Equation of motion of a particle in a potential proportional to square of second derivative of position w.r.t time in its Lagrangian

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Abstract:
I constructed a Lagrangian functional and equation of motion of a classical particle in a potential that is proportional to square of second derivative of its position with respect to time. Solution of equation of motion is identical to the solution of the equation of motion of the conventional simple harmonic oscillator in a potential that is proportional to square of its position.

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(1) Introduction:

(1.1) Definition of a harmonic oscillator

In classical mechanics, a harmonic oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force, F proportional to the displacement x:

$$\vec{F} = -k\vec{x}$$

where k is a positive constant.

If F is the only force acting on the system, the system is called a simple harmonic oscillator, and it undergoes simple harmonic motion: sinusoidal oscillations about the equilibrium point, with constant amplitude and a constant frequency (which does not depend on the amplitude).

Mechanical examples include pendulums (with small angles of displacement), masses connected to springs, and acoustical systems. Other analogous systems include electrical harmonic oscillators such as RLC circuits. The harmonic oscillator model is very important in physics, because any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations.

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Harmonic oscillators occur widely in nature and are exploited in many manmade devices, such as clocks and radio circuits. They are the source of virtually all sinusoidal vibrations and waves.

(1.2) Simple harmonic oscillator

A simple harmonic oscillator is an oscillator consists of a mass m, which experiences a single force, F which pulls the mass in the direction of the point $x = 0$ and depends only on the mass's position x and a constant k. Balance of forces (Newton's second law) for the system is

$$F = ma = m\frac{d^2x}{dt^2} = m\ddot{x} = -kx$$

Solving this differential equation, we find that the motion is described by the function

$$x(t) = x_0 \cos(\omega_0 t - \varphi),$$

where $x_0 = x(t = t_0), \omega_0 = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$

The motion is periodic, repeating itself in a sinusoidal fashion with constant amplitude x_0. In addition to its amplitude, the motion of a simple harmonic oscillator is characterized by its period T, the time for a single oscillation or its frequency $f = 1/T$, the number of cycles per unit time. The position at a given time t also depends on the phase, φ which determines the starting point on the sine wave. The period and frequency are determined by the size of the mass m and the force constant k, while the amplitude and phase are determined by the starting position and velocity.

The velocity and acceleration of a simple harmonic oscillator oscillate with the same frequency as the position but with shifted phases. The velocity is maximum for zero displacement, while the acceleration is in the opposite direction as the displacement.

The potential energy stored in a simple harmonic oscillator at position x is

$$U(x) = \frac{1}{2}kx^2$$

(1.3) Equivalent systems

Simple harmonic oscillators occurring in a number of areas of engineering are equivalent in the sense that their mathematical models are identical. Below is a table showing analogous quantities in four simple harmonic oscillator systems in mechanics and electronics. If analogous parameters on the same line in the table are given numerically equal values, the behaviors of the oscillators - their output waveform, resonant frequency - are the same.

Translational Mechanical	Rotational Mechanical	Series LC Circuit	Parallel LC Circuit
Position x	Angle θ	Charge q	Flux linkage ϕ
Velocity $\frac{dx}{dt}$	Angular velocity $\frac{d\theta}{dt}$	Current $\frac{dq}{dt}$	Voltage $\frac{d\phi}{dt}$
Mass m	Moment of inertia I	Inductance L	Capacitance C
Spring constant k	Torsion constant μ	Elastance $1/C$	Magnetic reluctance $1/L$
Differential equation:			
$m\ddot{x}+kx=0$	$I\ddot{\theta}+\mu\theta=0$	$L\ddot{q}+q/C=0$	$C\ddot{\phi}+\phi/L=0$

(1.4) Application to a conservative force

The problem of the simple harmonic oscillator occurs frequently in physics, because a mass at equilibrium under the influence of any conservative force, in the limit of small motions, behaves as a simple harmonic oscillator.

A conservative force is one that has a potential energy function. The potential energy function of a harmonic oscillator is:

$$U(x)=\frac{1}{2}kx^2$$

Given an arbitrary potential energy function $U(x)$, one can do a Taylor expansion in terms of $U(x)$ around an energy minimum $(x=x_0)$ to model the behavior of small perturbations from equilibrium.

$$U(x)=U(x_0)+(x-x_0)U'(x_0)+\frac{1}{2}(x-x_0)^2U''(x_0)+O(x-x_0)^3$$

Because $U(x_0)$ is a minimum, the first derivative evaluated at x_0 must be zero, so the linear term drops out:

$$U(x)=U(x_0)+\frac{1}{2}(x-x_0)^2U''(x_0)+O(x-x_0)^3$$

The constant term $U(x_0)$ is arbitrary and thus may be dropped, and a coordinate transformation allows the form of the simple harmonic oscillator to be retrieved:

$$U(x)\approx\frac{1}{2}x^2U''(0)=\frac{1}{2}kx$$

Thus, given an arbitrary potential energy function $U(x)$ with a non-vanishing second derivative, one can use the solution to the simple harmonic oscillator to provide an approximate solution for small perturbations around the equilibrium point.

(1.5) Examples:

(i) Simple pendulum

The differential equation governing a simple pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

If the maximum displacement of the pendulum is small, we can use the approximation $\sin(\theta) \approx \theta$ and instead consider the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0.$$

The solution to this equation is given by:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{l}{g}}t\right)$$

where θ_0 is the largest angle attained by the pendulum. The period, the time for one complete oscillation, is given by the expression

$$T_0 = 2\pi\sqrt{\frac{l}{g}},$$

which is a good approximation of the actual period when θ_0 is small.

(ii) Spring-mass system

When a spring is stretched or compressed by a mass, the spring develops a restoring force. Hooke's law gives the relationship of the force exerted by the spring when the spring is compressed or stretched a certain length:

$$F(t) = -kx(t)$$

where F is the force, k is the spring constant, and x is the displacement of the mass with respect to the equilibrium position. The minus sign in the equation indicates that the force exerted by the spring always acts in a direction that is opposite to the displacement (i.e. the force always acts towards the zero position), and so prevents the mass from flying off to infinity.

By using either force balance or an energy method, it can be readily shown that the motion of this system is given by the following differential equation:

$$F(t) = -kx(t) = m\frac{d^2}{dt^2}x(t) = ma.$$

...the latter being Newton's second law of motion.

If the initial displacement is x_0, and there is no initial velocity, the solution of this equation is given by:

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right).$$

Given an ideal mass-less spring, m is the mass on the end of the spring. If the spring itself has mass, its effective mass must be included in m.

(1.6) Energy variation in the spring system

In terms of energy, all systems have two types of energy, potential energy and kinetic energy. When a spring is stretched or compressed, it stores elastic potential energy, which then is transferred into kinetic energy. The potential energy within a spring is determined by the equation $U(x)=\frac{1}{2}kx^2$

When the spring is stretched or compressed, kinetic energy of the mass gets converted into potential energy of the spring. By conservation of energy, assuming the datum is defined at the equilibrium position, when the spring reaches its maximum potential energy, the kinetic energy of the mass is zero.
When the spring is released, it tries to return to equilibrium, and all its potential energy converts to kinetic energy of the mass.

(2) Discussion

(2.1) Second Variations of Calculus of Variations of scalar functions

It is well known that the Euler-Lagrange equation resulting from applying the second variations of the Calculus of Variations of a Lagrangian functional $L(t,q(t),\dot{q}(t),\ddot{q}(t))$ of a single independent variable $q(t)$, its first and second derivatives of the following action

$$I[q(t)]=\int L(t,q(t),\dot{q}(t),\ddot{q}(t))\ dt$$

when varied with respect to the arguments of integrand and the variation are set to zero, i.e.

$$0=\delta\ I[q(t)]=\delta\int L(t,q(t),\dot{q}(t),\ddot{q}(t))\ dt$$
$$=\int \delta[L(t,q(t),\dot{q}(t),\ddot{q}(t))]\ dt$$

- and assuming the Lagrangian functional doesn't depend on time explicitly (i.e. $L=L(q(t),\dot{q}(t),\ddot{q}(t))$), then $\frac{\partial L}{\partial t}=0$ and assume that the variations $\delta q=0,\delta\dot{q}=0$, at the end points of integration - then, the Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}+\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}}=0$$

(2.2) The Lagrangian of the simple harmonic oscillator

The Lagrangian of the conventional harmonic oscillator in one dimension may be written as

$$L=\frac{1}{2}m\ \dot{x}^2-\frac{1}{2}k\ x^2$$

In which the potential is proportional to the square of the position x .

The equation of motion is obtain from the Euler-Lagrange equation

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0$$

is given by

$$-kx-m\ddot{x}=0$$

(2.3) A Lagrangian of a particle in a potential that is proportional to square of second derivative of its position with respect to time

When a Lagrangian of a particle moving in one dimension is given by

$$L=\frac{1}{2}m\ \dot{x}^2-\frac{1}{2}J\ \ddot{x}^2$$

- where m ($[m]=kg$) is the mass of the particle which is assumed to be constant and J ($[J]=kg\ s^2$) is unknown constant ,where both constants are positive and have finite real values, and none have zero value (i.e. $m\neq 0, J\neq 0; m\neq\infty, J\neq\infty$) - then the equation as derived from Euler-Lagrange equation by performing the partial derivatives on the Lagrangian functional:

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}+\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}}=0$$

may be given by

$$-\frac{d}{dt}(m\dot{x})-\frac{d^2}{dt^2}(J\ \ddot{x})=0$$

Then, the equation of motion may be written as

$$-m\ddot{x}-J\ \ddot{\ddot{x}}=0$$

We assume a solution of the form

$$x=x_0\ e^{\pm i(\omega\ t-\varphi)}$$

where x_0 a constant measured in [m], $i=\sqrt{-1}$ is the imaginary unit, ω is a constant measured in [Hz], t is the time measured in [s] and φ is a phase factor measured in [radians].

Substituting in the equation of motion, we get

$$-m(-\omega^2)x-J(\omega^4)x=0$$

$$[-m(-\omega^2)-J(\omega^4)]x=0$$

Since, $x\neq 0$ then, we must have

$$[-m(-\omega^2)-J(\omega^4)]=0$$

$$-m(-\omega^2)-J(\omega^4)=0$$

$$+m\omega^2-J\omega^4=0$$

This has four solutions, using the general method for determining roots of a quadratic equation ($ax^2+bx+c=0, a\neq 0.$), they are:

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}; a = -J; b = +m; c = 0$$

The roots of the equation $+m\omega^2 - J\omega^4 = 0$ are,

$$\omega^2{}_{\pm} \equiv \omega_0{}^2{}_{\pm} = \frac{-m \pm \sqrt{m^2}}{2(-J)} = \frac{-m \pm \sqrt{m^2}}{-2J} = \frac{-m \pm m}{-2J} = \frac{-m \pm m}{-2J}$$

Then, the values of ω_0 are

$$\omega_0 = 0, \ \pm\sqrt{\frac{m}{J}}$$

(3) Conclusion:

The result above is remarkable. It tells us that the equation of motion of a particle in a potential that is proportional to the square of its position is exactly the same as in a potential that is proportional to the square of its acceleration. The Lagrangian functional of a simple harmonic oscillator is ambiguous in this formulation as long as the potential energy is concerned.

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