A COVARIANT RICCI FLOW

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Abstract: In this work, we discuss the possibility to formulate a covariant Ricci flow so that it satisfies the principle of relativity and therefore can be applied to all coordinate systems defined on a Riemannian manifold. Since the investigation may be considered to be in the domain of pure mathematics, which is outside our field of physical investigations, therefore there may be errors in mathematical arguments that we are unable to foresee.

In our previous works, we have examined geometrical and topological formulations that can be applied to physical theories, in particular spacetime structures of quantum particles [1,2]. In general, the geometrical approach to fundamental structures of quantum particles requires the construction of a space so that physical objects can be identified with the geometrical properties of that space. The physical description that is associated with the geometrical structures are expressed in the form of field equations, such as Einstein's field equations of general relativity. From this view point, geometry no longer exists independently of physics. Geometrical and topological methods applied to physical theories are a way of establishing a correspondence between mathematical concepts and physical objects. This can be achieved by searching for a mathematical structure that can provide a consistent description of dynamics and physical observables. Conversely, physical objects may provide an empirical realisation of the abstract concepts associated with geometry and topology. However, there are restrictions on the applicability of geometrical and topological methods in the construction of physical theories. A general feature of geometrical models applied to physics is that their mathematical realisation can only be described approximately since physical theories have been based on concepts, such as mass, which are poorly defined from the point of view of mathematics. Geometrical procedures that require the generalisation of such concepts may also lead to a poor description of physical phenomena. For this reason, Einstein's theory of general relativity cannot be considered to be a truly geometrical theory because the energy-momentum tensor which enters the theory is a non-geometrical quantity. As shown in our works [2,3], this blemish can be overcome by formulating the field equations of the gravitational field using the contracted Bianchi identities

$$\nabla_\beta R^{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} \nabla_\beta R \tag{1}$$

Equation (1) has a covariant form of the field equations of the electromagnetic field written in a covariant form as $\partial_\beta F^{\alpha\beta} = \mu j^\beta$, where the electromagnetic tensor $F^{\alpha\beta}$ is expressed in terms of the four-vector potential $A^\mu \equiv (V, \mathbf{A})$ as $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. The four-current j^μ is defined as $j^\mu \equiv (\rho_e, \mathbf{j}_e)$. From this similarity, it is possible to suggest that a four-current of some form of matter $j^\alpha = (\rho, \mathbf{j}_i)$ can be defined purely geometrical as

$$j^\alpha = \frac{1}{2} g^{\alpha\beta} \nabla_\beta R \tag{2}$$

For the case in which $g^{\alpha\beta} \nabla_\beta R = 0$, Equation (1) reduces to the equation

$$\nabla_\beta R^{\alpha\beta} = 0 \tag{3}$$

Firstly, since $\nabla_\mu g_{\alpha\beta} \equiv 0$ for a given metric tensor $g_{\alpha\beta}$, Equation (3) implies

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{4}$$

where Λ is an undetermined constant. If we consider a centrally symmetric field then the Schwarzschild solution can be found as

$$ds^2 = \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

Secondly, if we consider only coordinate transformations that are time-independent then from Equation (3) we can obtain the Ricci flow given by the evolution equation

$$\frac{\partial g_{\alpha\beta}}{\partial t} = k R_{\alpha\beta} \tag{6}$$

It is observed that even though the Ricci flow has been considered as a purely geometric process, whose purpose is to smooth out irregularities of a Riemannian manifold, the Ricci flow itself contains in it evolutionary processes that can be applied to explain physical phenomena. For example, consider the case of neckpinchings that arise from an evolution Ricci flow. It is a remarkable fact that the Ricci flow does not give rise to neckpinches on two-dimensional manifolds, but neckpinchings do happen in three-dimensional manifolds, as illustrated in two-dimensional space in Figure 1 below [4,5]

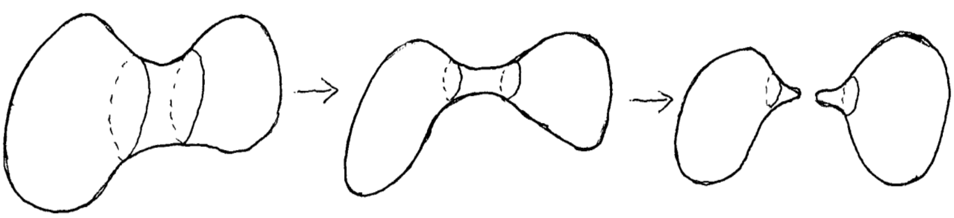

Figure 1

The neckpinching process can be used to explain the process of radiation of a photon from an atom as a geometrical process. Furthermore, if the radiation is a neckpinching process than quantum particles must exist as three-dimensional manifolds and this leads to the possibility to classify elementary particles as Thurston geometries [6,7]. However, there are problems with the form of the Ricci flow given in Equation (6). First, the Ricci flow given in Equation (6) is not a tensorial equation therefore it does not satisfy the principle of relativity and therefore cannot be applied to all coordinate systems. Second, the Ricci flow given in Equation (6) becomes a tensorial equation only when we restrict the coordinate transformations to those that are time-independent and this can only result in non-relativistic theories [3]. In the following we will discuss how this can be overcome by formulating a covariant Ricci flow that is invariant under all coordinate transformations instead of invariant only under coordinate transformations that are time-independent.

It is shown in differential geometry that besides the derivatives with respect to affine connections, Lie derivatives are also invariant under coordinate transformations on a differentiable manifold. For clarity, we first outline how the Lie derivatives are introduced into differential geometry [8]. On a manifold M, consider a congruence of curves given by $x^\mu = x^\mu(u)$ on which a tangent vector field dx^μ/du along the curve can be defined. If a tangent vector field can be defined for every curve in the congruence then a vector field X^μ can be established over the whole manifold. Inversely, a congruence of curves can be obtained from a given non-zero vector field defined over a differentiable manifold. Now a tensor field $T^{\alpha\cdots}_{\beta\ldots}(x)$ can be differentiated by using the vector field X^μ. First we use the congruence of curves to drag the tensor at some point P, $T^{\alpha\cdots}_{\beta\ldots}(P)$, along the curve passing through P to some neighbouring point Q and then compare the dragged-along tensor with the tensor already at Q, $T^{\alpha\cdots}_{\beta\ldots}(Q)$, as shown in Figure 2

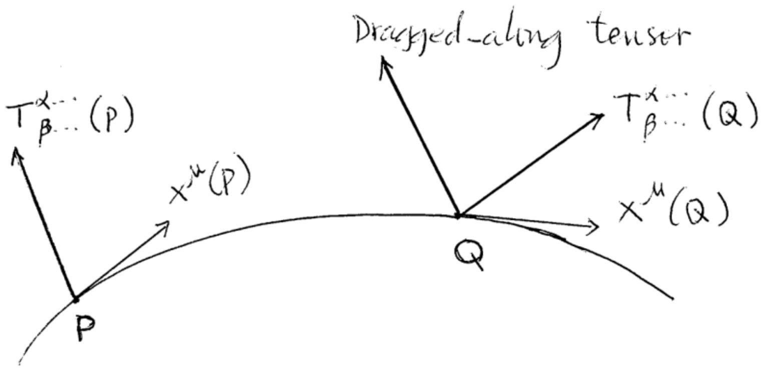

Figure 2

Therefore, a derivative can be defined by subtracting the dragged-along tensor $T'^{\alpha\cdots}_{\beta\ldots}(Q)$ and the tensor at Q, which is $T^{\alpha\cdots}_{\beta\ldots}(Q)$, as follows

$$L_X T^{\alpha\cdots}_{\beta\ldots}(Q) = \lim_{\delta u \to 0} \frac{T^{\alpha\cdots}_{\beta\ldots}(Q) - T'^{\alpha\cdots}_{\beta\ldots}(Q)}{\delta u} \tag{7}$$

The following results can be obtained:

The Lie derivative of a scalar field ϕ

$$L_X\phi = X^\mu \partial_\mu \phi \tag{8}$$

The Lie derivative of a covariant vector field Y_μ

$$L_X Y_\mu = X^\nu \partial_\nu Y_\mu + Y_\nu \partial_\mu X^\nu \tag{9}$$

The Lie derivative of a contravariant vector field Y^μ

$$L_X Y^\mu = X^\nu \partial_\nu Y^\mu - Y^\nu \partial_\nu X^\mu \tag{10}$$

The Lie derivative of a general tensor field $T^{\alpha\cdots}_{\beta\dots}$

$$L_X T^{\alpha\cdots}_{\beta\dots} = X^\mu \partial_\mu T^{\alpha\cdots}_{\beta\dots} - T^{\mu\cdots}_{\beta\dots}\partial_\mu X^\alpha - \cdots T^{\alpha\cdots}_{\mu\dots}\partial_\beta X^\mu + \cdots \tag{11}$$

In particular, for the case of a covariant metric tensor $g_{\alpha\beta}$ we have

$$L_X g_{\alpha\beta} = X^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\alpha}\partial_\beta X^\mu + g_{\mu\beta}\partial_\alpha X^\mu \tag{12}$$

Besides the important properties of being linear, satisfying the product rule for differentiation and commuting with contraction, the Lie differentiation with respect to a vector field X also preserves the type of a tensor. Since the Lie derivative of a covariant metric tensor of second rank is also a covariant tensor of second rank we may propose the following tensor equation

$$L_X g_{\alpha\beta} = -\kappa R_{\alpha\beta} \tag{13}$$

where κ is a dimensional constant. Using Equation (12), Equation (13) also can be written as

$$X^\mu \partial_\mu g_{\alpha\beta} + g_{\mu\alpha}\partial_\beta X^\mu + g_{\mu\beta}\partial_\alpha X^\mu = -\kappa R_{\alpha\beta} \tag{14}$$

As an illustration, consider the case when the vector field X^μ is the gradient of a scalar function f and satisfies the condition

$$X^\mu \partial_\mu g_{\alpha\beta} = \lambda g_{\alpha\beta} \tag{15}$$

then Equation (14) reduces to a homothetic Ricci soliton equation

$$\kappa R_{\alpha\beta} + \nabla_\alpha \nabla_\beta f + \lambda g_{\alpha\beta} = 0 \tag{16}$$

where λ is the homothetic constant. If $\kappa > 0$, then the soliton is shrinking if $\lambda < 0$, the soliton is expanding if $\lambda > 0$ and the soliton is static if $\lambda = 0$. An example of a static gradient Ricci soliton is the cigar soliton $f(x,y) = (x^2 + y^2)/2$ on the two-dimensional space R^2 with the metric is given by the relation $ds^2 = (dx^2 + dy^2)/(1 + x^2 + y^2)$ and the vector field $X = -(x\,\partial/\partial x + y\,\partial/\partial y)$ [4,5].

Mathematically, the covariant flow given in Equation (14) can be reduced to the Ricci flow given in Equation (6) if the vector field X^μ can be smoothly assigned values in the form

$X^\mu = (X^0, 0,0,0)$, where X^0 is a constant temporal component of the vector field. In this case we obtain the form of the Ricci flow given in Equation (6) as

$$X^0 \partial_0 g_{\alpha\beta} = -\kappa R_{\alpha\beta} \quad (17)$$

Physically, Equation (17) can be explained using commoving synchronous coordinate systems, as described in *Gravitation* [9]. First we choose a homogeneous three-dimensional spatial manifold S formed by some fluid substance, which is a hypersurface, and then assign a coordinate time t to all events on the manifold and set up a spatial coordinate system x^μ, $\mu = 1,2,3$ on S. These spatial coordinates propagate off S and throughout all spacetime by means of the world lines. The spatial coordinates are then considered to be comoving if they are assigned to events at which the world line intersects the hypersurface S. Because the hypersurfaces S are given by the condition $t = constant$, the spatial basis vector ∂/x^μ, $\mu = 1,2,3$ at any given event are tangent to the hypersurface and the temporal basis vector $\partial/\partial t$ is tangent to the world line. In this case, the temporal coordinate is the proper time of the world line and is the four-velocity of the motion of the fluid substance that form the three-dimensional spatial manifold.

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