How to Effect a Desired Rotation of a Vector about a Given Axis via Geometric (Clifford) Algebra

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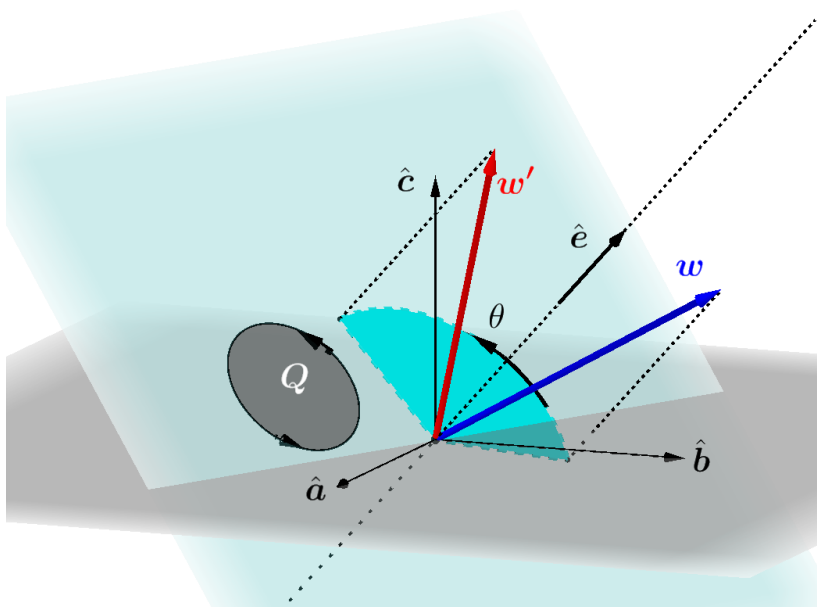
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Abstract

We show how to transform a "rotate a vector around a given axis" problem into one that may be solved via GA, which rotates objects with respect to bivectors. A sample problem is worked to show how to calculate the result of such a rotation conveniently via an Excel spreadsheet.

"Identify the new vector $\mathbf{w}'$ *produced by the rotation of vector* $\mathbf{w}$ *through the angle* θ, *around an axis given by the vector* $\hat{\mathbf{e}}$.

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Contents

1 Introduction

References [1] (pp. 280-286) and [2] (pp. 89-91) derive and explain the following formula for finding the new vector, $\mathbf{w}'$, that results from the rotation of a vector $\mathbf{w}$ through the angle θ with respect to a plane that is represented by the unit bivector $\mathbf{Q}$ to which that plane is parallel:

$$\mathbf{w}' = \left[exp\left(-\mathbf{Q}\frac{\theta}{2} \right) \right] [\mathbf{w}] \left[exp\left(\mathbf{Q}\frac{\theta}{2} \right) \right]. \tag{1.1}$$

That formula is convenient and efficient for manipulations of vectors that are represented abstractly as symbols, but what form does it take in a specific, concrete situation? For example, how do we use it when a client presents the vector $\mathbf{w}$ in terms of coordinates with respect to that client's chosen frame of reference, and wishes to know the coordinates of the vector that results when $\mathbf{w}$ is rotated through the angle θ about a given axis? What will we need to do to transform that problem into a form suitable for solution via Eq. (1.1), and what will the calculations "look like" as we work through them?

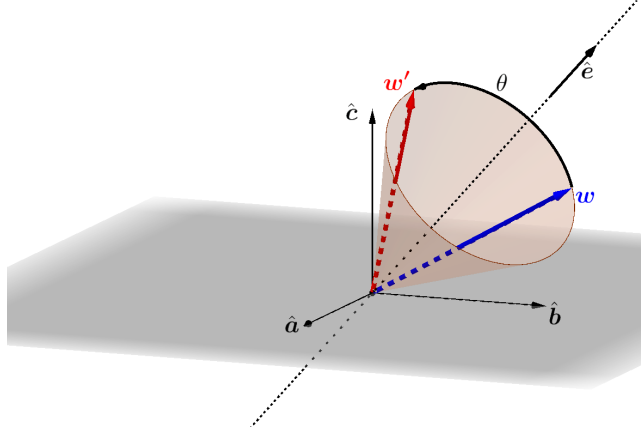   These are the questions that we will address here.

Figure 1: The new vector $\mathbf{w}'$ produced by the rotation of vector $\mathbf{w}$ through the angle θ, around an axis given by the vector $\hat{\mathbf{e}}$. The reference frame formed by vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ (taken in that order) is orthonormal.

2 Statement and Transformation of the Problem

2.1 Statement of the Problem

We state the problem as follows, with reference to Fig. 1:

The vector $\mathbf{w}$ $\left(= \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right)$ is rotated through the angle θ,

in the direction as indicated, about the unit vector $\hat{\mathbf{e}}$ $\left(= \hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)$.

Write the resulting vector $\mathbf{w}'$ in terms of the same basis vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$.

Note that the reference system is a right-handed orthonormal one (Reference [1], p. 53), and that the given rotation is a right-handed one about the vector $\hat{\mathbf{e}}$.

2.2 Transformation of the Problem

2.2.1 Why is a Transformation Necessary?

Rotations in three-dimensional Geometric Algebra ($\mathbb{G}^3$) are effected with respect to planes, rather than axes. Or to put it more correctly, with respect to bivectors rather than vectors. To use Eq. (1.1), we must identify the unit bivector that corresponds to the given axis of rotation, $\hat{\mathbf{e}}$. What do we mean by "corresponds to"? We'll answer that question in the next section.

2.2.2 Identifying the Unit Bivector that "Corresponds to" the Given Axis of Rotation

The rotation that we are asked to make is a right-handed one. Therefore —as may be inferred from a study of references [1] (p. (56, 63) and [2] (pp. 106-108) —the unit bivector $\mathbf{Q}$ that we seek is the one whose dual is $\hat{\mathbf{e}}$. That is, $\mathbf{Q}$ must satisfy the condition

$$\hat{\mathbf{e}} = \mathbf{Q}I_3^{-1};$$
$$\therefore \mathbf{Q} = \hat{\mathbf{e}}I_3. \tag{2.1}$$

Although we won't use that fact here, I_3^{-1} is I_3's negative: $I_3^{-1} = -\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$.

where I_3 is the right-handed pseudoscalar for $\mathbb{G}^3$. That pseudoscalar is the product, written in right-handed order, of our orthonormal reference frame's basis vectors: $I_3 = \hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ (and is also $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}$). Therefore, proceeding from Eq. (2.1),

To make this simplification, we use the following facts:

- The product of two perpendicular vectors (such as $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$) is a bivector;
- Therefore, for any two perpendicular vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{qp} = -\mathbf{qp}$; and
- (Of course) for any unit vector $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}\hat{\mathbf{p}} = 1$.

$$\begin{aligned}
\mathbf{Q} &= \hat{\mathbf{e}}I_3 \\
&= \left(\hat{\mathbf{a}}e_a + \hat{\mathbf{b}}e_b + \hat{\mathbf{c}}e_c\right)\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}} \\
&= \hat{\mathbf{a}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_a + \hat{\mathbf{b}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_b + \hat{\mathbf{c}}\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}e_c \\
&= \hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b. \tag{2.2}
\end{aligned}$$

In writing that last result, we've followed [2]'s convention (p. 82) of using $\hat{\mathbf{a}}\hat{\mathbf{b}}$, $\hat{\mathbf{b}}\hat{\mathbf{c}}$, and $\hat{\mathbf{a}}\hat{\mathbf{c}}$ as our bivector basis.

Two questions.

First, is $\mathbf{Q}$ a **unit** bivector, as Eq. (1.1) requires? Yes: for any bivector $\mathbf{B}$,

$$\|\mathbf{B}\| = \sqrt{\mathbf{B}\left(^-\mathbf{B}\right)}.$$

If we calculate $\|\mathbf{Q}\|$ according to that formula, using the expression in Eq. (2.2), we find (after expansion and simplification) that

$$\|\mathbf{Q}\| = \sqrt{e_a^2 + e_b^2 + e_c^2}\,,$$

which is equal to 1, because $\hat{\mathbf{e}}$ is a unit vector.

The second question is, "What would we have done if the required rotation were a left-handed one around $\hat{\mathbf{e}}$, rather than a right-handed one?" There are two reasonable ways to handle such a case. We could either (1) make the rotation a right-handed one around the vector $^-\hat{\mathbf{e}}$; or (2) recognize that a left-handed rotation through an angle ψ is a right-handed rotation through the angle $^-\psi$. Therefore, using the latter idea, we'd use the given vector $\hat{\mathbf{e}}$, but use $^-\psi$ as our angle instead of ψ itself.

Now that we've identified the unit bivector $\mathbf{Q}$ for our problem, we can re-state our problem in terms that will enable us to use Eq. (1.1).
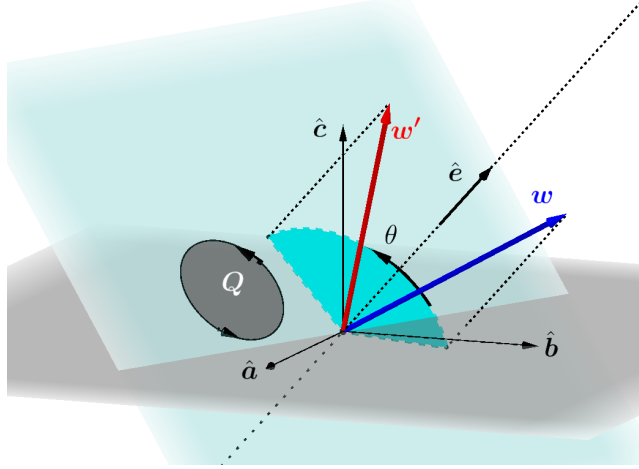
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Figure 2: The same situation as in Fig. 1, translated into GA terms. The unit bivector $\mathbf{Q}$ ($= \hat{\mathbf{e}}I_3$) is perpendicular to $\hat{\mathbf{e}}$. In GA, the angle of rotation would be the bivector $\mathbf{Q}\theta$ rather than the scalar θ.

2.3 Restatement of the Problem

When describing an angle of rotation in GA, we are often well advised—for sake of clarity—to write it as the product of the angle's scalar measure (in radians) and the bivector of the plane of rotation. In our present case, we would write that angle as $\mathbf{Q}\theta$. Therefore, using our expression for $\mathbf{Q}$ from Eq. (2.2), we restate our problem, with reference to Fig. 2, as

Recall that bivectors are "oriented areas": they do not possess the attribute "shape". Therefore, $\mathbf{Q}$ in Fig. 2 could have been drawn as any plane figure of unit area, and with the same orientation as shown.

The vector $\mathbf{w}$ $\left(= \hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right)$ is rotated through the angle $\left(\hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b\right)\theta$. Write the resulting vector $\mathbf{w}'$ in terms of the same basis vectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}$.

We're ready, now, to return to Eq. (1.1).

3 Calculating the Result, w′

To make use of Eq. (1.1), need (in addition to our expression for $\mathbf{Q}$ from Eq. (2.2)) the following identity, which holds for any unit bivector $\mathbf{B}$ and any angle ϕ (measured in radians):

$$exp\left(\mathbf{B}\phi\right) \equiv \cos\phi + \mathbf{B}\sin\phi.$$

Using this identity and Eq. (2.2), Eq. (1.1) becomes

$$\mathbf{w}' = \left[\cos\frac{\theta}{2} - \mathbf{Q}\sin\frac{\theta}{2}\right][\mathbf{w}]\left[\cos\frac{\theta}{2} + \mathbf{Q}\sin\frac{\theta}{2}\right]$$

$$= \left[\cos\frac{\theta}{2} - \left(\hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b\right)\sin\frac{\theta}{2}\right]\left[\hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right]\left[\cos\frac{\theta}{2} + \left(\hat{\mathbf{a}}\hat{\mathbf{b}}e_c + \hat{\mathbf{b}}\hat{\mathbf{c}}e_a - \hat{\mathbf{a}}\hat{\mathbf{c}}e_b\right)\sin\frac{\theta}{2}\right]. \quad (3.1)$$

If we expand the right-hand side of that result, we'll obtain 48 (!) terms, some of which will simplify to scalar multiples of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$, and others of which will simplify to scalar multiples of the trivector $\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$. The latter terms will cancel, leaving an expression for $\mathbf{w}'$ in terms of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$.

The prospect of carrying out that expansion and simplification is fairly terrifying, so before we dive into that task, we might want to think a bit about which tools we'd use to carry out the calculation in practice. In the absence of specialized GA software, we might use Excel to calculate the coordinates of $\mathbf{w}'$ in terms of $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$. With that end in mind, a reasonable step to take before expanding and simplifying the right-hand side of Eq. (3.1) is to define four scalar variables, which we'd use later in an Excel spreadsheet (Section 4):

- $f_o = \cos\dfrac{\theta}{2}$;

- $f_{ab} = e_c \sin\dfrac{\theta}{2}$;

- $f_{bc} = e_a \sin\dfrac{\theta}{2}$; and

- $f_{ac} = -e_b \sin\dfrac{\theta}{2}$.

Using these variables, Eq. (3.1) becomes

$$\mathbf{w}' = \left[f_o - \hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} - \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} - \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right]\left[\hat{\mathbf{a}}w_a + \hat{\mathbf{b}}w_b + \hat{\mathbf{c}}w_c\right]\left[f_o + \hat{\mathbf{a}}\hat{\mathbf{b}}f_{ab} + \hat{\mathbf{b}}\hat{\mathbf{c}}f_{bc} + \hat{\mathbf{a}}\hat{\mathbf{c}}f_{ac}\right].$$

After expanding and simplifying the right-hand side, we obtain

$$\mathbf{w}' = \hat{\mathbf{a}}\left[w_a\left(f_o^2 - f_{ab}^2 + f_{bc}^2 - f_{ac}^2\right) + w_b\left({}^-2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_c\left({}^-2f_of_{ac} + 2f_{ab}f_{bc}\right)\right]$$

$$+ \hat{\mathbf{b}}\left[w_a\left(2f_of_{ab} - 2f_{bc}f_{ac}\right) + w_b\left(f_o^2 - f_{ab}^2 - f_{bc}^2 + f_{ac}^2\right) + w_c\left({}^-2f_of_{bc} - 2f_{ab}f_{ac}\right)\right] \quad (3.2)$$

$$+ \hat{\mathbf{c}}\left[w_a\left(2f_of_{ac} + 2f_{ab}f_{bc}\right) + w_b\left(2f_of_{bc} - 2f_{ab}f_{ac}\right) + w_c\left(f_o^2 + f_{ab}^2 - f_{bc}^2 - f_{ac}^2\right)\right] .$$

But don't forget:

- The e's are for $\hat{\mathbf{e}}$ (a unit vector); and

- The angle θ is positive if in a right-handed sense with respect to the vector $\hat{\mathbf{e}}$, and negative in the contrary direction.
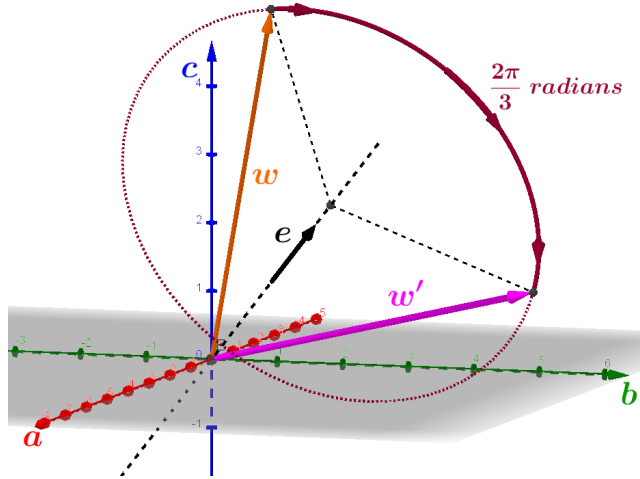
Figure 3: The vector $\mathbf{w} = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$ is rotated through $2\pi/3$ radians, in the direction shown, about an axis whose direction is given by the vector $\hat{\mathbf{e}} = \hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}$. What is the new vector, $\mathbf{w}'$, that results? Note that the axes $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually perpendicular.

4 A Sample Calculation

We'll solve the following problem, with reference to Figs. 3 and 4, keeping our eyes open for data that will need to be transformed so that we may use Eq. (3.2).

The vector $\mathbf{w}$ is given by $\mathbf{w} = \frac{4}{3}\hat{\mathbf{a}} + \frac{4}{3}\hat{\mathbf{b}} + \frac{16}{3}\hat{\mathbf{c}}$. It is rotated through $2\pi/3$ radians, in the direction shown, about an axis whose direction is given by the vector $\hat{\mathbf{e}} = \hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}$. What is the new vector, $\mathbf{w}'$, that results?

Examining Figs. 3 and 4, we see that the vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ (taken in that order) form an orthonormal reference frame, as required. However, the vector $\mathbf{e}$ is not a unit vector (its magnitude is $\sqrt{3}$), and the angle of rotation is in the direction defined as negative. Therefore, in our calculations we will use $\hat{\mathbf{e}}$ $\left(= \frac{1}{\sqrt{3}}\hat{\mathbf{a}} + \frac{1}{\sqrt{3}}\hat{\mathbf{b}} + \frac{1}{\sqrt{3}}\hat{\mathbf{c}} \right)$ as our "axis" vector, and $-2\pi/3$ as θ. A screen shot of the Excel spreadsheet ([3]) used for the calculation is shown in Fig. 5. From the perspective (Fig. 4) of someone who is looking at the origin along the direction $-\left(\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}\right)$, the rotation by $-2\pi/3$ brings $\mathbf{w}$ into alignment with the $\mathbf{b}$ axis. In the spreadsheet, that result is indicated by the fact that $\mathbf{w}'$'s $\hat{\mathbf{b}}$ coordinate is $\mathbf{w}$'s $\hat{\mathbf{c}}$ coordinate, and $\mathbf{w}'$'s $\hat{\mathbf{a}}$ coordinate is $\mathbf{w}$'s $\hat{\mathbf{c}}$ coordinate.
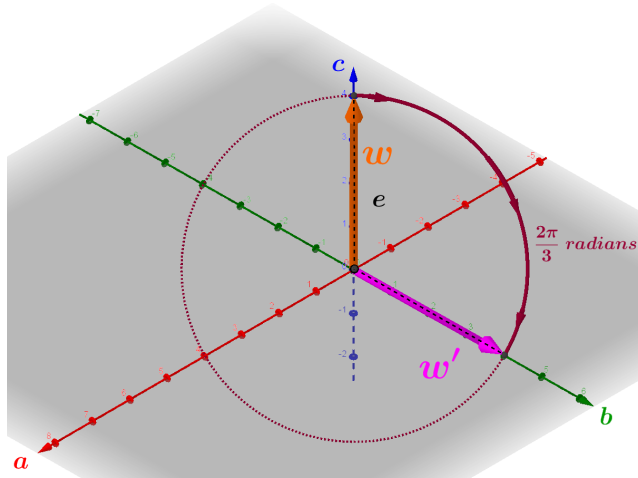
Figure 4: The same situation as in Fig. 3, but looking toward the origin along the direction $-\left(\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}}\right)$.

5 Summary

We have seen how to transform a "rotate a vector around a given axis" problem into one that may be solved via GA, which rotates objects with respect to bivectors. We have also seen how to calculate the result conveniently via an Excel spreadsheet. Two important cautions are (1) the axis must be expressed as a unit vector; and (2) the sign of the angle of rotation must be determined correctly.

References

[1] D. Hestenes, 1999, *New Foundations for Classical Mechanics*, (Second Edition), Kluwer Academic Publishers (Dordrecht/Boston/London).

[2] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[3] J. A. Smith, 2017, "Rotation of a Vector about an Axis" (an Excel spreadsheet), https://drive.google.com/file/d/0B2C4TqxB32RRNHBHV2tpSUhRTUk/view?usp=sharing.

Rotation of a Vector about an Axis

Gray fields are informational.

The vector, **w**, to be rotated
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.3333333	1.3333333	5.3333333

Angle of rotation In radians. Positive
sense is in right-hand sense about **e**

-2.094395102

Axis of rotation, **e**
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1	1	1

Axis expressed as a unit vector **ehat**
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
0.5773503	0.5773503	0.57735027

Components of the unit bivector, **Q**

a_hat^b_hat	b_hat^c_hat	a_hat^c_hat
0.5773503	0.5773503	-0.5773503

Check: Is **ehat** unitary?

||**ehat**||=	1

Calculated values of the factors f

f_o	f_{ab}	f_{bc}	f_{ac}
0.5	-0.5	-0.5	0.5

Check: Sum of the squares
of the f's = 1?

1

Result

The vector, **w'**, that results
from the rotation
Components **a_hat**, **b_hat**, **c_hat**

a_hat	b_hat	c_hat
1.3333333	5.3333333	1.3333333

Check: ||w'||=||w||?

||**w'**||=	32
||**w**||=	32

Figure 5: Screen shot of the Excel spreadsheet used to calculate the coordinates for the vector **w'**. Quantities shown in the spreadsheet are defined in the text.