^2(n), \ldots$$

always eventually enters a cycle and that there are only finitely many such cycles.
Clearly $T_d(n) = n$ for $-d/2 < n \leq d/2$. For $d = 3$, 6 and 10, there appears to be no other cycles. By replacing 2 by d, it given the $3x - 1$ conjecture will eventually enter a cycle. It is showed that the $3n + 1$ collatz conjecture when n is negative has finite cycles which terminates in -1 or -5 or -17 [9].
Thus if for every non zero $n \in Z$ the $3n + 1$ Collatz conjecture converges to $\{-17, -5, -1, 1\}$ and the $3n - 1$ collatz conjecture converges to $\{-1, 1, 5, 17\}$.

The $3n-1$ conjecture is a special case of the Lu Pei's generalization of the Collatz conjecture. The $3n-1$ conjecture is described here for clarity.

2.3. The $3n-1$ Conjecture

The $3n-1$ conjecture which is akin to the Collatz conjecture is proposed in this section. The $3n-1$ conjecture is as follows:
Take any arbitrary positive integer n. If n is even divide it by two and get $n/2$ if n is odd multiply it by 3 and subtract 1 and obtain $3n-1$, repeat this process indefinitely. We call this process as "Half Or Triple Minus One" or HOTMO. The conjecture states that immaterial of which number you begin with, you will eventually reach 1 or 5 or 17.

2.3.1. Statement of the Problem/Conjecture

On any arbitrary positive integer, consider the operation

- If the number is even, divide it by two
- Else triple it and subtract one

continue this process recursively. The $3n-1$ conjecture is that this process which will eventually reach either 1 or 5 or 17, regardless of which positive integer is selected at the beginning.
The smallest i such that $a_i = 1$ or 5 or 17 is called as the total stopping time of n. The $3n-1$ conjecture asserts that every n has a well defined total stopping time i. If for some n (any positive integer) such i (total stopping time) doesn't exist, then n has an infinite total stopping time then the conjecture is false. It can happen only because there is some starting number which gives a sequence that does not contain 1, 5 or 17. Such a sequence may have a repeating cycle that does not contain 1, 5 or 17 or it might increase without bounds. Till now such a sequence or number has not been found.
In simple modular arithmetic notation the $3n-1$ conjecture can be represented as

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 2) \\ \frac{3n-1}{2} & \text{if } n \equiv 1 (\text{mod } 2) \end{cases} \tag{4}$$

A sequence is formed by performing this operation repeatedly, it starts with any arbitrary positive integer and takes the result each step as the input for the next.

$$a_i = \begin{cases} n & \text{if } i = 0 \\ f(a_{i-1}) & \text{if } i \neq 0 \end{cases} \tag{5}$$

$a_i = f^i(n)$ that is a_i is the value of f applied to n recursively i times; n is the starting number and i at the end of the sequence is called the total stopping time.

Examples

The conjecture states that the sequence will reach 1, 5 or 17. The following repeated sequences / cycles happen for 1, 5 or 17.

1. $n = 1$; the repeated sequence is 4, 2, 1.
2. $n = 5$; the repeated sequence is 14, 7, 20, 10, 5.
3. $n = 17$; the repeated sequence is 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 136, 68, 34, 17.

We will illustrate this conjecture by some examples using the $3n - 1$ formula and taking numbers from 4 to 10. It is tabulated in Table 1

Table 1: Illustration of the $3n - 1$ conjecture

n	Sequence	i	Ends in
4	4, 2, 1	3	1
5	5, 14, 7, 20, 10, 5	1	5
6	6, 3, 8, 4, 2, 1	6	1
7	7, 20, 10, 5	4	5
8	8, 4, 2, 1	4	1
9	9, 26, 13, 38, 19, 56, 28, 14, 7, 20, 10, 5	12	5
10	10, 5	2	5

Similar to $3n + 1$ conjecture in $3n - 1$ conjecture also the powers of 2, converge quickly. Figure 2.3.2 gives the scatter plot that takes the starting number n from 1 to 1000 along the x-axis and the total stopping number i along the y-axis. Depending on which number the sequence ends, the colour is given. If the sequence ends in 1, then blue colour is given, if it ends in 5 then red colour is given and if it ends in 17 green colour is given.

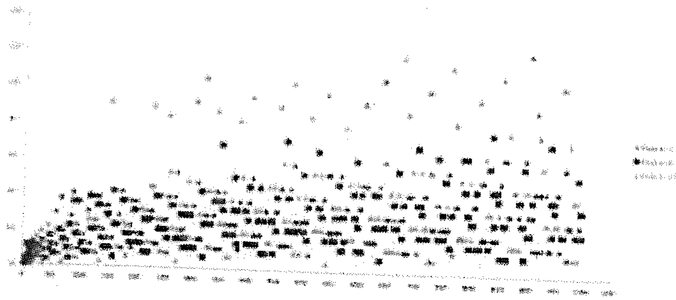

Figure 1. The scatter plot of first 1000 numbers and their stopping times

The $3n-1$ conjecture creates a sequence that ends in 3 different numbers with the sequence having a repeated sequence of

1. for any negative n the sequence ends in -1.
2. $n=1$; the repeated sequence is 4, 2, 1.
3. $n=5$; the repeated sequence is 14, 7, 20, 10, 5.
4. $n=17$; the repeated sequence is 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 136, 68, 34, 17.

The $3n-1$ conjecture and $3n+1$ conjecture are mirror functions. The $3n \pm p$ conjecture is defined in the next section.

3 THE $3n \pm p$ CONJECTURE

The $3n+p$ and $3n-p$ conjecture (or simply the $3 \pm p$ conjecture) is given in the following:

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 2) \\ \frac{3n \pm p}{2} & \text{if } n \equiv 1 (\text{mod } 2) \end{cases} \tag{6}$$

In simple modular arithmetic notation the $3n+p$ conjecture can be represented as

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 2) \\ \frac{3n+p}{2} & \text{if } n \equiv 1 (\text{mod } 2) \end{cases} \tag{7}$$

and the $3n-p$ conjecture can be represented as

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 2) \\ \frac{3n-p}{2} & \text{if } n \equiv 1 (\text{mod } 2) \end{cases} \quad (8)$$

It is clearly seen when $p = 1$, we see the sequence converges to $\{-17, -5, -1, 0, 1\}$ and when $p = -1$, the sequence converges to $\{-1, 0, 1, 5, 17\}$. When p is 1, it is Collatz conjecture and when p is -1 it is $3n - 1$ conjecture. We show for $3n + 5$ the sequence converges to $\{-85, -25, -5, -1, 0, 1, 5, 19, 23, 187, 407\}$ for any n in Z . For $3n - 5$ we get $\{-407, -187, -23, -19, -5, -1, 0, 1, 5, 25, 85\}$. $3n + 5$ and $3n - 5$ act like mirror functions.
In Table2 some $3n \pm p$ conjecture and their minimum cycle elements are listed.

Table 2: Illustration of the $3n \pm p$ conjecture

Sequence	Ends in
$3n + 3$	$\{-51, -5, -3, 0, 1, 3\}$
$3n - 3$	$\{-3, -1, 0, 3, 5, 51\}$
$3n + 5$	$\{-85, -25, -5, -1, 0, 1, 5, 19, 23, 187, 407\}$
$3n - 5$	$\{-407, -187, -23, -19, -5, -1, 0, 1, 5, 25, 85\}$
$3n + 7$	$\{-119, -35, -7, -1, 0, 1, 5, 7\}$
$3n - 7$	$\{-7, -5, -1, 0, 1, 7, 35, 119\}$
$3n + 11$	$\{-187, -55, -19, -11, -3, -1, 0, 1, 11, 13\}$
$3n - 11$	$\{-13, -11, -1, 0, 1, 3, 11, 19, 55, 187\}$
$3n + 13$	$\{-221, -65, -13, -1, 0, 1, 13, 131, 211, 227, 251, 259, 283, 287, 319\}$
$3n - 13$	$\{-319, -287, -283, -259, -251, -227, -211, -131, -13, -1, 0, 1, 13, 65, 221\}$

It is conjectured that for every prime p the $3n \pm p$ sequence will result in a finite cycle and there are finite number of such cycles.

4 RESULTS AND FURTHER STUDY

The proposed $3n \pm p$ conjecture is a new generalization of the $3n + 1$ conjecture or the Collatz conjecture. Given any starting number n, the conjecture states that the sequence will result in a finite cycle and there are finite number of such cycles. Cycles related to the $3n \pm p$, resulting hailstone numbers and parity sequence are left open for study.

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