Question 273: Nonlinear Equation , Bernoulli Numbers , Number Pi

Edgar Valdebenito

abstract

This note presents some formulas for π .

1. Introduction

- The nonlinear equation:

$$z(\alpha)-\alpha\left(1-e^{-z(\alpha)}\right)=0 \tag{1}$$

- Unknown: $z(\alpha)$
- Assumptions:

$$\alpha\in\mathbb{C}, z(\alpha)\neq 0, |z(\alpha)|<2\pi \tag{2}$$

- Particular cases:

$$\alpha=\left\{i,\frac{1+i}{\sqrt{2}},\frac{\sqrt{3}+i}{2},\frac{1+i\sqrt{3}}{2}\right\} \quad ,i=\sqrt{-1} \tag{3}$$

- Iterative method:

$$z_{n+1}=\frac{\alpha\left(1-(1+z_n)e^{-z_n}\right)}{1-\alpha e^{-z_n}} \quad ,n\in\mathbb{N} \tag{4}$$

$$z_1=z_1(\alpha) \text{ , initial point.} \tag{5}$$

$$z_n\rightarrow z(\alpha) \tag{6}$$

- Bernoulli numbers:

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \quad ,|x|<2\pi \tag{7}$$

$$B_0=1, B_1=-\frac{1}{2}, B_{2n+1}=0 \, , n\geq 0 \tag{8}$$

$$\{B_{2n}:n\in\mathbb{N}\}=\left\{\frac{1}{6},-\frac{1}{30},\frac{1}{42},-\frac{1}{30},\frac{5}{66},-\frac{691}{2730},...\right\} \tag{9}$$

2. Graphics and Formulas

❖ The function $f(x,y)$:

$$f(x,y) = x + y\,i - \alpha\left(1 - e^{-x-yi}\right) \tag{10}$$

❖ $\alpha = i$:

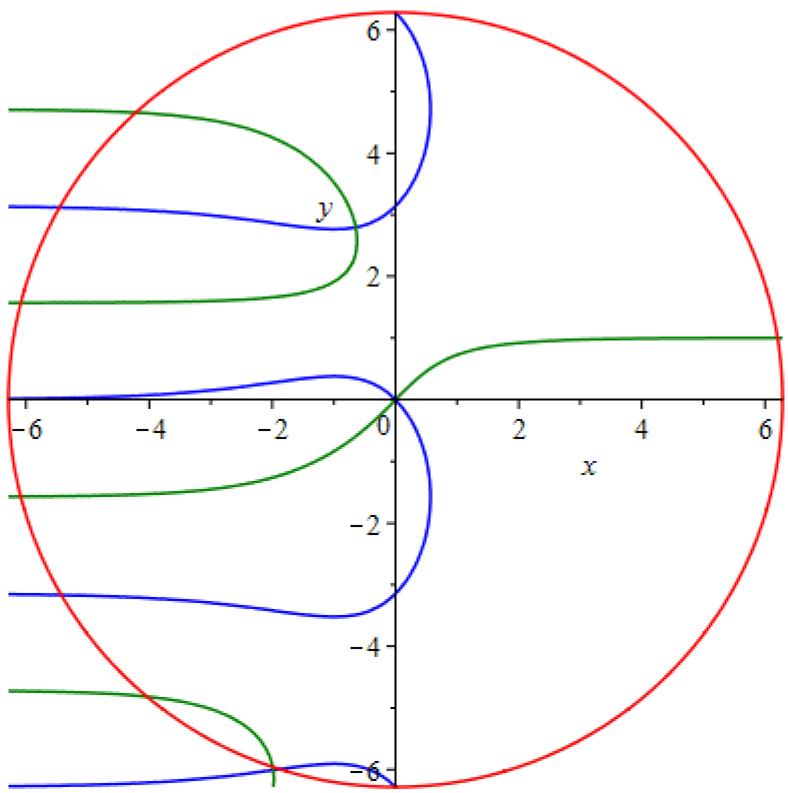

Fig. 1. $\alpha = i$, • $\mathrm{Re}(f) = 0$, • $\mathrm{Im}(f) = 0$, • $|x + yi| = 2\pi$.

$$z_1 = z_1(i) = -0.6 + 2.5i \tag{11}$$

$$z(i) = -0.6465206229101800... + i \times 2.7960686662872967... \tag{12}$$

$$\pi = 2i\sum_{n=1}^{\infty} \frac{B_n}{n \cdot n!}\left(z(i)\right)^n \tag{13}$$

❖ $\alpha = \frac{1+i}{\sqrt{2}}$:

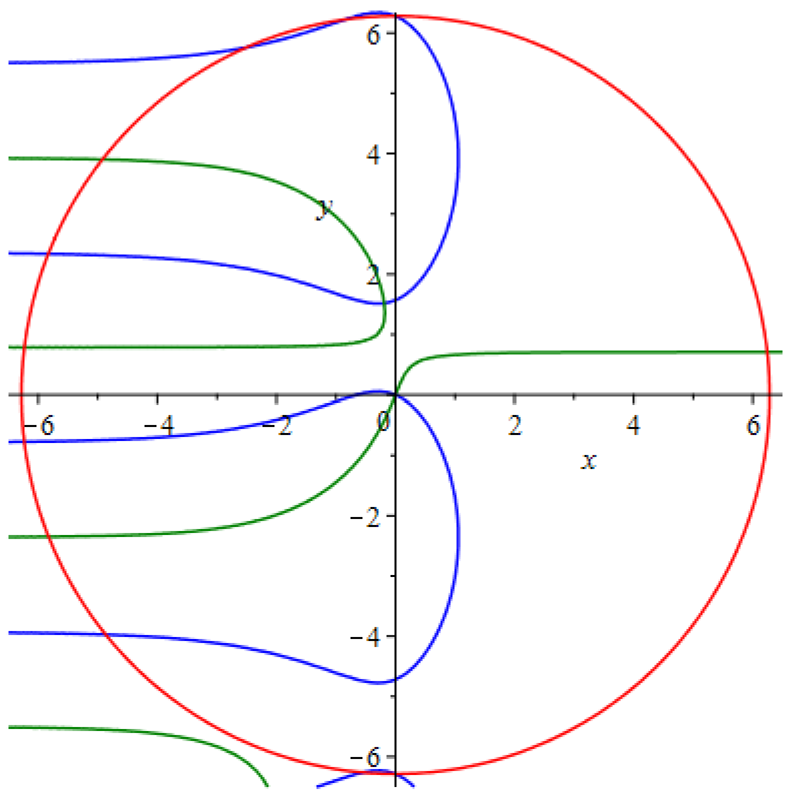

Fig. 2. $\alpha = \frac{1+i}{\sqrt{2}}$, • $\mathrm{Re}(f) = 0$, • $\mathrm{Im}(f) = 0$, • $|x + yi| = 2\pi$

$$z_1 = z_1\left(\frac{1+i}{\sqrt{2}}\right) = -0.2 + 1.5i \tag{14}$$

$$z\left(\frac{1+i}{\sqrt{2}}\right) = -0.1928840462381343... + i \times 1.5199782749161466... \tag{15}$$

$$\pi = 4i \sum_{n=1}^{\infty} \frac{B_n}{n \cdot n!} \left(z\left(\frac{1+i}{\sqrt{2}}\right) \right)^n \tag{16}$$

❖ $\alpha = \frac{\sqrt{3}+i}{2}$:

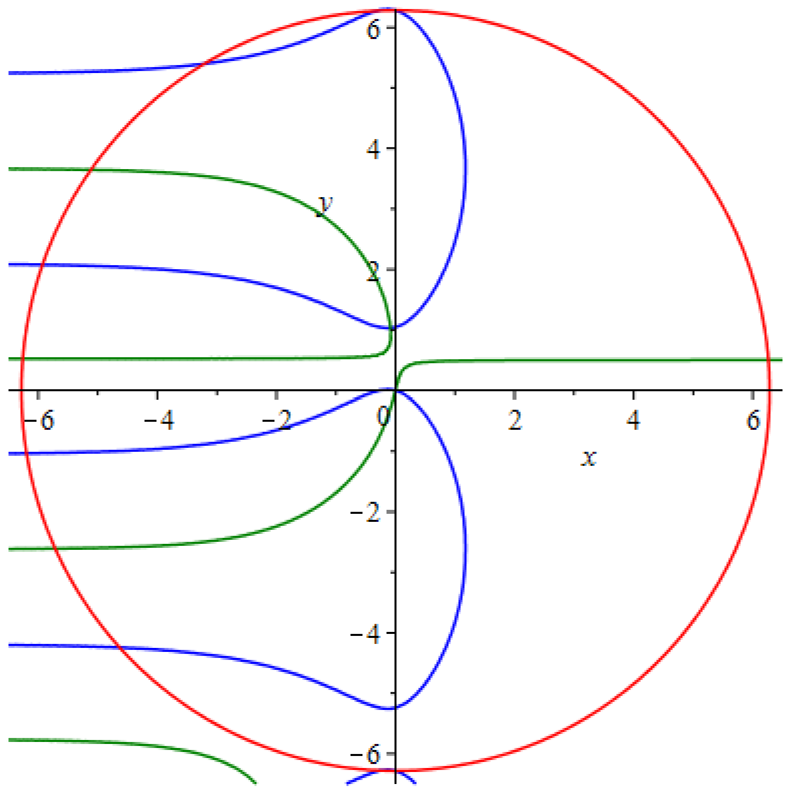

Fig. 3. $\alpha = \frac{\sqrt{3}+i}{2}$, $\bullet\,\mathrm{Re}(f)=0$, $\bullet\,\mathrm{Im}(f)=0$, $\bullet\,|x+yi|=2\pi$

$$z_1 = z_1\left(\frac{\sqrt{3}+i}{2}\right) = -0.1+i \tag{17}$$

$$z\left(\frac{\sqrt{3}+i}{2}\right) = -0.0887990604541688\ldots + i\times 1.0316535383920149\ldots \tag{18}$$

$$\pi = 6i\sum_{n=1}^{\infty}\frac{B_n}{n\cdot n!}\left(z\left(\frac{\sqrt{3}+i}{2}\right)\right)^n \tag{19}$$

❖ $\alpha = \frac{1+i\sqrt{3}}{2}$:

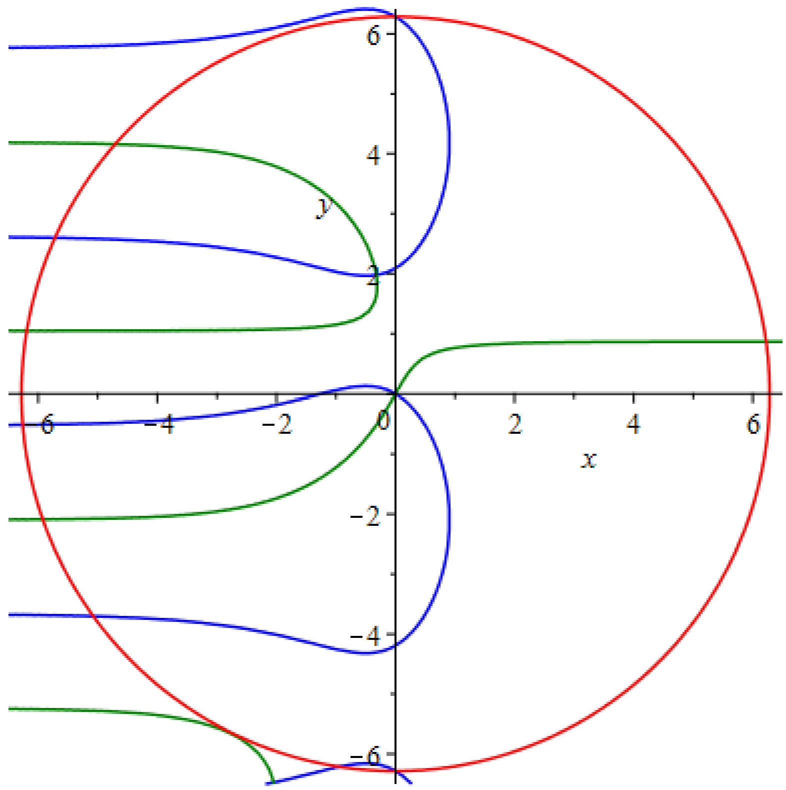

Fig. 4. $\alpha = \frac{1+i\sqrt{3}}{2}, \bullet \mathrm{Re}(f)=0, \bullet \mathrm{Im}(f)=0, \bullet |x+yi| = 2\pi$

$$z_1 = z_1\left(\frac{1+i\sqrt{3}}{2}\right) = -0.5+2i \tag{20}$$

$$z\left(\frac{1+i\sqrt{3}}{2}\right) = -0.3268293901494412\ldots + i\times 1.9790911559557430\ldots \tag{21}$$

$$\pi = 3i\sum_{n=1}^{\infty}\frac{B_n}{n\cdot n!}\left(z\left(\frac{1+i\sqrt{3}}{2}\right)\right)^n \tag{22}$$

3. Relations with Lambert W Function.

- Lambert $W(x)$ function:

$$x = W(x)e^{W(x)} \tag{23}$$

- The Lambert W function $W(x)$ is a set of solutions of the equation (23).
- $W(x)$ returns the principal branch of the Lambert W function.
- $W(k,x), k \in \mathbb{Z}$ is the kth branch of the Lambert W function.
- $W(x) = W(0,x)$.

- Formulas:

$$z(i) = i + W\left(1, -i\,e^{-i}\right) \tag{24}$$

$$z\left(\frac{1+i}{\sqrt{2}}\right) = \frac{1+i}{\sqrt{2}} + W\left(1, -\left(\frac{1+i}{\sqrt{2}}\right)e^{-(1+i)/\sqrt{2}}\right) \tag{25}$$

$$z\left(\frac{\sqrt{3}+i}{2}\right) = \frac{\sqrt{3}+i}{2} + W\left(1, -\left(\frac{\sqrt{3}+i}{2}\right)e^{-\left(\sqrt{3}+i\right)/2}\right) \tag{26}$$

$$z\left(\frac{1+i\sqrt{3}}{2}\right) = \frac{1+i\sqrt{3}}{2} + W\left(1, -\left(\frac{1+i\sqrt{3}}{2}\right)e^{-\left(1+i\sqrt{3}\right)/2}\right) \tag{27}$$

References

1. Abramowitz, M., and Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Applied Mathematical Series 55,National Bureau of Standards, Washington, DC; Repr. Dover, New York , 1965.
2. Apostol, T.: Mathematical Analysis. Addison-Wesley, Reading,Mass., 1957.
3. Dilcher, K.: A Bibliography of Bernoulli numbers. August 11,2003. Available at http//www.mscs.dal.ca/~ dilcher/Bernoulli.html