Hilbert's forgotten equation of velocity dependent acceleration in a gravitational field

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The principle of equivalence is used to argue that the known law of decreasing acceleration for high speed motion, in a low acceleration regime, produces the same result as found for a weak gravitational field, with subsequent implications for stronger fields. This result coincides with Hilbert's little explored equation of 1917, regarding the velocity dependence of acceleration under gravity. We derive this result, from first principles exploiting the principle of equivalence, without need for the full general theory of relativity.

I. INTRODUCTION

While it is accepted that general relativity provides the correct theory for macroscopic relativistic motion including gravity, nevertheless the special theory remains viable for describing accelerated motion as well as accelerated frames in flat space.

For example, Einstein reasoned that light would bend under gravity based on the principle of equivalence and acceleration arguments alone. He then calculated the more precise result using general relativity, taking into account the effect of curved spacetime. Using a similar approach we deduce that acceleration under gravity is velocity dependent using accelerating frames under special relativity, which we then confirm with the general theory, producing a result that coincides with one by Hilbert. We also find a similar discrepancy of a factor of two between the result using flat space accelerations and the full general relativistic analysis.

We can begin by defining a spacetime coordinate differential with a four-vector

$$dx^\mu = (cdt, dx, dy, dz), \tag{1}$$

with contribution from three spatial dimensions and t is the time in a particular reference frame and c is the invariant speed of light[1]. In this paper we are able to focus exclusively on one-dimensional motion and so we can suppress two of the space dimensions writing a spacetime vector $dx^\mu = (cdt, dx)$. We have the metric tensor $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ that defines the covariant vector $dx_\mu = g_{\mu\nu} dx^\nu = (cdt, -dx)$. In the co-moving frame we have $dx = 0$ and so $dx^\mu = (cd\tau, 0)$, which defines τ the local proper time. We define the proper velocity

$$v^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = (\gamma c, \gamma v), \tag{2}$$

where $v = dx/dt$ and

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{3}$$

We then have the magnitude of the spacetime velocity

$$\sqrt{v^\mu v_\mu} = \sqrt{\gamma^2 c^2 - \gamma^2 v^2} = c \tag{4}$$

that is a Lorentz invariant, where we have used the Einstein summation convention. We also have the proper acceleration

$$a^\mu = \frac{dv^\mu}{d\tau} = (\gamma^4 va/c, \gamma^4 a), \tag{5}$$

where we have produced the special case of one-dimensional motion in which v is parallel to a. We then find the magnitude of the spacetime acceleration

$$\sqrt{a^\mu a_\mu} = \sqrt{\gamma^8 v^2 a^2/c^2 - \gamma^8 a^2} = \gamma^3 a. \tag{6}$$

Now, in the momentarily co-moving frame (MCF) we have $v = 0$ giving the acceleration vector $a^\mu = (0, \alpha)$ and the velocity $v^\mu = (c, 0)$, which gives $\sqrt{a^\mu a_\mu} = \alpha$ and the expected orthogonality $v^\mu a_\mu = 0$. Hence, comparing the magnitudes of the proper acceleration in Eq. (6) with the magnitude in the MCF we find $\alpha = \gamma^3 a$ so that in an alternate non-comoving frame we observe an acceleration

$$a = \alpha/\gamma^3. \tag{7}$$

An alternative path to this result is to apply a Lorentz boost to the MCF proper acceleration $a^\mu = (0, \alpha)$, with the transformation $t' = \gamma(t + vx/c^2)$ and $x' = \gamma(x + vt)$. This produces $a^\mu = (\gamma v\alpha/c^2, \gamma\alpha)$ and so comparing this with Eq. (5) we have $\gamma\alpha = \gamma^4 a$ or $\alpha = \gamma^3 a$, confirming Eq. (7).

We now consider how a rocket's acceleration appears when viewed from different inertial reference frames each with different initial velocities. Then, using the principle of equivalence, we transfer our results to a gravitational setting.

A. Thought experiment

Consider a rocket out in space far from the effects of any gravitational influences. Within this, effectively flat region of space, we place small frames of reference that individually can measure the acceleration of passing objects. We will call these types of frames PG1 for particle group 1. The PG1 frames are currently at rest relative to the rocket and also with respect to each other and they are spread throughout the space surrounding the rocket. The rocket also has a hole at the top and bottom so that the PG1 can pass straight through allowing them to measure the acceleration of the rocket. The rocket also has an inbuilt mechanism so that, when the rocket is accelerating, it will drop a second group of particles, labeled PG2, from the top of the rocket, at predetermined fixed time intervals as measured by the rocket. Thus, PG2 can also measure the rocket's acceleration.

Now, for the sake of argument, let the rocket be accelerated at 9.8 ms^{-2} and as specified, PG2 will be dropping from the top of the rocket. The rocket now accelerates away from the PG2 frames with acceleration $\alpha = T/m = 9.8$ ms^{-2}, where m is the mass of the rocket and assuming T is an applied thrust in order to maintain a constant proper acceleration. The PG2, once released, comprise inertial objects not partaking in the rocket's acceleration. Additionally, as the rocket continues its acceleration it will encounter PG1 lying in its path that will enter the hole at the top of the rocket and while passing through measure the acceleration of the rocket. Now, as the rocket is maintaining a steady acceleration, clearly the velocity of the rocket will be steadily increasing. Hence the rocket will be encountering the PG1 at higher and higher relative velocities.

The question we now wish to consider is: *Will PG1 and PG2 measure the same acceleration for the rocket?*

Based on standard theory, we expect the answer to be in the negative. This is because special relativity asserts that, as viewed by PG1, the rocket's velocity will converge to the light speed upper bound, and so the acceleration will appear to decrease. Since, this physical setting is described by Eq. (7), the one-dimensional relativistic equation for acceleration a, as measured in the PG1 frames, can be written as

$$a = \frac{\alpha}{\gamma^3} = \frac{T}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}, \tag{8}$$

where α is the acceleration measured in the co-moving frames PG2, v is the velocity of the rocket relative to PG1.

Now, given this result, we can ask a pivotal question with respect to the physics of the situation: *Given the principle of equivalence will this result for accelerating observers be replicated under gravity?*

We presume for appropriately small regions of the field, based on the principle of equivalence, the answer must be in the affirmative.

B. Gravitational fields

The central role played by the equivalence principle in the general theory was stated by Einstein in 1907:

we [...] assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system.

Einstein's equivalence principle is based primarily on the well established equivalence of gravitational and inertial mass, also called the weak equivalence principle, which has been confirmed by experiment[2] to an accuracy better than 1×10^{-15}. It is now generally accepted that the full Einstein equivalence principle requires a curved spacetime metric theory of gravity in which particles follow geodesics within this space as described by Einstein in his general theory[3].

Hence, incorporating the equivalence principle, our current proposition is that since Eq. (8) pertains to a reference frame described above with an accelerating rocket then we also must have in a gravitational field

$$a = g\left(1 - \frac{v^2}{c^2}\right)^{3/2}, \tag{9}$$

where g is the acceleration due to gravity, which when stationary in gravity is a proper acceleration analogous to α. This shows that for gravity the rate of acceleration for free falling observers (equivalent to PG1) is velocity dependent.

We now confirm this conclusion by deriving a comparable result using the Schwarzschild solution of general relativity.

II. SCHWARZSCHILD SOLUTION

For a static, non-rotating, spherical mass the field equations of general relativity give the Schwarzschild solution[3] with the metric

$$c^2 d\tau^2 = \left(1 - \frac{2\mu}{r}\right) c^2 dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \cos^2\theta d\phi^2, \tag{10}$$

where $\mu = GM/c^2$ and r is measured from the center and outside the mass[3]. We therefore have $g_{rr} = -\left(1 - \frac{2\mu}{r}\right)^{-1}$ and $g_{tt} = \left(1 - \frac{2\mu}{r}\right)$. Now, we have the geodesic equation $a^\alpha = \frac{dv^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu} v^\mu v^\nu$ that can also be written in an equivalent form

$$\frac{d}{d\tau}\left(g_{\alpha\nu}\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{11}$$

This less common form of the geodesic equation can be convenient as the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$ do not need to be explicitly computed. So, setting the index α to the r coordinate, we produce

$$\frac{d}{d\tau}\left(g_{rr}\frac{dr}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{rr}}{\partial r}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{\partial g_{tt}}{\partial r}\left(\frac{dt}{d\tau}\right)^2, \tag{12}$$

utilizing the fact that we have a diagonal metric and the angular terms are zero for radial motion. We firstly calculate

$$\frac{\partial g_{rr}}{\partial r} = \left(1 - \frac{2\mu}{r}\right)^{-2} \frac{2\mu}{r^2}\,, \; \frac{\partial g_{tt}}{\partial r} = \frac{2\mu}{r^2}. \tag{13}$$

Also, dividing Eq. (10) through by $c^2 d\tau^2$ and removing the angular terms we have

$$\left(1 - \frac{2\mu}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1 - \frac{2\mu}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 1. \tag{14}$$

Substituting these results into Eq. (12), and after cancellations we find the well known result

$$\frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} = a. \tag{15}$$

An alternate, perhaps more direct derivation of this result is also shown in Appendix A. Note that a is the acceleration required to remain at rest at radius r and corresponds to the magnitude of the four-acceleration α calculated earlier. This shows a constant acceleration as assumed for the rocket frame as measured by PG2, referred to earlier as proper acceleration. This thus corresponds with Eq. (8) when $v = 0$. This implies the magnitude of the four-acceleration is

$$\sqrt{g_{\mu\nu} a^\mu a^\nu} = \sqrt{g_{rr}} \frac{GM}{r^2} = \frac{1}{\sqrt{1 - 2\mu/r}} \frac{GM}{r^2}. \tag{16}$$

We can write Eq. (15) as

$$\frac{d}{d\tau}\left(\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r}\right) = 0 \tag{17}$$

and so

$$\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2 - \frac{2GM}{c^2 r} = \text{constant} = \frac{E_0^2}{m^2 c^4} - 1, \tag{18}$$

where we assume a particle with initial energy E_0. Hence

$$\frac{dr}{d\tau} = c\sqrt{\frac{2\mu}{r} + \frac{E_0^2}{m^2 c^4} - 1}, \tag{19}$$

where $\frac{dr}{d\tau} \to \gamma v_0$ as $r \to \infty$, if we assume for large r that $E_0 = \frac{mc^2}{\sqrt{1 - v_0^2/c^2}}$.

Now $\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt}$ and so using Eq. (14) we determine

$$\frac{dt}{d\tau} = \frac{E_0}{mc^2\left(1 - \frac{2\mu}{r}\right)} \tag{20}$$

and so we find

$$\frac{dr}{dt} = c\left(1 - \frac{2\mu}{r}\right)\sqrt{1 - \frac{m^2 c^4}{E_0^2}\left(1 - \frac{2\mu}{r}\right)}, \tag{21}$$

where $\frac{dr}{dt} \to v_0$ as $r \to \infty$. Differentiating with respect to coordinate time, using the chain rule, $\frac{d^2 r}{dt^2} = \frac{d(dr/dt)}{dr}\frac{dr}{dt}$, we find

$$\frac{d^2 r}{dt^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(3\left(1 - \frac{2\mu}{r}\right)\frac{m^2 c^4}{E_0^2} - 2\right). \tag{22}$$

For a particle approaching a gravitational potential at a speed v_0 we have

$$\frac{d^2 r}{dt^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\left(3\left(1 - \frac{2\mu}{r}\right)\left(1 - \frac{v_0^2}{c^2}\right) - 2\right). \tag{23}$$

In the weak field we have $\frac{2\mu}{r} \to 0$ and so

$$\frac{d^2 r}{dt^2} = -\frac{\mu c^2}{r^2}\left(1 - \frac{3v_0^2}{c^2}\right), \tag{24}$$

a result first derived by Hilbert[4–6] in 1917, for particles moving radially in a gravitational potential.

Therefore we can see that the Schwarzschild solution also gives a velocity dependent acceleration for observers at rest with respect to the gravitational field coordinates. This implies an apparent weakening of the field strength in gravity, for radially moving objects, relative to stationary observers in weak gravitational fields. Indeed, to a first approximation, we have a velocity dependence from special relativity given in Eq. (8) of $1 - \frac{3v^2}{2c^2}\ldots$ compared with a Schwarzschild dependence, shown in Eq. (23) of $1 - \frac{3v^2}{c^2}$. This approximate confirmation of the result using the Schwarzschild solution suggests the basic principle to be sound enough to warrant experimental testing. This might be achieved in an earth bound frame, if there are accurate enough clocks to measure such deviations from current expected accelerations.

III. EXPERIMENTAL TESTS

Integrating the expression in Eq. (21), we can find the proper time taken between two heights as

$$\tau = \int_{r_0}^{r} \frac{dr}{c\sqrt{\frac{2\mu}{r} + \frac{E_0^2}{m^2 c^4} - 1}}. \tag{25}$$

This allows us to calculate the expected time difference for a falling particle based on velocity dependence v_0, and so allowing an experimental test of this principle[7].

Also, due to the rocket's mild acceleration rate, then inside the rocket frame itself, there will be extremely minor time dilation effects. This allows the stationary frame in gravity, to be the frame of reference to measure fairly accurately the rates of acceleration of PG1 and PG2. It is therefore proposed that this should be the reference frame for an experimental test of the principle. The maximum effect predicted in Eq. (8) will be for particles falling in the Earth's gravitational field at velocities approaching the speed of light.

IV. DISCUSSION

We show in this paper that by considering accelerating objects within the context of special relativity and using the equivalence principle, the behavior of weak uniform gravitational fields are predicted. Specifically, we have shown that acceleration due to gravity, is a function of radial particle velocity as shown in Eq. (8), a result first derived by Hilbert. This can also be interpreted as a weakening of the field. One way to intuitively understand this effect is that particles moving at high velocities in a gravitational field have clocks that are slowed and so effectively spend less time in the gravitational field and so experience lower acceleration than slow moving objects.

It could be claimed that this result shown in Eq. (24) of the velocity dependence of gravity is perhaps an artifact of the particular coordinates chosen, shown in Eq. (10). However, if we try the other common variants of the Schwarzschild metric, such as isotropic coordinates, Brillouin coordinates or indeed Schwarzschild's original metric, then the same result as shown in Eq. (24) is found. Refer to Appendix B for a list of these four common metrics.

We have shown there is no violation of the principle of equivalence since velocity dependance holds under both flat space accelerations and general relativity. There is also the issue though of the factor of two discrepancy between the result using acceleration under special relativity and that using the full general theory. However Einstein also found that the bending of starlight was twice the effect in GR when compared to using acceleration and the principle of equivalence. This difference is because the principle of equivalence holds for local regions of space and time which coincide with accelerating frames. Hence, for the bending of starlight we need to take account of the additional effect from the space curvature along the trajectory, which is present under gravity. Whereas, under acceleration the inertial observer observes the acceleration decreasing because the time over which the force acts appears to take longer and longer from his frame, therefore the entire effect is due changes in time not space. In an attempt confirm this, we redo our calculations for the Schwarzschild solution but set the spatial curvature to zero, then we find that we obtain a factor of two as opposed to three, which is much closer to our result in Eq. (24) and so appears to explain this discrepancy.

As noted, our result based on accelerating frames, leads to an expected effect about half that predicted by general relativity, as shown in Eq. (22). Hence it would make an interesting experiment to precisely measure this effect, and to verify the discrepancy between the two types of analysis and provide further confirmation of general relativity. This test would also thus allow a further verification of the Einstein principle of equivalence and Hilbert's equation.

Appendix A: Lagrangian approach to geodesics

A Lagrangian approach can also be used as an alternative to the geodesic equation and may be clearer for those less familiar with tensor algebra. It is also produces a shorter derivation of Eq. (15).

Now, we can rearrange the metric in Eq. (10) to define a Lagrangian

$$\mathcal{L} = \left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \frac{1}{c^2}\left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 = 1, \tag{A1}$$

where $\dot{t} = \frac{dt}{d\tau}$ and $\dot{r} = \frac{dr}{d\tau}$ and for purely radial motion we have assumed that the angular terms are zero.

We then have the action $S = \int d\tau = \int \mathcal{L} d\tau$ and so we can firstly maximize the action using Lagrange's equations for t, namely $\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0$, giving

$$\frac{d}{d\tau}\left(\left(1 - \frac{2\mu}{r}\right)c^2\dot{t}\right) = \frac{d\mathcal{L}}{dt} = 0. \tag{A2}$$

Hence we have a constant of the motion

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = \frac{E_0}{mc^2}, \tag{A3}$$

where E_0 can be shown to be the total energy for motion in a Schwarzschild metric. Substituting Eq. (A3) back into the metric we find

$$\frac{dr}{d\tau} = c\sqrt{\frac{E_0^2}{m^2c^4} - \left(1 - \frac{2\mu}{r}\right)}, \tag{A4}$$

in agreement with our previous result in Eq. (19). Also,

$$\frac{d^2r}{d\tau^2} = \frac{d}{dr}\left(\frac{dr}{d\tau}\right)\frac{dr}{d\tau} = \frac{-\mu c^2}{r^2}, \tag{A5}$$

in agreement with Eq. (15).

Now, multiplying the Lagrangian through by $\frac{d\tau^2}{dt^2}$, and solving for $\frac{dr}{dt}$ we find

$$\frac{dr}{dt} = c\left(1 - \frac{2\mu}{r}\right)\sqrt{1 - \frac{1}{\frac{dt^2}{d\tau^2}\left(1 - \frac{2\mu}{r}\right)}} \tag{A6}$$

Therefore, using Eq. (A3) we find

$$\frac{dr}{dt} = c\left(1 - \frac{2\mu}{r}\right)\sqrt{1 - \frac{m^2c^4}{E_0^2}\left(1 - \frac{2\mu}{r}\right)}, \tag{A7}$$

as shown in Eq. (21). This can be rearranged to give

$$E_0 = mc^2\left(1 - \frac{2\mu}{r}\right)^{\frac{1}{2}}\left(1 - \left(1 - \frac{2\mu}{r}\right)^{-2}\left(\frac{dr}{cdt}\right)^2\right)^{-\frac{1}{2}}, \tag{A8}$$

for the energy of the particle.

Now, differentiating Eq. (A7) with respect to time gives the coordinate acceleration shown in Eq. (22). Substituting for E_0 we find

$$\frac{d^2r}{dt^2} = -\frac{\mu c^2}{r^2}\left(1-\frac{2\mu}{r}\right)\left(1-\frac{3}{c^2}\left(\frac{dr}{dt}\right)^2\left(1-\frac{2\mu}{r}\right)^{-2}\right). \tag{A9}$$

Now if a particle at rest slowly enters the field with $r \to \infty$ then the particles' energy E_0 is approximately its rest energy mc^2, however if we wish to inject the particle into the field with velocity v_0 then $E_0 = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v_0^2/c^2}}$. This gives

$$\frac{dr}{d\tau} = c\sqrt{\frac{1}{1-v_0^2/c^2} - \left(1-\frac{2\mu}{r}\right)}. \tag{A10}$$

We can see that as $r \to \infty$ then $\frac{dr}{d\tau} \to v_0$ as required.

Appendix B: Common forms of the Schwarzschild metric

A general form of the Schwarzschild metric can be written as

$$c^2d\tau^2 = \left(1-\frac{2\mu}{C}\right)c^2dt^2 - (C')^2\left(1-\frac{2\mu}{C}\right)^{-1}dr^2 - C^2d\theta^2 - C^2\cos^2\theta d\phi^2. \tag{B1}$$

The four common variants, which are time independent, are: Schwarzschild's original metric with $C = (r^3+8\mu^3)^{1/3}$, isotropic coordinates with $C = r(1+\frac{\mu}{2r})^2$, Brillouin coordinates with $C = r+2\mu$ and the more common form of the metric with $C = r$.

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