# Double integrals and series for some classical constants

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ABSTRACT
This note presents a collection of double integrals
for some classical constants.

## INTRODUCTION

Some classical constants: $\pi$ , $\gamma$, $G$ , $\ln 2$ , $\ln 3$ , $\phi$ , $\zeta(2)$ , $\zeta(3)$ , $e$ ,...

1. $$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

2. $$\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n) \right)$$

3. $$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

4. $$\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad , \quad \ln(3) = 2 \sum_{n=0}^{\infty} \frac{2^{-2n-1}}{2n+1}$$

5. $$\phi = \frac{1+\sqrt{5}}{2}$$

6. $$\zeta(2) = \frac{\pi^2}{6} = \sum_{n=1}^{\infty} n^{-2} \quad , \quad \zeta(3) = \sum_{n=1}^{\infty} n^{-3}$$

7. $$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

In this note we give double integrals for some classical constants, Guillera-Sondow paper (2006) is "Master paper".

The following example appears in Ref.1(Guillera-Sondow,example 3.1,pag.7) :

8. $$G - \frac{\pi^2 i}{48} = \int_0^1 \int_0^1 \frac{1}{1 + x y i} \, dx \, dy$$

Using formula (8), we obtain:

9. $$\frac{\pi^2}{48} + i G = 2 \sum_{n=0}^{\infty} \left( \frac{1 + 2 i}{5} \right)^{n+1} \sum_{k=0}^{n} \binom{n}{k} (-2)^k (k+1)^{-2}$$

10. $$\frac{\pi^2}{48} + i G = \left( \frac{1 + 3 i}{5} \right) \sum_{n=0}^{\infty} \left( \frac{2 + i}{5} \right)^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1 + i)^k (k+1)^{-2}$$

11. $$\frac{\pi^2}{48} + i G = i \ln 2 + \sum_{n=1}^{\infty} \left( \frac{1 - (1 - i)^{n+1}}{n + 1} \right) \cdot \left( \ln 2 + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n+k} \left( \frac{1 - 2^{-n+k}}{n - k} \right) \right)$$

12. $$\frac{\pi^2}{48} + i G = \frac{\ln 2}{2} + \frac{\pi i}{4} + \sum_{n=1}^{\infty} \frac{1}{n+1} \left( \frac{\ln 2}{2} + \frac{\pi i}{4} + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^{n+k} \left( \frac{1 - (1 + i)^{-n+k}}{n - k} \right) \right)$$

13. $$\frac{\pi^2}{48} + i G = 2 i \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{n}{n-k} \binom{n-k}{m} \frac{(1 - i)^k (-1)^m 2^{k+m}}{(m + k + 1)^2 3^{n+1}}$$

14. $$\frac{\pi^2}{48} + i G = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \left( \frac{1 + i}{2} \right)^{n+1} F\left( \{1, n+1\}, \{n+2\}, \frac{1 + i}{2} \right)$$

remark: $F(\{a, b\}, \{c\}, z)$ , is the hypergeometric function.

15. $$G - \frac{\pi^2 i}{48} = 2 \int_0^1 \int_0^{\frac{\pi}{2}} \frac{r}{2 + i r^2 \sin(2\theta)} \, d\theta \, dr + \iint_R \frac{1}{1 + x y i} \, dx \, dy$$

16. $$R = \{0 < x < 1, 0 < y < 1, x^2 + y^2 > 1\}$$

17. $$\iint_{R1} \frac{1}{1 + x y i} \, dx \, dy = 2 \int_0^1 \int_0^{\frac{\pi}{2}} \frac{r}{2 + i r^2 \sin(2\theta)} \, d\theta \, dr = \frac{\pi}{2} \ln(\phi) - \frac{i}{2} (\ln(\phi))^2$$

18. $$R1 = \{0 < x < 1, 0 < y < 1, 0 < x^2 + y^2 < 1\}$$

19. $$\iint_R \frac{1}{1 + x y i}\, dx\, dy = G - \frac{\pi^2 i}{48} - \frac{\pi}{2}\ln(\phi) + \frac{i}{2}(\ln(\phi))^2$$

20. $$G - \frac{\pi \ln(\phi)}{2} = \iint_R \frac{1}{1 + x^2 y^2}\, dx\, dy$$

21. $$\frac{\pi^2}{48} - \frac{(\ln(\phi))^2}{2} = \iint_R \frac{x y}{1 + x^2 y^2}\, dx\, dy$$

22. $$\frac{\pi \ln(\phi)}{2} = \iint_{R1} \frac{1}{1 + x^2 y^2}\, dx\, dy$$

23. $$\frac{(\ln(\phi))^2}{2} = \iint_{R1} \frac{x y}{1 + x^2 y^2}\, dx\, dy$$

the next example (Guillera-Sondow,example 3.14,pag.11):

24. $$\ln(\pi) = \int_0^1\int_0^1 \frac{1 + x}{(1 + x y)(-\ln(x y))}\, dx\, dy$$

Using formula (24) , we obtain:

25. $$\ln(\pi) = \sum_{n=0}^{\infty} 3^{-n-1} \sum_{k=0}^{n} (-1)^k 2^{k+1} \binom{n}{k} \cdot \left( \frac{1}{k+1} + \ln\left( \frac{k+2}{k+1} \right) \right)$$

26. $$\pi = \prod_{n=0}^{\infty} \left\{ \prod_{k=0}^{n} \left( \frac{k+2}{k+1} e^{1/(k+1)} \right)^{(-1)^k 2^{k+1} \binom{n}{k}} \right\}^{3^{-n-1}}$$

the next example (Guillera-Sondow,example 3.16,pag.11):

27. $$\frac{\pi}{3\sqrt{3}} = \int_0^1\int_0^1 \frac{y}{1 - x^3 y^3}\, dx\, dy$$

Using formula (27), we obtain:

28. $$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{m=0}^{k} \binom{k}{m} \frac{1}{k+m+1}$$

29. $$\frac{\pi}{\sqrt{3}} - 1 = \int_0^1 \int_0^1 \frac{2y + xy^2}{1 + xy + x^2 y^2}\, dx\, dy$$

30. $$\frac{\pi}{\sqrt{3}} = 1 + \sum_{n=0}^{\infty} \left( \frac{2}{(3n+1)(3n+2)} - \frac{1}{(3n+2)(3n+3)} - \frac{1}{(3n+3)(3n+4)} \right)$$

31. $$\frac{\pi}{\sqrt{3}} = 1 + \sum_{n=0}^{\infty} \frac{c_n}{(n+1)(n+2)}$$

32. $$c_{3n} = 2\,,\; c_{3n+1} = c_{3n+2} = -1\,,\; n = 0, 1, 2, 3, \ldots$$

## DOUBLE INTEGRALS

33. $$\frac{\pi}{4} = \int_0^1 \int_0^1 \frac{1 + (1 + xy)^2}{(4 + x^4 y^4)(-\ln(xy))}\, dx\, dy$$

34. $$\frac{\ln(2)}{2} - \frac{\pi^2}{48} + i\left(\frac{\pi}{4} - G\right) = \int_0^1 \int_0^1 \frac{xy}{(1 + xyi)^2}\, dx\, dy$$

35. $$\frac{\pi}{4} - \frac{\ln(2)}{2} i = \int_0^1 \int_0^1 \frac{1}{(1 + xyi)^2}\, dx\, dy$$

36. $$\frac{\pi}{8} = \int_0^1 \int_0^1 \frac{(1 + x^4)\, y^6}{1 - x^8 y^8}\, dx\, dy$$

37. $$\pi = \int_0^1 \int_0^1 \frac{x^2 + 7y^2}{1 - x^4 y^4}\, dx\, dy$$

38. $$\pi = \int_0^1 \int_0^1 \frac{3x^2 + 5y^2}{1 - x^4 y^4}\, dx\, dy$$

39. $$\pi = \int_0^1 \int_0^1 \frac{9x^2 - y^2}{1 - x^4 y^4}\, dx\, dy$$

40. $$\frac{\pi}{4} = \int_0^1 \int_0^1 \frac{x^2 + y^2}{1 - x^4 y^4}\, dx\, dy$$

41. $$\frac{\pi}{8} = \int_0^1 \int_0^1 \frac{x^2}{1 - x^4 y^4}\, dx\, dy$$

42. $$\frac{\pi - 2}{4} = \int_0^1 \int_0^1 \frac{x^2}{1 + x^2 y^2}\, dx\, dy$$

43. $$\frac{\pi - 2}{2} = \int_0^1 \int_0^1 \frac{x^2 + y^2}{1 + x^2 y^2}\, dx\, dy$$

44. $$\pi - 2 = \int_0^1 \int_0^1 \frac{x^2 + 3y^2}{1 + x^2 y^2}\, dx\, dy$$

45. $$\pi - 2 = \int_0^1 \int_0^1 \frac{5x^2 - y^2}{1 + x^2 y^2}\, dx\, dy$$

46. $$\frac{\pi}{2} - \ln(2) = \int_0^1 \int_0^1 \frac{1}{1 + x^2 y}\, dx\, dy$$

47. $$\pi \left( \frac{1}{\sqrt{3}} - \frac{1}{2} \right) + \ln(2) = \int_0^1 \int_0^1 \frac{1}{1 + x^2 y^3}\, dx\, dy$$

48. $$\frac{\pi}{3\sqrt{3}} - \ln\left( \frac{4}{3} \right) = \int_0^1 \int_0^1 \frac{1}{3 + x^2 y}\, dx\, dy$$

49. $$\frac{\pi}{8} = \int_0^1 \int_0^1 \frac{-\ln(x)}{(1 + x^2 y^2)(\ln(x y))^2}\, dx\, dy$$

50. $$\frac{\pi^2}{24} - \frac{(\ln(2))^2}{4} = \int_0^1\int_0^1 \frac{\ln(x)}{(2 - x y)(\ln(x y))}\, dx\, dy$$

51. $$\frac{\ln(2)}{2} = \int_0^1\int_0^1 \frac{-\ln(x)}{(2 - x y)(\ln(x y))^2}\, dx\, dy$$

52. $$\frac{7\,\zeta(3)}{8} - \frac{\pi^2 \ln(2)}{12} + \frac{(\ln(2))^3}{6} = \int_0^1\int_0^1 \frac{-\ln(x)}{2 - x y}\, dx\, dy$$

53. $$\frac{G}{2} = \int_0^1\int_0^1 \frac{\ln(x)}{\left(1 + x^2 y^2\right)(\ln(x y))}\, dx\, dy$$

54. $$\frac{\pi^3}{32} = \int_0^1\int_0^1 \frac{-\ln(x)}{1 + x^2 y^2}\, dx\, dy$$

55. $$\frac{G}{2} - \frac{\pi^2}{96} i = \int_0^1\int_0^1 \frac{\ln(x)}{(1 + x y i)(\ln(x y))}\, dx\, dy$$

56. $$2\ln(2) - 1 = \int_0^1\int_0^1 \frac{\ln(1 + x y)}{-\ln(x y)}\, dx\, dy$$

57. $$4\,G + \pi - 2\ln(2) = \int_0^1\int_0^1 \frac{-\ln(x y)}{(1 + x y)^2 \sqrt{x}}\, dx\, dy$$

58. $$\frac{7\,\pi^4}{240} = \int_0^1\int_0^1 \frac{\ln(x)\,\ln(x y)}{1 + x y}\, dx\, dy$$

59. $$\frac{\pi^4}{30} = \int_0^1\int_0^1 \frac{\ln(x)\,\ln(x y)}{1 - x y}\, dx\, dy$$

60. $$\frac{i\,\pi}{4\left(1 - e^{i\pi/4}\right)} = \int_0^1\int_0^1 \frac{1}{\left(1 - x y + x y\, e^{i\pi/4}\right)\ln(x y)}\, dx\, dy$$

61. $$\pi = 16 \int_0^1 \int_0^1 \frac{4 - 2\,(x\,y)^3 - (x\,y)^4 - (x\,y)^5}{\left(16 - x^8\,y^8\right)(-\ln(x\,y))}\, dx\, dy$$

62. $$\pi = 16 \int_0^1 \int_0^1 \frac{6\,x^3 + 4\,x^4 + 5\,x^5}{16 - x^8\,y^8}\, dx\, dy$$

63. $$\frac{\pi^2}{6} = \int_0^1 \int_0^1 \frac{(x + y)(-\ln(x\,y))}{1 - x^2\,y^2}\, dx\, dy$$

64. $$\frac{\pi^2}{12} - \frac{(\ln(2))^2}{2} = \int_0^1 \int_0^1 \frac{1}{1 + x + y - x\,y}\, dx\, dy$$

65. $$\ln(2) + \frac{(\ln(2))^2}{2} - \frac{\pi^2}{12} = \int_0^1 \int_0^1 \frac{x\,y}{(2 - x\,y)^2}\, dx\, dy$$

66. $$\frac{\ln(2)}{2} = \int_0^1 \int_0^1 \frac{1}{(2 - x\,y)^2}\, dx\, dy$$

67. $$\frac{G}{2} - \frac{\pi}{8} = \int_0^1 \int_0^1 \frac{x^2\,y^2}{\left(1 + x^2\,y^2\right)^2}\, dx\, dy$$

68. $$2\,G - \frac{\pi^2}{24} = \int_0^1 \int_0^1 \frac{(x + y)(-\ln(x\,y))}{1 + x^2\,y^2}\, dx\, dy$$

69. $$\frac{G}{2} + \frac{\pi^2}{32} = \int_0^1 \int_0^1 \frac{-x\ln(x\,y)}{1 - x^4\,y^4}\, dx\, dy$$

70. $$\frac{\pi^2}{30} = \int_0^1 \int_0^1 \frac{\phi - 3\,x\,y}{\phi^2 - x^2\,y^2}\, dx\, dy$$

71. $$\frac{\pi^2}{30} = \int_0^1 \int_0^1 \frac{1}{(\phi - x\,y)\left(\phi^2 - x\,y\right)}\, dx\, dy$$

72. $$\frac{\pi^2}{12\,\phi} - \frac{3(\ln(\phi))^2}{4\,\phi} = \int_0^1\int_0^1 \frac{1}{\phi^2 - x^2 y^2}\,dx\,dy$$

73. $$\frac{\pi^2\phi^2}{10} - \phi^2(\ln(\phi))^2 = \int_0^1\int_0^1 \frac{\phi^3 + x\,y}{(\phi - x\,y)(1 + \phi\,x\,y)}\,dx\,dy$$

74. $$\frac{\pi^2}{20} - \frac{(\ln(\phi))^2}{2} = \int_0^1\int_0^1 \frac{x}{(\phi - x\,y)(x + y)}\,dx\,dy$$

75. $$\frac{\pi^2}{30} - \frac{(\ln(\phi))^2}{2} = \int_0^1\int_0^1 \frac{x}{(\phi^2 - x\,y)(x + y)}\,dx\,dy$$

76. $$\ln(\phi) = \int_0^1\int_0^1 \frac{1}{(\phi - x\,y)(\phi^2 - x\,y)(-\ln(x\,y))}\,dx\,dy$$

77. $$\frac{3\ln(\phi)}{2\,\phi} = \int_0^1\int_0^1 \frac{1}{(\phi^2 - x^2 y^2)(-\ln(x\,y))}\,dx\,dy$$

78. $$\frac{\pi\sqrt{3}}{9} + \frac{\ln(2)}{3} = \int_0^1\int_0^1 \frac{1}{(1 + x^3 y^3)(-\ln(x\,y))}\,dx\,dy$$

79. $$\frac{\pi\sqrt{3}}{9} - \frac{\ln(2)}{3} = \int_0^1\int_0^1 \frac{x\,y}{(1 + x^3 y^3)(-\ln(x\,y))}\,dx\,dy$$

80. $$\frac{\pi\sqrt{2}}{8} + \frac{\sqrt{2}\,\ln(1 + \sqrt{2})}{4} = \int_0^1\int_0^1 \frac{1}{(1 + x^4 y^4)(-\ln(x\,y))}\,dx\,dy$$

81. $$e - 1 = \int_0^1\int_0^1 \frac{e^{xy}}{-\ln(x\,y)}\,dx\,dy$$

82. $$\frac{\pi}{4} = \int_0^1\int_0^1 \frac{1}{(1 + (1 - x\,y)^2)(-\ln(x\,y))}\,dx\,dy$$

83. $$\frac{\pi^2}{12} = \int_0^1\int_0^1 \frac{1}{1+xy}\, dx\, dy$$

84. $$\frac{\pi^2}{12} = \int_0^1\int_0^1 \frac{1}{(1+x)(2x-xy+y)}\, dx\, dy$$

85. $$\frac{\pi^2}{8} = \frac{3}{4}\zeta(2) = \int_0^1\int_0^1 \frac{1}{1-x^2y^2}\, dx\, dy$$

86. $$\frac{\pi}{12\sqrt{3}} + \frac{\pi^2}{72} = \int_0^1\int_0^1 \frac{xy^2(x+y)}{1-x^6y^6}\, dx\, dy$$

87. $$\frac{\pi}{12\sqrt{3}} - \frac{\pi^2}{72} = \int_0^1\int_0^1 \frac{xy^2(y-x)}{1-x^6y^6}\, dx\, dy$$

88. $$\frac{\pi}{4} = \int_0^1\int_0^1 \frac{1-x^2y^2}{\left(1+x^2y^2\right)^2}\, dx\, dy$$

89. $$\frac{\pi}{4} = \int_0^1\int_0^1 \frac{1}{\left(1+x^2y^2\right)(-\ln(xy))}\, dx\, dy$$

90. $$\frac{\pi}{6\sqrt{3}} = \int_0^1\int_0^1 \frac{1}{\left(3+x^2y^2\right)(-\ln(xy))}\, dx\, dy$$

91. $$\frac{\pi}{8\left(\sqrt{2}+1\right)} = \int_0^1\int_0^1 \frac{1}{\left(3+2\sqrt{2}+x^2y^2\right)(-\ln(xy))}\, dx\, dy$$

92. $$\frac{\pi}{2} = \int_0^1\int_0^1 \frac{1}{\sqrt{xy}\left(1+xy\right)(-\ln(xy))}\, dx\, dy$$

93. $$\frac{\pi}{3\sqrt{3}} = \int_0^1\int_0^1 \frac{1}{\left(1+3x^2y^2\right)(-\ln(xy))}\, dx\, dy$$

94. $$\frac{\pi}{20} = \int_0^1\int_0^1 \frac{6+x^2y^2}{(4+x^2y^2)(9+x^2y^2)(-\ln(xy))}\,dx\,dy$$

95. $$\frac{2\pi}{3\sqrt{3}} = \int_0^1\int_0^1 \frac{x+y}{1-x^3y^3}\,dx\,dy$$

96. $$\frac{\pi^2}{18} = \int_0^1\int_0^1 \frac{\sqrt{xy}}{1-x^3y^3}\,dx\,dy$$

97. $$\frac{10\pi^3}{81\sqrt{3}} = \int_0^1\int_0^1 \frac{-\ln(xy)}{1-xy+x^2y^2}\,dx\,dy$$

98. $$\frac{7\pi^3\sqrt{2}}{128} = \int_0^1\int_0^1 \frac{-\ln(xy)}{1-\sqrt{2}\,xy+x^2y^2}\,dx\,dy$$

99. $$\frac{55\pi^3}{648} = \int_0^1\int_0^1 \frac{-\ln(xy)}{1-\sqrt{3}\,xy+x^2y^2}\,dx\,dy$$

100. $$\frac{\pi^2}{18} = \int_0^1\int_0^1 \frac{1-2xy}{1-xy+x^2y^2}\,dx\,dy$$

101. $$\frac{11\pi^2\sqrt{2}}{192} = \int_0^1\int_0^1 \frac{1-\sqrt{2}\,xy}{1-\sqrt{2}\,xy+x^2y^2}\,dx\,dy$$

102. $$\frac{13\pi^2}{72} = \int_0^1\int_0^1 \frac{\sqrt{3}-2xy}{1-\sqrt{3}\,xy+x^2y^2}\,dx\,dy$$

103. $$(\ln(2))^2 = \int_0^1\int_0^1 \frac{xy}{(1-xy)(2-xy)}\,dx\,dy$$

104. $$\zeta(3) = \int_0^1\int_0^1 \frac{1}{1-xy}\ln\left(\frac{2\sqrt{2}}{2-x}\right)dx\,dy$$

105. $$\zeta(3)=\int_0^1\int_0^1 \frac{1}{1-xy}\ln\left(\frac{2\sqrt{2}}{1+xy}\right)\,dx\,dy$$

106. $$\zeta(3)-\frac{\pi^2\ln(2)}{12}=-\int_0^1\int_0^1 \frac{1}{1-xy}\ln\left(1-\frac{x}{2}\right)\,dx\,dy$$

107. $$\frac{\pi^2\ln(2)}{6}-\frac{5}{8}\zeta(3)=-\int_0^1\int_0^1 \frac{1}{1-xy}\ln\left(1-\frac{xy}{2}\right)\,dx\,dy$$

108. $$\zeta(3)=\int_0^1\int_0^1 \frac{1}{1-xy}\ln\left(\frac{1-xy}{1-x}\right)\,dx\,dy$$

109. $$2\,\zeta(3)=\int_0^1\int_0^1 \frac{-\ln(x)}{1+xy-y}\,dx\,dy$$

110. $$\frac{1}{2}\zeta(3)=\int_0^1\int_0^1 \frac{-x\ln(x)}{1+x^2y-y}\,dx\,dy$$

111. $$\ln\left(\frac{\pi}{2}\right)=\int_0^1\int_0^1 \frac{x}{(1+xy)(-\ln(xy))}\,dx\,dy$$

112. $$\ln(\pi)=\int_0^1\int_0^1 \frac{2x^2y+3x-1}{(1+xy)(-\ln(xy))}\,dx\,dy$$

113. $$\ln(\pi)=\int_0^1\int_0^1 \frac{3x^3+3x^4y+x-1}{(1+xy)(-\ln(xy))}\,dx\,dy$$

114. $$\frac{\pi^2\ln(2)}{4}-\zeta(3)=\int_0^1\int_0^1 \frac{\ln(1+x)}{1+xy-y}\,dx\,dy$$

115. $$\frac{\pi^2\ln(2)}{4}-\zeta(3)=\int_0^1\int_0^1 \frac{\ln(1+x)}{x+y-xy}\,dx\,dy$$

116. $$\ln(2) = \int_0^1\int_0^1 \frac{x}{-\ln(x\,y)}\,dx\,dy$$

117. $$2\ln(3) = \int_0^1\int_0^1 \frac{x^2}{-\ln(x\,y)}\,dx\,dy$$

118. $$\ln(3) = \int_0^1\int_0^1 \frac{2}{(3-2\,x\,y)(-\ln(x\,y))}\,dx\,dy$$

119. $$\ln(2) = \int_0^1\int_0^1 \frac{1}{(1+2\,x\,y)(-\ln(x\,y))}\,dx\,dy$$

120. $$\ln(3) = \int_0^1\int_0^1 \frac{2}{(1+3\,x\,y)(-\ln(x\,y))}\,dx\,dy$$

121. $$\frac{\pi}{4} - \frac{\ln(2)}{2} = \int_0^1\int_0^1 \frac{\tan^{-1}(x\,y)}{-\ln(x\,y)}\,dx\,dy$$

122. $$\frac{\pi}{6} - \frac{\sqrt{3}}{2}\ln\left(\frac{4}{3}\right) = \int_0^1\int_0^1 \frac{1}{-\ln(x\,y)}\tan^{-1}\left(\frac{x\,y}{\sqrt{3}}\right)dx\,dy$$

123. $$\frac{\pi}{8} - \frac{\ln\left(4-2\sqrt{2}\right)}{2\left(\sqrt{2}-1\right)} = \int_0^1\int_0^1 \frac{\tan^{-1}\left(\left(\sqrt{2}-1\right)x\,y\right)}{-\ln(x\,y)}\,dx\,dy$$

124. $$\frac{\pi}{2} + \ln(2) - 2 = \int_0^1\int_0^1 \frac{\ln\left(1+x^2y^2\right)}{-\ln(x\,y)}\,dx\,dy$$

125. $$\frac{\pi}{\sqrt{3}} - 2 + 2\ln(2) = \int_0^1\int_0^1 \frac{\ln\left(3+x^2y^2\right)}{-\ln(x\,y)}\,dx\,dy$$

126. $$\frac{\pi}{4\left(\sqrt{2}-1\right)} - 2 + \ln\left(4-2\sqrt{2}\right) = \int_0^1\int_0^1 \frac{\ln\left(1+\left(3-2\sqrt{2}\right)x^2y^2\right)}{-\ln(x\,y)}\,dx\,dy$$

127. $$\frac{\pi}{3\sqrt{3}} = \sum_{n=0}^{\infty} \frac{a^{3n+1}\left(1+a-a^{3n+2}\right)}{(3n+1)\cdot(3n+2)} + \frac{1}{2}\int_a^1\int_a^1 \frac{x+y}{1-x^3y^3}\,dx\,dy$$
$$0 \le a \le 1$$

128. $$G = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}\left(2-a^{2n+1}\right)}{(2n+1)^2} + \int_a^1\int_a^1 \frac{1}{1+x^2y^2}\,dx\,dy$$
$$0 \le a \le 1$$

129. $$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{a^{2n+1}\left(2-a^{2n+1}\right)}{(2n+1)^2} + \int_a^1\int_a^1 \frac{1}{1-x^2y^2}\,dx\,dy$$
$$0 \le a \le 1$$

130. $$\int_0^1\int_0^1 \frac{1}{9-x^2y^2}\,dx\,dy = \frac{\pi^2}{6} - \frac{(\ln(3))^2}{2} - \frac{1}{36}\sum_{n=0}^{\infty} \frac{3^{-2n}}{(n+1)^2}$$

131. $$\int_0^1\int_0^1 \frac{xy}{9-x^2y^2}\,dx\,dy = \frac{\pi^2}{6} - \frac{(\ln(3))^2}{2} - \sum_{n=0}^{\infty} \frac{3^{-2n}}{(2n+1)^2}$$

132. $$\int_0^1\int_0^1 \frac{1}{\phi^3 - xy}\,dx\,dy = \frac{\pi^2}{12} - \frac{3}{2}(\ln(\phi))^2 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}\phi^{-3n}$$

133. $$\int_0^1\int_0^1 \frac{1}{\phi^3 + xy}\,dx\,dy = \frac{\pi^2}{12} - \frac{3}{2}(\ln(\phi))^2 - \sum_{n=1}^{\infty} \frac{1}{n^2}\phi^{-3n}$$

134. $$\int_0^1\int_0^1 \frac{1}{64-x^6y^6}\,dx\,dy = \frac{\pi^2}{1152} +$$
$$\sum_{n=0}^{\infty} 2^{-6n-10}\left(\frac{24}{(6n+2)^2} + \frac{8}{(6n+3)^2} + \frac{6}{(6n+4)^2} - \frac{1}{(6n+5)^2}\right)$$

135. $$\int_0^1\int_0^1 \frac{1}{1-xy}\tan^{-1}\left(\frac{xy}{\sqrt{2}-\left(\sqrt{2}-1\right)xy}\right)dx\,dy = \frac{\pi^3}{24} +$$
$$\sum_{n=1}^{\infty} \frac{H_{n,2}}{n}\,\mathrm{Im}\left(\left(1-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)^n\right)$$

136. $$\int_0^1\int_0^1 \frac{1}{1-xy}\tan^{-1}\left(\frac{xy}{2-\left(2-\sqrt{3}\right)xy}\right)\,dx\,dy = \frac{\pi^3}{36} + \sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,\mathrm{Im}\left(\left(1-\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^n\right)$$

137. $$\int_0^1\int_0^1 \frac{1}{1-xy}\tan^{-1}\left(\frac{(1-a)xy}{1-axy}\right)\,dx\,dy = \frac{\pi^3}{24} - \sum_{n=1}^{\infty}\frac{H_{n,2}}{n}\,\mathrm{Im}\left((a+(1-a)i)^n\right)$$

$$0 < a < 1$$

138. $$\int_0^1\int_0^1 \frac{1}{1-xy}\tan^{-1}\left(\frac{xy}{2-xy}\right)\,dx\,dy = \frac{\pi^3}{24} - \sum_{n=1}^{\infty}\frac{H_{n,2}}{n\,2^n}\,\mathrm{Im}\left((1+i)^n\right)$$

remark: $H_{n,2} = \sum_{k=1}^{n} k^{-2}$

139. $$G = n^2\int_0^1\int_0^1 \frac{(xy)^{n-1}}{1+(xy)^{2n}}\,dx\,dy \quad , n > 0$$

140. $$G = m\,n\int_0^1\int_0^1 \frac{x^{n-1}\,y^{m-1}}{1+x^{2n}\,y^{2m}}\,dx\,dy \quad , m, n > 0$$

141. $$\zeta(2) = n^2\int_0^1\int_0^1 \frac{(xy)^{n-1}}{1-(xy)^n}\,dx\,dy \quad , n > 0$$

142. $$\frac{\pi}{4} = n\int_0^1\int_0^1 \frac{(xy)^{n-1}}{\left(1+(xy)^{2n}\right)(-\ln(xy))}\,dx\,dy \quad , n > 0$$

143. $$\pi = 4\sum_{k=0}^{n}\frac{(-1)^k}{2k+1} + 4(-1)^{n+1}\int_0^1\int_0^1 \frac{(xy)^{2n+2}}{\left(1+(xy)^2\right)(-\ln(xy))}\,dx\,dy \quad , n \in \mathbb{N}\cup\{0\}$$

144. $$\frac{\pi}{3\sqrt{3}} = \int_1^{\infty}\int_1^{\infty} \frac{x}{x^3y^3-1}\,dx\,dy$$

145. $$\frac{\pi^3}{16} = \int_1^\infty \int_1^\infty \frac{\ln(x\,y)}{x^2 y^2 + 1}\, dx\, dy$$

146. $$\frac{\pi^2}{12} - \frac{(\ln(2))^2}{2} = \int_1^\infty \int_1^\infty \frac{1}{x\,y\,(2\,x\,y - 1)}\, dx\, dy$$

147. $$G - \frac{\pi^2\, i}{48} = \int_1^\infty \int_1^\infty \frac{1}{x\,y\,(\,x\,y + i)}\, dx\, dy$$

148. $$G - \frac{\pi^2}{48} = \int_1^\infty \int_1^\infty \frac{\ln(x\,y)}{x\,(x^2 y^2 + 1)}\, dx\, dy$$

149. $$\frac{\pi^2}{12} = \int_1^\infty \int_1^\infty \frac{\ln(x\,y)}{x\,(x^2 y^2 - 1)}\, dx\, dy$$

150. $$\frac{3}{4}\sqrt{\pi} = \int_0^\infty \int_0^\infty e^{-x-y}\sqrt{x + y}\, dx\, dy$$

151. $$\frac{3}{4}\sqrt{\pi} = \sum_{n=0}^{\infty} (n+1)\,e^{-n} \int_0^1 \int_0^1 e^{-x-y}\sqrt{n + x + y}\, dx\, dy$$

152. $$\frac{1}{2}\sqrt{\pi} = \int_0^\infty \int_0^\infty \frac{e^{-x-y}}{\sqrt{x + y}}\, dx\, dy$$

153. $$\frac{1}{2}\sqrt{\pi} = \sum_{n=0}^{\infty} (n+1)\,e^{-n} \int_0^1 \int_0^1 \frac{e^{-x-y}}{\sqrt{n + x + y}}\, dx\, dy$$

154. $$\frac{\pi^2}{12} - \frac{(\ln(2))^2}{2} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} (n - k + 1)\,2^{-k}\left(\frac{1 - e^{-k}}{k}\right)^2 e^{-k(n-k)}$$

155. $$G - \frac{\pi^2}{48} + \frac{\pi^3}{32} = 2\sum_{n=0}^{\infty} (-1)^n\,(n+1)\,\left((2\,n+1)^{-3} - (2\,n+2)^{-3}\right)$$

156. $$G - \frac{\pi^2}{48} + \frac{\pi^3}{32} = \frac{8}{9}\sum_{n=0}^{\infty} (n+1)\,3^{-n}\sum_{k=0}^{n}\binom{n}{k}(-2)^k\left((2\,k+1)^{-3} - (2\,k+2)^{-3}\right)$$

157. $$\zeta(3) = \frac{8}{5} \sum_{n=1}^{\infty} \frac{3^{-n}}{n} \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1} 2^k H_k}{k}$$

158. $$\frac{\pi}{2} - i \ln(2) = 2 \sum_{n=0}^{\infty} \frac{(-i)^n}{n+1} = \sum_{n=0}^{\infty} 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-i)^k}{k+1}$$

159. $$\frac{\ln(2)}{2} - \frac{\pi^2}{48} + i\left(\frac{\pi}{4} - G\right) = \sum_{n=0}^{\infty} \frac{(-i)^n (n+1)}{(n+2)^2} = \sum_{n=0}^{\infty} 2^{-n-1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-i)^k (k+1)}{(k+2)^2}$$

160. $$\ln(\pi) = \frac{3-i}{5} \sum_{n=0}^{\infty} \left(\frac{2+i}{5}\right)^n \sum_{k=0}^{n} \binom{n}{k} (-1+i)^k \left(\frac{1}{k+1} + \ln\left(\frac{k+2}{k+1}\right)\right)$$

161. $$\pi = \prod_{n=0}^{\infty} \left( \prod_{k=0}^{n} \left( \frac{k+2}{k+1} e^{1/(k+1)} \right)^{(-1+i)^k \binom{n}{k}} \right)^{(3-i)(2+i)^n 5^{-n-1}}$$

162. $$\frac{\pi}{3\sqrt{3}} = \int_0^1 \int_0^1 \frac{e^{-x-2y}}{1 - e^{-3x-3y}} \, dx \, dy + \sum_{m=0}^{\infty} \sum_{n,\, k \geq 0,\ (n,\, k) \neq (0,\, 0)} e^{-n-2k-3m(n+k)} \left( \frac{\left(1 - e^{-3m-1}\right)\left(1 - e^{-3m-2}\right)}{(3m+1)(3m+2)} \right)$$

163. $$\frac{\pi\sqrt{3}}{81} = \int_0^1 \int_0^1 \frac{54x + 27x^3 + 60x^5 + 18x^6 + 14x^7 + 9x^9}{729 - x^{12} y^{12}} \, dx \, dy$$

# SOME TRIPLE INTEGRALS

164. $$\frac{\pi^3}{32} = \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 + x^2 y^2 z^2} \, dx \, dy \, dz$$

165. $$2\,\zeta(3) = \int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - x + xyz} \, dx \, dy \, dz$$

166. $$\frac{5}{8}\,\zeta(3) = \int_0^1\int_0^1\int_0^1 \frac{1}{1+x-xyz}\,dx\,dy\,dz$$

167. $$\frac{5}{8}\,\zeta(3) = \int_0^1\int_0^1\int_0^1 \frac{y}{(1+xy)\cdot(1-yz)}\,dx\,dy\,dz$$

168. $$\zeta(3) = \int_0^1\int_0^1\int_0^1 \frac{yz}{(1-xyz)\cdot(1-yz)}\,dx\,dy\,dz$$

169. $$2\,\zeta(3) = \int_0^1\int_0^1\int_0^1 \frac{y}{(1-xy)\cdot(1-yz)}\,dx\,dy\,dz$$

170. $$\frac{\pi^2}{12} = \int_0^1\int_0^1\int_0^1 \frac{y}{(1-x+xyz)\cdot(1+yz)}\,dx\,dy\,dz$$

171. $$\frac{\pi^3}{16} = \int_0^1\int_0^1\int_0^1 \frac{1-yz}{(1-x+xyz)\cdot\left(1+y^2z^2\right)}\,dx\,dy\,dz$$

172. $$\frac{\pi}{4} - \frac{\ln(2)}{2} = \int_0^1\int_0^1\int_0^1 \frac{yz}{\left(1+x^2y^2z^2\right)(-\ln(yz))}\,dx\,dy\,dz$$

173. $$\frac{\pi}{4} - \ln(2) = \int_0^1\int_0^1\int_0^1 \frac{(xy)^{z^2}-(xy)^z}{-\ln(xy)}\,dx\,dy\,dz$$

174. $$\frac{\pi}{6\sqrt{3}} = \int_0^\infty\int_0^\infty\int_0^\infty \frac{e^{-z-(x+y)\left(3+e^{-2z}\right)}}{x+y}\,dx\,dy\,dz$$

175. $$\frac{\pi}{4} = \int_0^\infty\int_0^\infty\int_0^\infty \frac{\left(2+3e^{-5x-5y}\right)e^{-z-(x+y)\left(4+e^{-2z}\right)}}{x+y}\,dx\,dy\,dz$$

176. $$\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + ... + \ln(n) \right) = \lim_{n \to \infty} \left( H_n - \ln(n) \right)$$

177. $$H_n = \sum_{k=1}^{n} k^{-1} \ , H_0 = 0$$

178. $$\Gamma(x) = \int_0^{\infty} \mathrm{e}^{-t} t^{x-1} \, \mathrm{d}t \ , x > 0$$

179. $$\Gamma(x+1) = x \, \Gamma(x)$$

180. $$\Gamma(x+1) = x!$$

181. $$\gamma = - \frac{\ln(\Gamma(1+x))}{x} + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{x} \ln\left(1 + \frac{x}{n}\right) \right)$$

182. $$\gamma = - \lim_{x \to 0} \frac{\ln(\Gamma(1+x))}{x}$$

183. $$\gamma = - \lim_{x \to 0} \frac{\ln(x!)}{x}$$

184. $$\gamma = - \lim_{x \to 0} \ln\left( \sqrt[x]{x!} \right)$$

185. $$\gamma = - \lim_{n \to \infty} \ln\left( \left( \left( \frac{1}{n} \right)! \right)^n \right)$$

186. $$\gamma = - \lim_{n \to \infty} \left( n \ \ln\left( \Gamma\left( 1 + \frac{1}{n} \right) \right) \right)$$

187. $$\gamma = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \ln\left( 1 + \frac{1}{n} \right) \right)$$

188. $$\gamma = \ln\left( \frac{4}{\pi} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{n} - 2 \ln\left( 1 + \frac{1}{2n} \right) \right)$$

189. $$\gamma = \ln(\pi) + \sum_{n=1}^{\infty} \left( \frac{1}{n} + 2 \ln\left( 1 - \frac{1}{2n} \right) \right)$$

190. $$\gamma = \ln(2) + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \ln\left( \frac{2n+1}{2n-1} \right) \right)$$

191. $$\gamma = \ln\left( \frac{2\pi}{\sqrt{3}} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{n} + \ln\left( 1 - \frac{1}{n} + \frac{2}{9n^2} \right) \right)$$

192. $$\gamma = \ln\left( \pi\sqrt{2} \right) + \sum_{n=1}^{\infty} \left( \frac{1}{n} + \ln\left( 1 - \frac{1}{n} + \frac{3}{16n^2} \right) \right)$$

193. $$\gamma = \left( \ln\left( \frac{4}{\pi} \right) \right) \prod_{n=2}^{\infty} \left( \frac{(n+1)\ln\left( \Gamma\left( 1 + \frac{1}{n+1} \right) \right)}{n\ln\left( \Gamma\left( 1 + \frac{1}{n} \right) \right)} \right)$$

194. $$\gamma = \left( \ln\left( \frac{4}{\pi} \right) \right) \prod_{n=1}^{\infty} \left( \frac{2\ln(\Gamma(1 + 2^{-n-1}))}{\ln(\Gamma(1 + 2^{-n}))} \right)$$

195. $$\gamma = (\ln(\pi)) \prod_{n=1}^{\infty} \left( \frac{2\ln(\Gamma(1 - 2^{-n-1}))}{\ln(\Gamma(1 - 2^{-n}))} \right)$$

196. $$\gamma = \ln\left( \frac{4}{\pi} \right) + \sum_{n=1}^{\infty} 2^n \ln\left( \frac{\Gamma(1 + 2^{-n})}{(\Gamma(1 + 2^{-n-1}))^2} \right)$$

197. $$\gamma = \ln(\pi) - \sum_{n=1}^{\infty} 2^n \ln\left( \frac{\Gamma(1 - 2^{-n})}{(\Gamma(1 - 2^{-n-1}))^2} \right)$$

198. $$\gamma = 1 - \lim_{x \to 1} \frac{\ln(\Gamma(1 + x))}{x\ln(x)}$$

199. $$\gamma = 1 - \lim_{x \to 1} \frac{\sqrt[x]{x!}}{\ln(x)}$$

200. $$\gamma = 1 - \lim_{x \to 1} \log_x\left( \sqrt[x]{x!} \right)$$

Continued fraction:

201. $$\gamma = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}} = [0; 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1, 1, 8, 2, 2, 4, 1, 1, 40, 1, 11, \dots]$$

Fractions (convergents):

202. $\{c_n : n \in \mathbb{N} \cup \{0\}\} = \left\{0, 1, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{11}{19}, \frac{15}{26}, \frac{71}{123}, \frac{228}{395}, \frac{3035}{5258}, \frac{15403}{26685}, \frac{18438}{31943}, \ldots\right\}$

203. $$\gamma = c_n + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{v_n}\ln\left(1 + \frac{v_n}{k}\right)\right) \quad , \; n \in \mathbb{N}$$

204. $$c_n v_n + \ln\left(\Gamma(1 + v_n)\right) = 0 \quad , n \in \mathbb{N}$$

| $n$ | $v_n$ |
|---|---|
| 1 | −0.40323166 ... |
| 2 | 0.09829979 ... |
| 3 | −0.02733163 ... |
| 4 | 0.00706043 ... |
| 5 | −0.00210334 ... |
| 6 | 0.00035580 ... |
| 7 | −0.00002444 ... |

# J. SONDOW INTEGRAL AND RELATED FORMULAS

205. $$\gamma = \int_0^1\int_0^1 \frac{1 - x}{(1 - x y)\,(-\ln(x y))}\, \mathrm{d}x\, \mathrm{d}y$$

## Related formulas

206. $$\gamma = \int_1^\infty \int_1^\infty \frac{x-1}{x^2 y\,(x y - 1)\ln(x y)}\,\mathrm{d}x\,\mathrm{d}y$$

207. $$\gamma = \int_0^\infty \int_0^\infty \frac{1-\mathrm{e}^{-x}}{\left(1-\mathrm{e}^{-x-y}\right)(x+y)}\,\mathrm{d}x\,\mathrm{d}y$$

208. $$\gamma = \frac{1}{2}\int_0^1\int_0^1 \frac{2-x-y}{(1-x y)(-\ln(x y))}\,\mathrm{d}x\,\mathrm{d}y$$

209. $$\gamma = \frac{1}{2}\int_0^1\int_0^1 \frac{x+y}{(x+y-x y)(-\ln((1-x)(1-y)))}\,\mathrm{d}x\,\mathrm{d}y$$

210. $$\gamma = \sum_{k=1}^{n}\left(\frac{1}{k} - \ln\left(\frac{k+1}{k}\right)\right) + \int_0^1\int_0^1 \frac{(1-x)(x y)^n}{(1-x y)(-\ln(x y))}\,\mathrm{d}x\,\mathrm{d}y \quad , n \in \mathbb{N} \cup \{0\}$$

211. $$\gamma = H_n - \ln(n+1) + \int_0^1\int_0^1 \frac{(1-x)(x y)^n}{(1-x y)(-\ln(x y))}\,\mathrm{d}x\,\mathrm{d}y \quad , n \in \mathbb{N} \cup \{0\}$$

212. $$\gamma = n\int_0^1\int_0^1 \frac{\left(1-x^n\right)(x y)^{n-1}}{\left(1-(x y)^n\right)(-\ln(x y))}\,\mathrm{d}x\,\mathrm{d}y \quad , n > 0$$

213. $$\gamma = \ln(2) - \int_0^1\int_0^1 \frac{x\,(1+x y-2 y)}{\left(1-x^2 y^2\right)(-\ln(x y))}\,\mathrm{d}x\,\mathrm{d}y$$

Reference.

[1] Guillera J. and Sondow J.: Double Integrals and Infinite Products for Some Classical Constants via Analytic Continuations of Lerch's Transcendent.
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