Question 357: Some Infinite Products

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ABSTRACT

This note presents some infinite products involving pi

1. Introduction

The number pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265... \tag{1}$$

2. Infinite Products

$$\pi\sqrt[4]{8}\sqrt{\sqrt{2}+1} = \frac{64}{7}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{8n}\right)^{-1}\left(1+\frac{7}{8n}\right)^{-1} \tag{2}$$

$$\pi\left(\sqrt{2}-1\right) = \frac{16}{9}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{8n}\right)^{2}\left(1+\frac{1}{4n}\right)^{-2}\left(1+\frac{3}{8n}\right)^{-2} \tag{3}$$

$$\pi\sqrt{2} = \frac{128}{25}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{8n}\right)^{-2}\left(1+\frac{5}{8n}\right)^{-2}\left(1+\frac{1}{4n}\right)^{2} \tag{4}$$

$$\pi\sqrt[3]{4} = \frac{27}{4}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{3n}\right)^{-4}\left(1+\frac{1}{6n}\right)^{2} \tag{5}$$

$$\pi\sqrt{2}\sqrt{1-\frac{1}{\sqrt{5}}} = \frac{25}{6}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{5n}\right)^{-1}\left(1+\frac{3}{5n}\right)^{-1} \tag{6}$$

$$\pi\sqrt{2}\sqrt{1+\frac{1}{\sqrt{5}}} = \frac{25}{4}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{5n}\right)^{-1}\left(1+\frac{4}{5n}\right)^{-1} \tag{7}$$

$$\pi\sqrt[5]{2}=\frac{200}{49}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{5n}\right)^{-2}\left(1+\frac{7}{10n}\right)^{-2}\left(1+\frac{2}{5n}\right)^{2} \tag{8}$$

$$\frac{\pi\sqrt[5]{4}}{5+\sqrt{5}}=\frac{25}{32}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{5n}\right)^{-2}\left(1+\frac{2}{5n}\right)^{-2}\left(1+\frac{1}{10n}\right)^{2} \tag{9}$$

$$\frac{\pi\sqrt{2}\left(\sqrt{3}-1\right)}{\sqrt[4]{3}}=\frac{81}{25}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{5}{12n}\right)^{-2}\left(1+\frac{1}{3n}\right)^{-2}\left(1+\frac{1}{4n}\right)^{2} \tag{10}$$

$$\frac{\pi\left(3-\sqrt{5}\right)}{\sqrt[5]{2}}=\frac{25}{9}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{5n}\right)^{-2}\left(1+\frac{3}{10n}\right)^{-2}\left(1+\frac{1}{5n}\right)^{2} \tag{11}$$

$$\frac{\pi\sqrt{2}}{\sqrt[4]{27}\left(\sqrt{3}+1\right)}=\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{3n}\right)^{-2}\left(1+\frac{1}{4n}\right)^{-2}\left(1+\frac{1}{12n}\right)^{2} \tag{12}$$

References

1. Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tabes. Applied Mathematical Series 55, National Bureau of Standards, Washington, DC; Repr. Dover, New York, 1965.
2. Gradshteyn , I. S., and Ryzhik, I. M.: Table of Integrals, Series and Products. 5th ed., ed. Alan Jeffrey. Academic Press,1994.