Neutrino Oscillations with Hopf algebras

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Abstract

The Bogoliubov transformation in quantum field theory is used [1] to refine neutrino oscillation transition formulae by a deformation parameter, appearing in a noncommutative spectral model for neutrinos. We consider Hopf algebras in mixing from a more mathematical perspective, justified by the motivic nature of quantum field theory. Experimental constraints on mixing angles are considered.

1 Introduction

A spectral model from noncommutative geometry was used in [1] to study the Hopf algebra structure of neutrino mixing, introducing a new deformation parameter into transition probabilities coming from the Bogoliubov transformation of thermal field theory [2]. The Bogoliubov map, which acts on a two dimensional noncommutative operator space built from creation and annihilation operators, is the 2×2 quantum Fourier transform [3]

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}$$

Both operator and mixing algebras are given a Hopf structure, begging for a model independent study of mixing based on motivic ideas.

The tribimaximal mixing [4], which is a first order ansatz for neutrino mixing when $\delta_{13} = 0$, is given by the matrix $F_3 F_2$, where

$$F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \overline{\omega} \\ 1 & \overline{\omega} & \omega \end{pmatrix} \tag{2}$$

is the 3×3 Fourier transform for ω the cubed root of unity. In any dimension n, the quantum Fourier matrix diagonalises 1-circulants, matrices with $A_{ij} = A_{1(j-i+1 \mod n)}$. For n prime, there is a special set of $n-1$ 1-circulants along with F_n that is a quantum model for multiplication in the finite field with n elements.

The set of all $n \times n$ circulants over the complex numbers is the group Hopf algebra for the permutation group S_n. Hopf algebras are more universal objects than groups themselves, and in the deformed case of quantum groups there actually is no underlying group. For a generic semisimple Lie algebra $\mathcal{L}$, or for the central elements in the oscillator case, the coproduct map Δ is primitive

$$\Delta(h) = 1 \otimes h + h \otimes 1. \tag{3}$$

The Lie bracket $[x, y]$ in the free tensor algebra is $x \otimes y - y \otimes x$ and one may check that $\Delta([x, y]) = [\Delta(x), \Delta(y)]$. In the tensor algebra, the bracket is encoded as a product $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$. On the other hand, for fermionic creation and annihilation operators we want the anticommutator $\{a, a^\dagger\}$ to be constant, in the field of scalars in the tensor algebra. This is also a basic operation in the category of algebras, but not a product. Here we insist that a sits in $\mathcal{L}$ while $a^\dagger$ is in $\mathcal{L}^\vee$, although the dual is usually incorporated into the same algebra. We think of natural categorical maps like

$$\epsilon : \mathcal{L} \otimes \mathcal{L}^\vee \to \mathbb{F} \tag{4}$$
$$\epsilon^\vee : \mathcal{L}^\vee \otimes \mathcal{L} \to \mathbb{F}$$

and a braiding arrow γ on $\mathcal{L} \otimes \mathcal{L}^\vee$. The Bogoliubov transformation requires an addition of the a and $a^\dagger$, and so must act on a larger algebra such as $(\mathcal{L} \otimes \mathcal{L}^\vee)^{\otimes 2}$. In this picture, the initial Fock space is built in a braided category with duals and the oscillator coproducts have a beautiful representation in ribbon diagrams, using Δ on $\mathcal{L}$ for a and $\Delta^\vee$ for $a^\dagger$. Since $\{a, a^\dagger\}$ essentially lands in $\mathbb{F}$, we view it as a map $(a, a^\dagger) \mapsto 1$ in ϵ, and

$$a \otimes a^\dagger + a^\dagger \otimes a = (1 + \gamma)(1 \otimes \vee)(a \otimes a), \tag{5}$$

where γ is the braiding and $a \otimes a$ is a grouplike coproduct for a Hopf algebra $\mathcal{A}$.

Although a working model of quantum gravity does not yet exist, it is possible that a simple derivation of neutrino mass eigenvalues exists in a motivic context, especially given the well studied connection between modern methods in QCD ($N = 4$ SYM) and its gravitational cousin. We suppose that the nonassociativity of color, often related to octonion algebras, appears in quasi-Hopf algebras for motives in a higher dimensional categorical setting.

The neutrino mixing angles are δ_{ij} for $i, j \in 1, 2, 3$. An additional Bogoliubov angle θ gives a deformation parameter q, and there are mass phases in the Koide formalism, but all these parameters are probably related. We will show that deformation parameters should correspond directly to mixing angles, rather than being nested as in the standard Bogoliubov picture. The next section introduces the Bogoliubov mixing, and section 2 the Hopf algebras underlying 3×3 mixing, which is given with the Brannen-Koide rules in section 4.

2 2×2 Bogoliubov Mixing

In QFT, mixing acts on fermionic fields rather than states [2]. There are two relevant sets of deformed creation and annihilation operators: $a(\theta)$, $a^\dagger(\theta)$ and $\tilde{a}(\theta)$, $\tilde{a}^\dagger(\theta)$ for a deformation parameter θ. We stick to the fundamental fermionic representation, where $h = 1/2$, $N = a^\dagger a$ and the basic anticommutator

$$\{a, a^\dagger\} = 2h = 1 \tag{6}$$

is independent of the fermion deformation parameter $q = \exp 2i\theta$. Let

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

be the quantum ordinal. The deformed coproduct [2][1] is

$$\Delta(a) = e^{-i\theta} \otimes a + a \otimes e^{i\theta}, \tag{7}$$

$$\Delta(a^\dagger) = e^{-i\theta} \otimes a^\dagger + a^\dagger \otimes e^{i\theta},$$

$$\Delta(h) = 1 \otimes h + h \otimes 1.$$

Define the operators

$$A_\theta = e^{i\theta}(a \otimes 1) + e^{-i\theta}(1 \otimes a) \tag{8}$$

$$B_\theta = e^{i\theta}(a \otimes 1) - e^{-i\theta}(1 \otimes a)$$

and similarly $A_q^\dagger$, $B_q^\dagger$. Let

$$A(\theta) = \frac{1}{2\sqrt{2}}(A_\theta + A_{-\theta} + A_\theta^\dagger - A_{-\theta}^\dagger) \tag{9}$$

$$B(\theta) = \frac{1}{2\sqrt{2}}(B_\theta + B_{-\theta} - B_\theta^\dagger + B_{-\theta}^\dagger).$$

Then the Bogoliubov transformation is

$$\begin{pmatrix} a(\theta) \\ \tilde{a}(\theta) \end{pmatrix} \equiv F_2 \begin{pmatrix} A(\theta) \\ B(\theta) \end{pmatrix}, \tag{10}$$

where $a(\theta) = \cos\theta(a \otimes 1) - i\sin\theta(1 \otimes a^\dagger)$. We write this in the form

$$\begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix} = F_2 \begin{pmatrix} A & B \\ B & A \end{pmatrix} F_2^\dagger. \tag{11}$$

Compute the deformed anticommutator

$$\frac{1}{[2]_q}\{A_\theta, B_\theta^\dagger\} = \tan 2\theta \tag{12}$$

using $(1 \otimes x)(y \otimes 1) = -(y \otimes 1)(1 \otimes x)$. The Bogoliubov generator is

$$\mathbf{G} \equiv a^\dagger \tilde{a}^\dagger - a\tilde{a} \tag{13}$$

so that $a(\theta) = \exp(i\theta\mathbf{G})a\exp(-i\theta\mathbf{G})$. The deformation gives a thermal vacuum state [2] $\exp(i\theta\mathbf{G})|0,0\rangle$.

Now let

$$\begin{pmatrix} \cos\delta_{12} & \sin\delta_{12} \\ -\sin\delta_{12} & \cos\delta_{12} \end{pmatrix} \tag{14}$$

rotate two neutrino mass states ν_1, ν_2 into the flavor pair. The time dependent generator $\mathbf{G}(t)$ of this rotation is given [1] as

$$\mathbf{G}(t) = \exp[\frac{\delta_{12}}{2}\int d^3x(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x))] \tag{15}$$

where

$$\nu_i(x) = \sum_{\pm}\int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x}(u_i^\pm(t)\alpha_i^\pm(k) + v_i^\pm(t)\alpha_i^{\dagger\pm}(-k)). \tag{16}$$

Here the momentum k is always directed in a single direction, and we let $E_i = \sqrt{k^2 + m_i^2}$ be the neutrino energies. In [1], the oscillation probabilities P_{ee} and $P_{e\mu}$ are deformed from the standard formulae by a dual dependence on δ_{12} and θ, justified by a consideration of the mass state creation and annihilation operators. We argue instead that the experimentally successful standard formulae should be maintained exactly, and the θ parameters become identified with the δ_{ij} in some deeper theory of Hopf algebras for a nonperturbative regime.

3 Hopf Algebras

Let $\mathbb{F}$ be a characteristic zero field. For a finite group G, the group algebra $\mathcal{A}(G)$ [5][6] over $\mathbb{F}$ is the set of all sums $\sum_{g \in G} c_g g$ over the group, where $c_g \in \mathbb{F}$. The product $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the obvious termwise multiplication, which is noncommutative for nonabelian groups, with unit map η. There is also a coproduct $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, defined on group elements by

$$\Delta(g) = g \otimes g. \tag{17}$$

The counit $\epsilon : \mathcal{A} \to \mathbb{F}$ sends g to 1. $\mathcal{A}$ has the further structure of a Hopf algebra, since the group inverse defines an antipode map $S : \mathcal{A} \to \mathcal{A}$

$$S(g) = g^{-1} \tag{18}$$

and these operations obey the Hopf axiom

$$m(S \otimes 1)\Delta = \eta\epsilon. \tag{19}$$

For neutrino mixing, we start with 3×3 matrices in the group algebra $\mathcal{A}$ for S_3, the permutation group on three objects. A general such matrix has the form

$$P = \begin{pmatrix} a+x & b+y & c+z \\ c+y & a+z & b+x \\ b+z & c+x & a+y \end{pmatrix} \tag{20}$$

for six complex variables. To construct a unitary such matrix, use a product of three Euler factors $U_\nu = R_{12}(r_1)R_{23}(r_2)R_{13}(r_3)$ with

$$U_\nu = \begin{pmatrix} r_1 & i & 0 \\ i & r_1 & 0 \\ 0 & 0 & r+i \end{pmatrix} \begin{pmatrix} r_2+i & 0 & 0 \\ 0 & r_2 & i \\ 0 & i & r_2 \end{pmatrix} \begin{pmatrix} r_3 & 0 & i \\ 0 & r_3+i & 0 \\ i & 0 & r_3 \end{pmatrix}, \tag{21}$$

up to a normalisation factor. The real spectral parameters $r_i = \tan \delta_{ij}$ make U_ν vaguely resemble a Yang-Baxter equation, although there are no tensor products of spaces on which the factors act. That is, an equivalent mixing matrix is obtained on exchanging the outer two factors

$$R_{12}R_{23}R_{13} = \pi(R_{13}R_{23}R_{12}). \tag{22}$$

Below we will construct a proper $\mathbf{R}$-matrix for the Drinfeld double Hopf algebra.

In the next section, experimental values for the neutrino r_i are given. The matrix P_ν of (20) is recovered under the change of variables

$$-ia = r_1 r_2, \quad -ib = r_2 r_3, \quad -ic = r_1 r_3 - 1, \tag{23}$$

$$x = -r_1 - r_3, \quad y = -r_2, \quad z = r_1 r_2 r_3.$$

In this form, U_ν has a maximal CP phase of $\sim 3\pi/2$ in agreement with current experimental hints [7]. Our justification for restricting $U(3)$ to S_3 is as follows. In motivic scattering theory, S_3 is a Galois group that naturally acts on a canonical three dimensional space which carries abelian structure, most simply that of the Cartan subalgebra for either $sl(4)$, a part of D_4, or affine $sl(3)$. In this view the 2×2 factors of U_ν act on the Cartan subalgebra for $SU(3)$, the gauge group for color. We expect a similar parameterisation for quarks.

Rather than thinking of the associated $SU(3) \times SU(2)$, on the Hopf side there is a 24 dimensional adjoint representation for $su(3) \otimes su(2)$, and probably a Leech lattice picture of octonion triplets [8] related to color triality, where the circulant neutrino mass matrix is given as a 3×3 Hermitian element of a Jordan algebra.

The second Hopf algebra associated to a finite group is the functions $\mathcal{F}(G)$ on G with basis f_g for $g \in G$. This is commutative under pointwise multiplication but not cocommutative, since

$$\Delta f(g,h) = f(gh). \tag{24}$$

The Hopf algebras $\mathcal{A}$ and $\mathcal{F}$ are associative and coassociative. Another canonical Hopf structure is the Drinfeld double $\mathcal{D}(G)$ [9][5][10], namely $\mathcal{F}(G) \otimes \mathcal{A}(G)$, which is neither commutative nor cocommutative. Let $\delta_{g,k}$ denote the delta function that is 1 when $g = k$. Then the multiplication for $\mathcal{D}(G)$ is

$$(f_g \otimes u)(f_h \otimes v) = \delta_{gu,uh} f_g \otimes uv \tag{25}$$

and the coproduct

$$\Delta(f_g \otimes u) = \sum_{k \in G} (f_k \otimes u) \otimes (f_k^{-1} f_g \otimes u). \tag{26}$$

The counit sends f_g to $\delta_{g,1}$. $\mathcal{D}(G)$ has a unique, large $\mathbf{R}$-matrix given by

$$\mathbf{R} \equiv \sum_{g \in G} (f_g \otimes 1) \otimes (1 \otimes g). \tag{27}$$

The representation category for $\mathcal{D}(G)$ is a modular braided tensor category, where the modular group generators S and T are indexed by irrep objects.

In motivic QFT, we need representations of the so called cosmic Galois Hopf algebra. Like the Drinfeld double, this is a crossed product involving an action of the cocommutative part. Neutrino mixing looks at the adjoint case for S_3. Motivic methods [11] also involve other Hopf algebras, starting with the Butcher-Connes-Kreimer algebra [12][13][14] of rooted planar trees $\mathcal{H}_R$ for perturbative renormalisation, generalised to an algebra on labelled Feynman graphs. Rooted trees are a universal algebraic object, since for any other algebra B with unit and any morphism $b : B \to B$, there exists a unique map $\rho : \mathcal{H}_R \to B$ such that $\rho \cdot R = b \cdot \rho$, where R is the operation of adding a new root to a forest in $\mathcal{H}_R$, turning any forest into a tree with one extra node. In a category of algebras, for $\mathcal{H}$ a Hopf algebra, the set of morphisms $f, g : \mathcal{H} \to B$ forms a Butcher group with convolution $m_B(f \otimes g)\Delta_{\mathcal{H}}$. The antipode for $\mathcal{H}$ is the convolution inverse of the identity $1 : \mathcal{H} \to \mathcal{H}$.

The cyclic subgroup of 1-circulants in the fundamental representation of C_3 generates a subalgebra of $\mathcal{A}(S_3)$, in which the Koide mass matrix sits. Consider the double $\mathcal{D}(C_3)$. The Fourier transform F_3 diagonalises

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \overline{X} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{28}$$

to give

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix} = F_3^\dagger X F_3, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{\omega} & 0 \\ 0 & 0 & \omega \end{pmatrix} = F_3^\dagger \overline{X} F_3. \tag{29}$$

These eigenvalue diagonals are columns of F_3, that is functions (characters) on C_3. The next section shows that any diagonal should be interpreted as a function on C_3, and the change of basis F_3 represents functions as circulants in $\mathcal{A}$. So there is a representation of $\mathcal{D}(C_3)$ which looks like $\mathcal{A} \otimes \mathcal{A}$. Pointwise products of such functions f are preserved by F_3, and $\Delta(f) = f \otimes f$ is diagonalised by F_9. Since 1-circulants always commute, a trivial $\mathbf{R}$-matrix is easily computed. The usual $\mathbf{R}$-matrix starts with a function basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{30}$$

of non circulants, but the transform of these matrices should also be interpreted as an $\mathbf{R}$-matrix for $\mathcal{A}^{\otimes 4}$. The 16×16 case for C_2 is easy to check.

4 Neutrino Mixing

Circulants for mixing are motivated by the quantum representations for finite fields that occur in quantum information theory. We first define the Brannen-Koide mass triplets [16][17] for active neutrinos, neglecting for now the question of non local mirror states [18], although these may be relevant to mass generation.

The three components of the circulant Hermitian matrix $\sqrt{3}K(\delta)$ at $r = \sqrt{2}$ are

$$\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} + \begin{pmatrix} 0 & e^{i\delta} & 0 \\ 0 & 0 & e^{i\delta} \\ e^{i\delta} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & e^{-i\delta} \\ e^{-i\delta} & 0 & 0 \\ 0 & e^{-i\delta} & 0 \end{pmatrix}. \tag{31}$$

In the group algebra of S_3, each factor is an operation on the root lattice for $sl(4)$. $K(\delta)$ has eigenvalues

$$\lambda_1 = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\cos(\delta), \quad \lambda_2 = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\cos(\delta+\omega), \tag{32}$$

$$\lambda_3 = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\cos(\delta+\overline{\omega}),$$

which give a function $(\lambda_1, \lambda_2, \lambda_3)$ on C_3. The Brannen-Koide charged lepton and neutrino mass triplets are written in the form

$$m_i = \frac{\mu}{6}\lambda_i^2, \tag{33}$$

where μ is the scale. The λ_i triplet is the diagonalisation $F_3 K(\delta) F_3{}^\dagger$. For neutrinos, empirically, $\mu = 0.03$ eV and for the charged leptons μ is the proton mass m_p. The original Koide relation for charged leptons [19][20], near $\delta = 0.222$ rad, was used to predict the τ mass. Note that μ is proportional to $m_1 + m_2 + m_3$. The neutrino triplet is assumed to be a normal hierarchy at $r = \sqrt{2}$, but a candidate inverted hierarchy is obtained at $r = 2.4$ and $\delta = 1.039$, which may better suit the measured maximal CP phase. Using the cosine formula, the λ_i may be written

$$\lambda_1 = \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}}\cos(\delta) + 0, \tag{34}$$

$$\lambda_2 = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\cos(\delta) - \frac{1}{\sqrt{2}}\sin(\delta),$$
$$\lambda_3 = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\cos(\delta) + \frac{1}{\sqrt{2}}\sin(\delta),$$

where the nine coefficients form the unitary tribimaximal mixing matrix, and the sum along rows is from the product

$$\frac{\sqrt{2}}{\sqrt{3}}\begin{pmatrix} 1 & \cos(1) & \sin(1) \\ 1 & \cos(\omega) & \sin(\omega) \\ 1 & \cos(\overline{\omega}) & \sin(\overline{\omega}) \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & \cos(\delta) & 0 \\ 0 & 0 & \sin(\delta) \end{pmatrix}. \tag{35}$$

Recall that F_3 is a character table for the cyclic basis of K, with the column $\chi_1 = 1, \omega, \overline{\omega}$. The columns of the left factor in (35) are respectively χ_0, $\cos\chi_1$ and $-\sin\chi_2$, real projections of the complex characters. A unitary tribimaximal matrix is alternatively defined in the group algebra as

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1+i \end{pmatrix} \begin{pmatrix} \sqrt{2}+i & 0 & 0 \\ 0 & \sqrt{2} & i \\ 0 & i & \sqrt{2} \end{pmatrix}. \tag{36}$$

In this form one easily sees the zero coming from the missing R_{31} factor in (21). Including the observed non zero δ_{13} in $R_{31}(r_3)$, using the experimental values [7] for the normal hierarchy,

$$\delta_{12} = 34.50\,\text{deg}, \quad \delta_{23} = 41.00\,\text{deg}, \quad \delta_{13} = 8.44\,\text{deg}, \tag{37}$$

we obtain a current estimate for the neutrino mixing norms

$$|U_\nu/N| = \begin{pmatrix} 0.6320 & 0.5551 & 0.5408 \\ 0.7690 & 0.5201 & 0.3717 \\ 0.0963 & 0.6491 & 0.7546 \end{pmatrix}, \quad (38)$$

where the inserted normalisation factor is given by

$$N^2 = r_1^2 r_2^2 r_3^2 + r_1^2 r_2^2 + r_2^2 r_3^2 + r_1^2 r_3^2 + r_1^2 + r_2^2 + r_3^2 + 1 \quad (39)$$

and we use (23).

Consider K now as an element of the Jordan algebra of 3×3 complex matrices, embedded in the exceptional Jordan algebra over the octonions. The symmetric action of triality [21] on K is restricted to left, right or bimultiplication by $\exp(i2\delta/3)$. For the matrix $K(2/9)$, which approximates the precise charged lepton masses [16], this triality phase is 8.49 deg, a candidate for δ_{13} in neutrino mixing. That is, $\delta_{13} \sim \frac{4}{27}$ rad somehow defines the lepton δ phase in (35), for either a normal or inverted mixing matrix [7].

Observe that this use of δ_{13} in (35) distinguishes δ_{13} from the larger two mixing angles in U_ν, and suggests a fundamental role for the tribimaximal matrix.

The neutrino masses require a phase offset of $\delta = \pi/12$ from the charged lepton phase of $\delta = 0.222$ [16]. The triality fraction of this is 10 deg, which happens to be the difference between the tribimaximal 45 deg and the mixing angle of 35 deg. There is a similar difference with the third mixing angle in the normal hierarchy [7] and the other tribimaximal phase.

5 Summary

We have shown that Hopf algebraic structures may illuminate neutrino oscillations in the context of a motivic QFT, even without a working model of quantum gravity. In the motives underlying Feynman amplitudes in perturbative QFT [11], allowed mass parameters are given a priori and used to extend rational bases to a fixed algebraic field, but conceptually the quantisation of lepton mass is analogous to the quantisation of spin, which is easily specified by the eigenvalues of simple operators.

The success of the standard oscillation transition formulae for three active neutinos indicates that any Bogoliubov mixing coming from the spectral model in noncommutative geometry [1] appears only in the small θ limit. We suggest instead that θ should represent a mixing angle δ_{ij}, since in the fermion algebra the A and B operators effectively reduce to $(\cos\theta)a$ and $(\sin\theta)a^\dagger$. There is no place for mass state creation and annihilation operators, because mass generation is not accounted for by the Dirac fields of the Standard Model. Rather, the mixing algebra operates on one copy of the Dirac equation.

In the Drinfeld double $\mathcal{D}(C_3)$ one pairs functions and elements of the group algebra, but we have a map from functions to matrices in the group algebra, so that the double algebra looks like $\mathcal{A} \otimes \mathcal{A}$. Quantum Fourier transforms are inherently arithmetic in nature, and it is not surprising that they should in some way illuminate the working of Drinfeld **R**-matrices in an arbitrary dimensional vector space. Other structures in the modular Hopf category will presumably also appear in the neutrino phenomenology. Elsewhere we have considered the phase $\delta = 4/27$ in mass matrices for quarks, so it appears to be fundamental.

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