DEMONSTRATION OF EVEN GAP AND POLIGNAC'S CONJECTURES

25 juillet 2017

BADO Olivier Idriss

(*ISE* 2)
Ecole Nationale SupÃ©rieure de Statistique et d'Economie.
08 BP 03 Abidjan 08, COTE D'IVOIRE.
E-mail: virostake@gmail.com

Abstract

In this paper we will first establish that there are many prime p such that $p+n$ is also prime for even integer n by using Chebotarev-Artin theorem
Mertens third formula and the principle inclusion-exclusion of Moivre
With these tools we get a fonction whose count the number of prime p such that $p+n$ is prime less than x for even integer n and for $n = \inf\{m \in 2\mathbb{N} : p+m \in \mathbb{P}\}$
we deduce Polignac's conjecture

1 demonstration of even gap conjecture and Polignac's conjecture

In number theory,Polignac's conjecture was made by Alphonse de Polignac in 1849 and states For any positive even number n,there are many cases of two consecutive prime numbers with difference n. In other words for any even integer n ,it exists infinitely many primes p such that p+n are consecutive primes. The object of this paper is to demonstrate this old conjecture. We propose here an elegant and original proof by proving this conjecture for even number n it exists a prime p such that $p+n$ is prime
We are going to call it even gap conjecture

2 Principle of the demonstration

To prove the conjecture of polignac's, we will first establish the formula giving the cardinal of the set of prime p such that p+n is also prime ,less than or equal to x+n where $x \geq 5$ we find $\alpha_n(x) = b_n(x) \times \frac{x}{ln^2(x)} + \bigcirc(\frac{x}{ln^3(x)})$ where $b_n(x)$ is a fonction such that $\lim_{x \to +\infty} b_n(x) = b_n$ is a constant defined by $b_n = 4\exp(-\gamma)C_n$ where $C_n = C_2 \prod_{p \in P, p \geq 3, p/n} \frac{p-1}{p-2}$ where C_2 and γ are respectively the twin prime constant and Euler-Mascheroni constant. To do that ,we decompose $C_x = \{9, 15, 21, 25, 27, 33....\}$ that is the set of the composed odd integers of $[9, x]$,via the aritmetic sequences $A_{2p,p\geq 3} = \{3p, 5p, 7p....(1 + 2p\lfloor \frac{x-p}{2p} \rfloor)p\}$ whose first element is $3p$ and of reason $2p$; where $p \in \mathcal{P}_{\sqrt{x}}$ and such that all its terms are less than x. We can then evaluate the quantity of prime numbers inside the set And then ,by applying,the Chebotarev-Artin theorem,before conclude to each the set of composed odd integers of $[9, x]$ Let the bijective mapping be $\begin{array}{cccc} f_n & : & C_x & \rightarrow & C_x + n \\ & & m & \mapsto & m+n \end{array}$

2.1 definition

Let's partionned the following set $C_x + n = IC_{\leq x+n} \cup G_{\leq x+n}$ where:

$IC_{\leq x+n}$ is the subset of $C_x + n$ formed of the composed old integers and $G_{\leq x+n}$ the subset of $C_x + n$ composed of prime numbers Let $p \in \mathcal{P}_{\leq x+n}$ the set of prime numbers less than $x + n$

2.2 lemma1

for any even integer n and for any prime $p \in \mathcal{P}_{\leq x+n} \backslash G_{\leq x+n}$ such that $p \geq n+1$ so $p - n$ is a prime number

2.3 proof of lemma 1

let$n$ be a given even integer for any $p \in \mathcal{P}_{\leq x+n} \backslash G_{\leq x+n}$ such that $p \geq n+1$ we get two situation: or $p - n < 9$ or $p - n \geq 9$ as $p - n$ is old so in the first situation $p - n$ is prime obvious manner and in the second situation $p - n \notin C_x$ so $p - n \in [9, x] \backslash C_x$ which permit us to conclude

2.4 definition

Denote by $\delta_n(x) = card(G_{\leq x+n})$, $\alpha_n(x) = card(p \in \mathcal{P}_{\leq x+n} \backslash G_{\leq x+n} : p \geq n+1)$ and $\Pi(x+n) = card(\mathcal{P}_{\leq x+n})$ .So we have $\Pi(x+n) = \delta_n(x) + \alpha_n(x) + \Pi(n+1)$

Without loss of generality,observe that each number $m \in C_x$ is divisible by at least one prime $p \leq \sqrt{x}$ Let $\mathcal{P}_{\leq \sqrt{x}} = \{p_1, p_2, p_3, ...p_r\}$ where $p_1 = 2, p_2 = 3, p_3 = 5, ...p_r = max(\mathcal{P}_{\leq \sqrt{x}})$

Each element of C_x has at least one divisor in that set $\mathcal{P}_{\leq \sqrt{x}}$ Let consider the $\lfloor \frac{x-p}{2p} \rfloor$ first element of arithmetic sequences :

$A_{2p, p \geq 3} = \{3p, 5p, 7p....(1 + 2p\lfloor \frac{x-p}{2p} \rfloor)p\} \subset C_x$ where $p \in \mathcal{P}_{\leq \sqrt{x}}$ consisting of p without p and $2p$ and less than x

2.5 remarque

The first element of $A_{2p,p\geq 3}$ is $3p$, the last element is $(1 + 2p\lfloor \frac{x-p}{2p} \rfloor)p\}$ and whose reason is $2p$ which permit us to write $C_x = \bigsqcup_{p \in \mathcal{P}_{\sqrt{x}}}(A_{2p,p\geq 3})$ so $C_x + n = \bigsqcup_{p \in \mathcal{P}_{\sqrt{x}}}(A_{2p,p\geq 3} + n)$. in the following we are going to apply Chebotarev-Artin's theorem in one hand and the other hand the principle inclusion-exclusion of Moivre, in order to evaluate the prime numbers of $\bigsqcup_{p \in \mathcal{P}_{\sqrt{x}}}(A_{2p,p\geq 3} + n)$

2.6 THEOREM 1,cf lectures on nx(p), Jean Pierre Serre

Let $a, b > 0$ integers such that$\gcd(a, b) = 1$.

Let $\Pi(x, a, b) = card(p \leq x, p \equiv a[b])$ so $\exists c > 0$ such that:

$\Pi(x, a, b) = \frac{L_i(x)}{\phi(b)} + \bigcirc(cx \exp(-\sqrt{lnx}))$

where $L_i(x) = \int_0^x \frac{dt}{lnt}$
According to the prime numbers theorem's we have $\Pi(x) \sim_\infty \frac{x}{lnx}$
so $\Pi(x,a,b) = \frac{\Pi(x)}{\phi(b)} + \bigcirc(cx\exp(-\sqrt{lnx}))$

2.7 THEOREM 2

Let $a, b > 0$ such that $\gcd(a,b) = 1$ .Let $\Pi(x,a,b) = card(p \leq x, p \equiv a[b])$ so we have $\frac{\pi(x,a,b)}{\pi(x)} = \frac{1}{\phi(b)} + \bigcirc(clnx\exp(-\sqrt{lnx}))$ In probabilistic point of view,the probability of prime numbers less than a given real number x on arithmetic progression of reason b such that $\gcd(a,b) = 1$ is $\frac{1}{\phi(b)} + \bigcirc(clnx\exp(-\sqrt{lnx}))$ in the following we are going to vindicate the application of Chebotarev-Artin's theorem to the sets $\bigcap_{j=1}^k A_{2p_{i_j}+n, p_{i_j} \leq x}$ for the integers $1 \leq i_1 \leq i_2 \leq i_3 \leq ..... \leq i_k \leq r$

2.8 REMARKS

It is easy to see that $\bigcap_{j=1}^k A_{2p_{i_j}+n, p_{i_j} \leq x}$ is the set of multiple $\prod_{j=1}^k p_{i_j}$ without $\prod_{j=1}^k p_{i_j}$ and $2\prod_{j=1}^k p_{i_j}$ we pull without problem that $\bigcap_{j=1}^k A_{2p_{i_j}+n, p_{i_j} \leq x} = \{i\prod_{j=1}^k p_{i_j} + n | 3 \leq i \leq \lfloor \frac{x - \prod_{j=1}^k p_{i_j}}{2\prod_{j=1}^k p_{i_j}} \rfloor\}$ we see that $\bigcap_{j=1}^k A_{2p_{i_j}+n, p_{i_j} \leq x}$ is an arithmetic sequence of reason $2\prod_{j=1}^k p_{i_j}$ and the first term is $3\prod_{j=1}^k p_{i_j} + n$. for vindicating the hyphothesis of Chebotarev -Artin theorem's it will be question to show that $\gcd(3\prod_{j=1}^k p_{i_j} + n, 2\prod_{j=1}^k p_{i_j}) = 1$ which easy because $\prod_{j=1}^k p_{i_j}$ don't divide n

3 Polignac's conjecture proof

3.1 THEOREM of even gap conjecture

Let $x > 0$ an arbitrarily real number, n an even integer ,$\alpha_n(x)$ the number of prime number less than x,γ Euler-Mascheroni constant C_2 twin prime constant .So it exists a fonction $b_n(x)$ such that $\lim_{n\to\infty} b_n(x) = 4\exp(-\gamma)C_n$
where $C_n = C_2 \prod_{p \geq 3, p/n} \frac{p-1}{p-2}$ such that:$\alpha_n(x) = \frac{xb_n(x)}{(lnx)^2} + \bigcirc(\frac{x}{(lnx)^3})$

3.2 useful lemma

Let $a_1, a_2, ...a_r$ r non-negative real numbers so
$1 - \sum_{i=1}^r \frac{1}{a_i} + \sum_{1 \leq i < j \leq r} \frac{1}{a_i a_j} + ... + \frac{(-1)^r}{a_1 a_2 ...... a_r} = \prod_{i=1}^r \frac{a_i - 1}{a_i}$

3.3 proof of the lemma

Consider the polynomial $P(x) = \prod_{i=1}^r (x - \frac{1}{a_i})$ .According to the relations roots-coefficients $P(x) = x^n + \sum_{k=1}^r \sum_{1 \leq i_1 < i_2 < .... < i_k \leq r} \frac{(-1)^k x^{n-k}}{\prod_{j=1}^k a_{i_j}}$ for $x = 1$ we obtain the result

3.4 proof of theorem

According to the principle of inclusion-exlusion of Moivre we have :
$\varrho(\bigcup_{j=2}^r A_{2p_j+n}, p_j \nmid n) = \sum_{k=2}^r (-1)^{k-1} \sum_{2 \leq i_1 < i_2 < ... < i_k \leq r} \varrho(\bigcap_{j=2}^k A_{p_{i_j}}, p_{i_j} \nmid n)$
where ϱ represent the probability of prime numbers and $r = max\{i | p_i \leq \sqrt{x}\}$
$\varrho(C_x + n) = \varrho(\bigcup_{j=2}^r A_{2p_j+n}, p_j \nmid n) = \frac{\delta_n(x)}{\pi(x+n)}$

According to the Chebotarev-Artin's theorem :
we have $\varrho(\bigcap_{j=2}^{k} A_{p_{i_j}}, p_{i_j} \nmid n) = \frac{1}{\phi(2\prod_{j=1}^{k} p_{i_j})} + h(x+n)$
where $h(x+n) = \bigcirc(cln(x+n)exp(-\sqrt{ln(x+n)}))$ so
$\frac{\delta_n(x)}{\Pi(x+n)} = h(x+n) + \sum_{k=2}^{r} \sum_{2 \leq i_1 < i_2 < ..... < i_k \leq r} \frac{(-1)^{k-1}}{\phi(2\prod_{j=2}^{k} p_{i_j}, p_{i_j} \nmid n)}$
$\frac{\delta_n(x)}{\pi(x+n)} = h(x+n) + \sum_{k=2}^{r} \sum_{2 \leq i_1 < i_2 < ..... < i_k \leq r} \frac{(-1)^{k-1}}{\prod_{j=2}^{k}(p_{i_j}-1), p_{i_j} \nmid n}$
According to the useful lemma we can write :
$\frac{\delta_n(x)}{\pi(x+n)} = h(x+n) + (1 - \prod_{i=2, p_i \nmid n}^{r} \frac{p_i - 2}{p_i - 1})$
$\delta_n(x) = \pi(x+n) - \alpha_n(x) - \pi(n+1)$
So $\alpha_n(x) = \pi(x+n) - \delta_n(x) - \pi(n+1)$ finally
$\alpha_n(x) = \pi(x+n) \prod_{i=2, p_i \nmid n}^{r} \frac{p-2}{p-1} - \pi(n+1) - \pi(x+n)h(x+n)$
as $r = \max\{i | p_i \sqrt{x}\}$ so $\alpha_n(x) = \pi(x+n) \prod_{3 \leq p \leq \sqrt{x}, \nmid n} \frac{p-2}{p-1} - \pi(n+1) - \pi(x+n)h(x+n)$
we going now to apply Merten's third formula in order to evaluate $c_n(x) = \prod_{3 \leq p \leq \sqrt{x}, p \nmid n} \frac{p-2}{p-1}$
As
$\prod_{3 \leq p \leq \sqrt{x}} \frac{p-2}{p-1} = \prod_{3 \leq p \leq \sqrt{x}, p \nmid n} \frac{p-2}{p-1} \prod_{3 \leq p \leq \sqrt{x}, p | n} \frac{p-2}{p-1}$
we deduce that $c_n(x) = \prod_{3 \leq p \leq \sqrt{x}} \frac{p-2}{p-1} \prod_{3 \leq p \leq \sqrt{x}, p | n} \frac{p-1}{p-2}$

The formula of Mertens can been expressed by:
$\prod_{p \leq x}(1 - \frac{1}{p}) = \frac{exp(-\gamma)}{lnx}(1 + \bigcirc(\frac{1}{lnx}))$
So $\prod_{p \leq \sqrt{x}}(1 - \frac{1}{p}) = \frac{2exp(-\gamma)}{lnx}(1 + \bigcirc(\frac{1}{lnx}))$
Let $c_2(x) = \prod_{3 \leq p \leq \sqrt{x}} \frac{p(p-2)}{(p-1)^2}$
$c_2(x) = \prod_{3 \leq p \leq \sqrt{x}} \frac{p}{p-1} \prod_{3 \leq p \leq \sqrt{x}} \frac{p-2}{p-1}$
So $c_n(x) = \prod_{3 \leq p \leq \sqrt{x}}(1 - \frac{1}{p}) c_2(x) \prod_{3 \leq p \leq \sqrt{x}, p | n} \frac{p-1}{p-2}$
so $c_n(x) = 2\prod_{p \leq \sqrt{x}}(1 - \frac{1}{p}) c_2(x) \prod_{3 \leq p \leq \sqrt{x}, p | n} \frac{p-1}{p-2}$
With the formula of Mertens we deduce that :
$c_n(x) = \frac{4c_2(x)exp(-\gamma)}{lnx} \prod_{3 \leq p \leq \sqrt{x}, p | n} \frac{p-1}{p-2}[1 + \bigcirc(\frac{1}{lnx})]$
As $\pi(x+n) = \frac{x+n}{ln(x+n)}[1 + \bigcirc(\frac{1}{ln(x+n)})]$
then $\alpha_n(x) = \frac{4xc_2(x)exp(-\gamma)}{ln^2(x)} \prod_{3 \leq p \leq \sqrt{x}, p | n} \frac{p-1}{p-2}[1 + \bigcirc(\frac{1}{lnx})] - \pi(x+n)h(x+n) - \pi(n+1)$
But in obvious manner we prove that $\pi(x+n)h(x+n) = \bigcirc(\frac{x}{(ln(x))^3})$
for x an arbitrarily real number and an integer n such that $n \ll x$ we can conclude
for $n = \inf\{m \in 2\mathbb{N} : p + m \in \mathbb{P}\}$ we deduce Polignac's conjecture

4 Acknowledgments

The author wish to express their appreciation and sincere thanks to Professor Pascal Adjamogbo for his encouragements
and advices.

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