Double integrals involving π^2 and π^3

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abstract

In this note we give some double integrals involving π^2 and π^3 .

1. Introduction

The numbers $s(n), c(n), n \in \mathbb{N} = \{1,2,3,4,...\}$:

$$s(n) = 2\sin\left(\frac{\pi}{2^{n+1}}\right) = \underbrace{\sqrt{2-\sqrt{2+...+\sqrt{2}}}}_{n-radicals} \tag{1}$$

$$c(n) = 2\cos\left(\frac{\pi}{2^{n+1}}\right) = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n-radicals} \tag{2}$$

Examples:

$$s(1) = \sqrt{2}\,,\, s(2) = \sqrt{2-\sqrt{2}}\,,\, s(3) = \sqrt{2-\sqrt{2+\sqrt{2}}}\,,\, s(4) = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}} \tag{3}$$

$$c(1) = \sqrt{2}\,,\, c(2) = \sqrt{2+\sqrt{2}}\,,\, c(3) = \sqrt{2+\sqrt{2+\sqrt{2}}}\,,\, c(4) = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} \tag{4}$$

Remark: $\lim_{n\to\infty} s(n) = 0 \quad , \quad \lim_{n\to\infty} c(n) = 2$.

We give double integrals involving $\pi^2, \pi^3, s(n), c(n)$.

2. Double Integrals

For $n \in \mathbb{N} = \{1,2,3,4,...\}$:

$$\int_0^1\int_0^1 \frac{\overbrace{\sqrt{2-\sqrt{2+...+\sqrt{2}}}}^{n-rad.} + 2xy}{1 + xy\underbrace{\sqrt{2-\sqrt{2+...+\sqrt{2}}}}_{n-rad.} + x^2y^2}\, dxdy = \pi^2\left(\frac{1}{24} + 2^{-n-2} - 2^{-2n-3}\right) \tag{5}$$

$$\int_0^1\int_0^1 \frac{\overbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}^{n-rad.}+2xy}{1+xy\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\pi^2\left(\frac{1}{6}-2^{-2n-3}\right) \tag{6}$$

$$\int_0^1\int_0^1 \frac{\overbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}^{n-rad.}-2xy}{1-xy\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\pi^2\left(\frac{1}{3}-2^{-n-1}+2^{-2n-3}\right) \tag{7}$$

$$\int_0^1\int_0^1 \frac{\overbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}^{n-rad.}-2xy}{1-xy\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\pi^2\left(-\frac{1}{24}+2^{-n-2}+2^{-2n-3}\right) \tag{8}$$

$$\int_0^1\int_0^1 \frac{-\ln(xy)}{1+xy\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\frac{\pi^3 2^{-n-1}}{3\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}}\left(1-2^{-2n-2}\right) \tag{9}$$

$$\begin{aligned}\int_0^1\int_0^1 \frac{-\ln(xy)}{1+xy\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\\ =\frac{\pi^3}{3\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}}\left(\frac{1}{4}-2^{-2n-2}\right)\left(\frac{3}{2}-2^{-n-1}\right)\end{aligned} \tag{10}$$

$$\int_0^1\int_0^1 \frac{-\ln(xy)}{1-xy\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\frac{\pi^3\left(2^{n+1}-1\right)\left(2^{n+2}-1\right)2^{-3n-3}}{3\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}} \tag{11}$$

$$\int_0^1\int_0^1 \frac{-\ln(xy)}{1-xy\underbrace{\sqrt{2-\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}+x^2y^2}\,dxdy=\frac{\pi^3\left(2^{2n}-1\right)\left(3\cdot 2^{n}+1\right)2^{-3n-3}}{3\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n-rad.}} \tag{12}$$

Examples , $n=4$ in (6) and (9):

$$\int_0^1\int_0^1 \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}+2xy}{1+xy\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}+x^2y^2}\,dxdy=\frac{1021\pi^2}{6144} \tag{13}$$

$$\int_0^1\int_0^1 \frac{-\ln(xy)}{1+xy\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}+x^2y^2}\,dxdy=\frac{341\pi^3}{32768\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}} \tag{14}$$

Examples , $n\to\infty$:

$$\frac{\pi^2}{48}=\int_0^1\int_0^1 \frac{xy}{1+x^2y^2}\,dxdy \tag{15}$$

$$\frac{\pi^2}{12}=\int_0^1\int_0^1 \frac{1}{1+xy}\,dxdy \tag{16}$$

$$\frac{\pi^2}{6}=\int_0^1\int_0^1 \frac{1}{1-xy}\,dxdy \tag{17}$$

$$\frac{\pi^2}{3}=\int_0^1\int_0^1 \frac{-\ln(xy)}{(1-xy)^2}\,dxdy \tag{18}$$

$$\frac{\pi^2}{6}=\int_0^1\int_0^1 \frac{-\ln(xy)}{(1+xy)^2}\,dxdy \tag{19}$$

$$\frac{\pi^3}{16}=\int_0^1\int_0^1 \frac{-\ln(xy)}{1+x^2y^2}\,dxdy \tag{20}$$

References

1. G. Boros and V. Moll, Irresistible Integrals: Symbolics, Analysis, and Experiments in the Evaluation of Integrals, Cambridge University Press, Cambridge, 2004.
2. J. Guillera and J. Sondow, Double Integrals and Infinite Products for some Classical Constants via Analytic Continuations of Lerch's Transcendent. arXiv:math/0506319v3[math:NT] 5 Aug 2006.