Describing a Fluid Motion with 3-D Rectangular Coordinates

Valdir Monteiro dos Santos Godoi
valdir.msgodoi@gmail.com

Abstract: We describe a fluid in three-dimensional motion with at most one spatial variable by rectangular coordinate, beyond time, and conclude on the breakdown of Euler and Navier-Stokes solutions and the necessity of use of vector pressure.

Keywords: Euler equations, Navier-Stokes equations, Lagrangian description, Eulerian description, Bernoulli's law, breakdown solutions, vector pressure.

1 – Introduction

In [1] we showed that the three-dimensional Euler ($\nu = 0$) and Navier-Stokes equations in rectangular coordinates need to be adopted as

(1) $$\frac{\partial p}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3} \nu \frac{\partial}{\partial x_i} (\nabla \cdot u) + f_i,$$

for $i = 1,2,3$, where $\alpha_j = \frac{dx_j}{dt}$ is the velocity in Lagrangian description and u_i and the partial derivatives of u_i are in Eulerian description, as well as the scalar pressure p and density of external force f_i. The coefficient of viscosity is ν and by ease we prefer to use the mass density $\rho = 1$ (otherwise substitute p by p/ρ and ν by ν/ρ).

An alternative equation is

(2) $$\frac{\partial p_i}{\partial x_i} + \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} = \nu \nabla^2 u_i + \frac{1}{3} \nu \frac{\partial}{\partial x_i} (\nabla \cdot u) + f_i,$$

thus making the pressure a vector: $p = (p_1, p_2, p_3)$. In both equations is valid

(3) $$\frac{Du_i}{Dt} = \frac{Du_i^E}{Dt} = \frac{Du_i^L}{Dt} = \left(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} \right) \Big|_L,$$

where the upper letter E refers to Eulerian velocity (u) and L to Lagrangian velocity (α). The symbol $|_L$ means the respective calculation in Lagrangian description, substituting each x_i as a function of time, initial value and eventually some parameters. With the notation $\frac{D}{Dt}$ we want, in principle, to make explicit that we are calculating a total derivative in relation to time, and the result is a function exclusively of time (and possibly a set of parameters), without the spatial coordinates x, y, z, but when for some reason we need to leave the result as a function of the spatial coordinates we can also do it.

A condition indicated by us in [1] were

$$(4)\qquad \begin{cases} \frac{\partial u_i}{\partial x_j} = 0, \ i \neq j, \\ \partial x_i = u_i \partial t \end{cases}$$

because we have, by definition,

$$(5)\qquad u_i = \frac{dx_i}{dt},$$

in Lagrangian description, and for this reason the velocity u_i, *a priori*, is not dependent of others variables x_j, with $x_j \neq x_i$. More than a rigorous mathematical proof, this is a practical approach, which simplifies the original system.

It is very easy to accept the first equation of (4) when there is no link between the spatial coordinates during the movement of the fluid over time, but in a circular motion, for example, it seems to be no longer valid. In order to show how it is possible to describe a motion with a single independent spatial variable by rectangular coordinate, $u_i = \varphi_i(x_i, t)$, we will describe in section 2 a circular motion and in section 3 a quite general movement.

The section 4 will be our Conclusion, concluding on the breakdown solutions and the necessity of use of vector pressure.

2 – Circular Motion

Let a circular motion of radius R, centered at $(x_C,\ y_C)$ and with constant angular velocity $\omega > 0$ described by the equations:

$$(6)\qquad \begin{cases} x = x_C + R\cos(\theta_0 + \omega t) \\ y = y_C + R\sin(\theta_0 + \omega t) \end{cases}$$

and consequently

$$(7)\qquad (x - x_C)^2 + \left(y - y_C\right)^2 = R^2.$$

Then the velocity components are

$$(8)\qquad \begin{cases} \alpha_1 = u_1^L = \dot{x} = -\omega R\sin(\theta_0 + \omega t) = -\omega(y - y_C) = u_1^E \\ \alpha_2 = u_2^L = \dot{y} = +\omega R\cos(\theta_0 + \omega t) = +\omega(x - x_C) = u_2^E \end{cases}$$

and the acceleration components are

$$(9) \qquad \begin{cases} \frac{Du_1^L}{Dt} = \ddot{x} = -\omega^2 R\cos(\theta_0 + \omega t) = -\omega^2(x - x_C) = \frac{Du_1^E}{Dt} \\ \frac{Du_2^L}{Dt} = \ddot{y} = -\omega^2 R\sin(\theta_0 + \omega t) = -\omega^2(y - y_C) = \frac{Du_2^E}{Dt} \end{cases}$$

Supposing that the particles of fluid obey the motion described by (6) to (9), we have

$$(10) \qquad \begin{cases} \frac{\partial u_1}{\partial y} = -\omega, \quad \frac{\partial u_1}{\partial x} = 0 \\ \frac{\partial u_2}{\partial x} = +\omega, \quad \frac{\partial u_2}{\partial y} = 0 \end{cases}$$

apparently in disagree with (4) if $\omega \neq 0$. But, as x is a function of y and reciprocally, in this circular motion according (7), again (4) turns valid, for any signal of x and y. For to complete a three-dimensional description, we define $z = z_0$, without dependence of time, and $u_3 = 0$.

This is a motion of velocity without potential, because $\frac{\partial u_i}{\partial x_j} \neq \frac{\partial u_j}{\partial x_i}$ for some $i \neq j$, but if $f = (f_1, f_2, f_3)$ has potential we have $\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$ for all $i, j = 1,2,3$, with

$$(11) \qquad S_i = -\frac{\partial u_i}{\partial t} - \sum_{j=1}^{3} \alpha_j \frac{\partial u_i}{\partial x_j} + \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$$

then the system (1) has solution.

A calculation for the scalar pressure of this motion is

$$(12) \qquad \begin{aligned} p &= \int_L (S_1, S_2, S_3) \cdot dl = \int_L \left(-\frac{Du}{Dt} + f\right) \cdot dl \\ &= \omega^2 \left[\left(\frac{x^2}{2} - x_C x\right)\Big|_{x_0}^{x} + \left(\frac{y^2}{2} - y_C y\right)\Big|_{y_0}^{y}\right] + U - U_0 + \theta(t) \\ &= \omega^2 \left[\left(\frac{x^2}{2} - x_C x\right) - \left(\frac{{x_0}^2}{2} - x_C x_0\right) + \left(\frac{y^2}{2} - y_C y\right) - \left(\frac{{y_0}^2}{2} - y_C y_0\right)\right] + \\ &\quad U - U_0 + \theta(t), \end{aligned}$$

where $f = \nabla U$, $U_0 = U(x_0, y_0, z_0, t)$ and L is any smooth path linking a point (x_0, y_0, z_0) to (x, y, z). We can ignore the use of x_0, y_0, z_0 and U_0, and use only the free function for time, $\theta(t)$, which on the other hand can include the terms in x_0, y_0 and z_0, and nevertheless this solution shows us that the pressure is not uniquely well determined, therefore we get to the negative answer to Smale's 15th problem, according already seen in [2] and [3], even if we assign the velocity value on some surface that we wish and even if $\theta(t)$ and U does not depend explicitly on the variable time t. In this motion the pressure is dependent, besides of x, y and U, without any problematic question, and x_C, y_C and ω, specific parameters of the movement conditions of a particle, of $\theta(t)$, U_0 and more three parameters, x_0, y_0 and z_0, then there is not uniqueness of solution.

Another calculation for pressure is possible due to fact that we can describe the acceleration $\frac{Du}{Dt}$ of a particle of fluid as a function only of time, $\frac{D\alpha}{Dt}$, without the variables x, y, z, and then

$$(13)\quad \begin{aligned} p &= -\frac{D\alpha}{Dt}\cdot\int_L dl + U - U_0 + \theta(t) \\ &= +\omega^2 R[\cos(\theta_0+\omega t)(x-x_0) + \sin(\theta_0+\omega t)(y-y_0)] \\ &\quad + U - U_0 + \theta(t), \end{aligned}$$

with

$$(14)\quad \begin{cases} \frac{\partial p}{\partial x} = +\omega^2 R\cos(\theta_0+\omega t) + f_1 = +\omega^2(x-x_C) + f_1 \\ \frac{\partial p}{\partial y} = +\omega^2 R\sin(\theta_0+\omega t) + f_2 = +\omega^2(y-y_C) + f_2 \\ \frac{\partial p}{\partial z} = f_3 \end{cases}$$

in fact derivatives such as can be obtained from (12).

Note that in order to continue using the traditional form of the Euler and Navier-Stokes equations we will have non-linear equations, which can make it difficult to obtain the solutions and bring all the difficulties that we know. To make sense to use the velocity in Eulerian description rather than the Lagrangian description in α_j it is necessary that, for all $t \geq 0$,

$$(15)\quad u^E(x(t), y(t), z(t), t) = \alpha(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = u^L(t),$$

omitting the use of possible parameters of motion, then nothing more natural than the definitive substitution of the terms $\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 \alpha_j \frac{\partial u_i}{\partial x_j}$, as well as $\frac{\partial u_i}{\partial t} + \sum_{j=1}^3 u_j \frac{\partial u_i}{\partial x_j}$ in the traditional form, by $\frac{Du_i^L}{Dt}$ or $\frac{D\alpha_i}{Dt}$. This brings a great and important simplification in the equations, and to return to having the position as reference it is enough to use the conversion or definition adopted for $x(t), y(t)$ and $z(t)$, including the possible additional parameters, for example, substituting initial positions in function of position and time, etc.

Thus, more appropriate Euler ($\nu = 0$) and Navier-Stokes equations with scalar pressure are, in index notation,

$$(16)\quad \frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu\nabla^2 u_i + \frac{1}{3}\nu\frac{\partial}{\partial x_i}(\nabla\cdot u) + f_i.$$

3 – Generic three-dimensional motion

Suppose that a particle of fluid moves according to equation

(17) $$x_i = A_i(t)x_i^0 + B_i(t),$$

$A_i(0) = 1,\ B_i(0) = 0,\ A_i, B_i \in C^\infty\big([0,\infty)\big),\ i = 1,2,3,\ (x_1, x_2, x_3) \equiv (x, y, z)$, where $(x_1^0, x_2^0, x_3^0) \equiv (x_0, y_0, z_0)$ is the initial position of this particle in relation to three-orthogonal system of reference considered at rest.

Your velocity in relation to this system is, for $i = 1,2,3$,

(18) $$\dot{x}_i = \frac{d}{dt}x_i = u_i^L = \alpha_i = A_i'(t)x_i^0 + B_i'(t),$$

with acceleration

(19) $$\ddot{x}_i = \frac{d}{dt}\dot{x}_i = \frac{D}{Dt}u_i^L = \frac{D}{Dt}\alpha_i = A_i''(t)x_i^0 + B_i''(t).$$

We are using both the superior point $(\dot{x})$ and the prime mark (A'), and respective repetitions, for indicate differentiations in relation to time.

We are going to transform Lagrangian velocity into Eulerian velocity through transformation

(20) $$x_i^0 = \frac{x_i - B_i(t)}{A_i(t)},$$

which results in

(21) $$u_i^E = A_i'(t)x_i^0 + B_i'(t) = A_i'(t)\frac{x_i - B_i(t)}{A_i(t)} + B_i'(t)$$
$$= \frac{A_i'(t)}{A_i(t)}x_i - \frac{A_i'(t)B_i(t)}{A_i(t)} + B_i'(t)$$

and

(22) $$\frac{Du_i^E}{Dt} = A_i''(t)x_i^0 + B_i''(t) = A_i''(t)\frac{x_i - B_i(t)}{A_i(t)} + B_i''(t)$$
$$= \frac{A_i''(t)}{A_i(t)}x_i - \frac{A_i''(t)B_i(t)}{A_i(t)} + B_i''(t).$$

We see that both u_i^E and $\frac{Du_i^E}{Dt}$ are linear functions in x_i or only functions of time if $A_i(t) \equiv 1$. We still want the limits $\lim\frac{A_i'(t)}{A_i(t)}$ and $\lim\frac{A_i''(t)}{A_i(t)}$ to be finite for all $t \geq 0$, otherwise we will have infinite velocities or accelerations in these instants of infinity if the corresponding values in Lagrangian description also are. When $A_i(t) = 0$ the values respect to Eulerian description are equal to the corresponding Lagrangian description.

The expression (22) is also obtained through the chain rule

$$\text{(23)} \qquad \frac{Du_i^E}{Dt} = \frac{\partial u_i^E}{\partial t} + \sum_{j=1}^{3} \alpha_j \frac{\partial u_i^E}{\partial x_j},$$

being

$$\text{(24)} \qquad \frac{\partial u_i^E}{\partial t} = \frac{A_i'' A_i - \left(A_i'\right)^2}{A_i^2} x_i - \left(\frac{A_i' B_i}{A_i}\right)' + B_i'',$$

$$\text{(25)} \qquad \left(\frac{A_i' B_i}{A_i}\right)' = \frac{A_i'' B_i + A_i' B_i'}{A_i} - \left(\frac{A_i'}{A_i}\right)^2 B_i$$

and

$$\text{(26)} \qquad \sum_{j=1}^{3} \alpha_j \frac{\partial u_i^E}{\partial x_j} = \alpha_i \frac{\partial u_i^E}{\partial x_i} = \left(A_i' x_i^0 + B_i'\right) \frac{A_i'}{A_i}.$$

With movements where there is some linear relation between the spatial coordinates, as

$$\text{(27)} \qquad x_i = A_{i1}(t) x_1^0 + A_{i2}(t) x_2^0 + A_{i3}(t) x_3^0 + B_i(t),$$

$A_{ij}(t), B_i(t) \in C^\infty([0,\infty))$ for $i, j = 1,2,3$, we can transform

$$\text{(28.1)} \quad A_i(t) \mapsto A_{ii}(t) x_i^0$$
$$\text{(28.2)} \quad B_i(t) \mapsto A_{ij}(t) x_j^0 + A_{ik}(t) x_k^0 + B_i(t)$$

into the previous equations (17) to (26), with $j < k,\ i \neq j \neq k,\ i, j, k = 1,2,3$, and we will arrive at results similar to those already obtained.

If the relation between the coordinates is more complicated, not just linear, for example when the particles need follow a specific family of surfaces of type $z = g(x, y)$ (omitting other possible parameters), for g smooth function, then we can abandon the dependency of position, at least in one coordinate, as

$$\text{(29)} \qquad z = g(x, y) = g\big(x(t), y(t)\big) = h(t),$$

and therefore

$$\text{(30)} \qquad \begin{cases} u_1 = \varphi_1(x, t) \\ u_2 = \varphi_2(y, t) \\ u_3 = \varphi_3(z, t) = \varphi_3(h(t), t) = \alpha_3(t) \end{cases}$$

Thus, (4) holds in an infinity of cases and the Euler and Navier-Stokes equations has solution in this way (if f is conservative).

Note that in both examples, sections 2 and 3, the solutions for velocity are at most linear in relation to spatial coordinates, and then there is no necessity of calculation of second derivatives of velocity, i.e., $\nabla^2 u = 0$ for any viscosity coefficient and the Navier-Stokes equations are reduced to the Euler equations. In general terms we have, from (21),

$$(31) \quad u_i^0 = \frac{A_i'(0)}{A_i(0)} x_i - \frac{A_i'(0) B_i(0)}{A_i(0)} + B_i'(0),$$

where we suppose that $\lim_{t \to 0} \frac{A_i'(t)}{A_i(t)}$ is finite for $i = 1,2,3$. If it is necessary that $\nabla \cdot u = \nabla \cdot u^0 = 0$ (incompressible fluids) then it must be valid, for all $t \geq 0$, the relation

$$(32) \quad \frac{A_1'(t)}{A_1(t)} + \frac{A_2'(t)}{A_2(t)} + \frac{A_3'(t)}{A_3(t)} = 0.$$

In all functions of time $A_i(t)$, $A_{ij}(t)$ and $B_i(t)$ are implicit the inclusion of constant parameters of movement, as $R, \theta_0, \omega, x_C, y_C$, etc.

The scalar pressure is equal to

$$(33) \quad p = \int_L \ (S_1, S_2, S_3) \cdot dl = \int_L \ \left(-\frac{Du}{Dt} + f\right) \cdot dl$$
$$= \sum_{i=1}^{3} \int_{x_i^0}^{x_i} \left(-\frac{Du_i}{Dt}\right) dx_i + U - U_o + \theta(t),$$

if f is a conservative external force, $f = \nabla U$, with

$$(34) \quad S_i = -\frac{Du_i}{Dt} + f_i$$

and

$$(35) \quad \frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}, \text{ for } i, j = 1,2,3, \text{ i.e., } \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i},$$

and then there is solution for Euler equations in this case.

As we have seen previously, the calculation of pressure is not unique and we can use $\frac{Du_i}{Dt}$ as a function of x_i and t or only of t. The simpler calculation gives

$$(36) \quad p = -\sum_{i=1}^{3} \left[A_i''(t) x_i^0 + B_i''(t)\right]\left(x_i - x_i^0\right) + U - U_o + \theta(t),$$

using

(37) $\frac{Du_i}{Dt} = \frac{D\alpha_i}{Dt} = A_i''(t)x_i^0 + B_i''(t),$

according (19). The pressure is not dependent only of position and time, but also initial position, although there is a one-to-one correspondence between initial position with time and position, according (17) and (20).

See that we use $A_i(t) \not\equiv 0$ because any particle start from some position and it is not possible all particles start from the same position, but if $A_i(t) = 0$ for some $t > 0$ use for (18) to (37) the results equivalents to $A_i'(t) = A_i''(t) = 0$ and $A_i(t) = 1$, except (20) which is no sense in this case, and (17) will be $x_i = B_i(t)$.

Another calculation for scalar pressure gives, from (33) and using

(38) $\frac{Du_i}{Dt} dx_i = \frac{dx_i}{dt} du_i = u_i du_i,$

the interesting result

(39) $$p = -\sum_{i=1}^{3} \frac{1}{2}\left(u_i^2 - u_i^{0\,2}\right) + U - U_o + \theta(t)$$
$$= -\frac{1}{2}(u^2 - u^{0\,2}) + U - U_o + \theta(t),$$

as the Bernoulli's law with $\frac{\partial \phi}{\partial t} = 0,\ u = \nabla \phi$.

4 – Conclusion

From equation (16),

(40) $\frac{\partial p}{\partial x_i} + \frac{D\alpha_i}{Dt} = \nu \nabla^2 u_i + \frac{1}{3}\nu \frac{\partial}{\partial x_i}(\nabla \cdot u) + f_i,$

we realize that if $\nu = 0$ and f is not conservative then there is no solution for Euler equations, as well as if u is conservative and f is not conservative there is no solution for Navier-Stokes equations, which now it is very clear to see and it is complementing [4]. More specifically, if u^0, the initial velocity, is conservative (irrotational or potential flow) and f is not conservative then there is no solution for Navier-Stokes equations, because it is impossible to obtain the pressure. This then solve [5] for the cases (C) and (D), the breakdown of solutions, for both u^0 and f belonging to Schwartz Space in case (C), and smooth functions with period 1 in the three orthogonal directions e_1, e_2, e_3 in case (D). As u^0 need obey to the incompressibility condition, $\nabla \cdot u^0 = 0$, with $\nabla \times u^0 = 0$ and $u^0 = \nabla \varphi^0$, where φ^0 is the potential of u^0, we have $\nabla^2 u^0 = 0$ and $\nabla^2 \varphi^0 = 0$, i.e., u^0 and φ^0 are harmonic functions, unlimited functions except the constants, including zero. As u^0 need be limited, we choose $u^0 = 0$ for case (C) (where it is necessary that $\int_{\mathbb{R}^3} |u^0|^2 dx\, dy\, dz$ is finite) and any constant for case (D), of spatially periodic solutions. In case (D) the external force need belonging to Schwartz Space with relation to time.

Note that the application of a non conservative force in fluid is naturally possible and there will always be some movement, even starting from rest. So that this is not a paradoxical situation it seems certain that the pressure in this case cannot be scalar, but rather vector, and thus the equation returns to solution in all cases (assuming all derivatives are possible, etc.). It is as indicated in (2), or substituting p by p_i in (16).

According to what we saw in this article, solve the Navier-Stokes equations can be synonymous to solve the Euler equations and we can take advantage of this facility. For the time being, I do not know any reason for having to a more complicated solution than the one described here, when the use of $\nabla^2 u \neq 0$ is necessary, except if the compromise with the motion of particles is forgotten or we intend to describe a spatially periodic solution in Fourier series or the pressure is given and is not $\nabla p = f$ or, the worst, the velocity is not smooth (C^∞) and there are boundary conditions. Nevertheless, even in the most complicated cases, the movement of particles can be transformed into functions exclusively of time. Perhaps naval or aeronautical engineers have other motives, but with a greater rigor, involving temperature and the collision of particles, other equations must be constructed.

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