Restraining general covariance

Le chant du Cygne

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Abstract

The Postulate of General covariance, a last choice that Einstein introduced in his General relativity theory of gravitation, endows the theory with an excessive generality that needs to be restrained. Otherwise it is easy to check that, beyond the first order of approximation, several space-time models, all of them derived from the original Schwarzschild's initial solution of Einstein's field equations corresponding to a static spherical source, would predict different results to two fundamental experiments: the measure of the force acting on a given passive test mass at a distance R of the center of the source, or equivalently, its initial acceleration when falling from rest, and the comparison of two way transit times of light traveling into an optic fiber along a vertical direction and along a meridian. The last section describes how to find numeric, static and spherically symmetric, interior models with pre-selected mass m and radius R with $m/R < 1$ or $m/R > 1$ solving the horizon problem.

1 Introduction

The line-element of any static spherically symmetric space-time vacuum solution, can be written using a variety of systems of "polar space coordinates" as follows:

$$ds^2 = -(Adt + A^{-1}fdr)^2 + A^{-2}d\bar{s}^2, \quad d\bar{s}^2 = B^2dr^2 + BCr^2d\Omega^2 \qquad (1)$$

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$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 \tag{2}$$

where A, B, C, f are functions of r. I shall tell below why I used quotation marks in mentioning the polar coordinates condition being used. Later also we shall see why it make sense to distinguish the space metric $d\bar{s}^2$ from the quotient metric $d\hat{s}^2 = A^{-2}d\bar{s}^2$. This was done for the first time in [2] and has been used since then in many other papers, among them [11],[12],[13],[16],[20], giving in the process a physical meaning to it as a 3-dimensional physical metric.

The vector with components $\xi^0 = 1$ and $\xi^i = 0$, i=1,2,3, is the Killing vector defining the frame of reference of all the models that I considera nd can be derived one from another by a coordinate transformation such as:

$$r' = r'(r), \quad t' = t + \beta(r) \tag{3}$$

To be more specific, the models that I consider are : the Isotropic model on one hand, and on the other hand the Fock model, the Droste-Hilbert model, the Brillouin model, the Whitehead-Kerr-Schild model, the Eddington-Finkelstein model, and the Gullstrand-Painlevé model that they are all related to one another by simple adapted coordinate transformation of type (3):

$$r' = r + \mu m, \quad t' = t + \beta(r) \tag{4}$$

m being the mass of the source and μ any constant.

Using temporarily suffixes as indices of the variable r these are some of the models that could be considered:

Isotropic model

$$A_i^2 = \frac{(2r_i - m)^2}{(2r_i + m)^2}, \quad f_i = 0, \quad d\bar{s}_i^2 = \left(1 - \frac{m^2}{4r_i^2}\right)(dr_i^2 + r_i^2 d\Omega^2) \tag{5}$$

Fock model

$$A_f^2 = \frac{r_f - m}{r_f + m}, \quad f_f = 0, \quad d\bar{s}_f^2 = dr_f^2 + \left(1 - \frac{m^2}{r_f^2}\right) r_f^2 d\Omega^2 \tag{6}$$

Droste-Hilbert model

$$A_d^2 = 1 - \frac{2m}{r_d}, \quad f_d = 0, \quad d\bar{s}_d^2 = dr_d^2 + \left(1 - \frac{2m}{r_d}\right) r_d^2 d\Omega^2 \tag{7}$$

Brillouin

$$A_b^2 = \frac{r_b}{r_b + 2m}, \quad f_b = 0, \quad d\bar{s}_b^2 = dr_b^2 + \left(1 + \frac{2m}{r_b}\right) r_b^2 d\Omega^2 \tag{8}$$

Whitehead or Kerr-Schild

$$A_w^2 = 1 - \frac{2m}{r_w}, \quad f_w = \pm\frac{2m}{r_w}, \quad d\bar{s}_w^2 = dr_w^2 + \left(1 - \frac{2m}{r_w}\right) r_w^2 d\Omega^2 \tag{9}$$

Eddington-Finkelstein

$$A_e^2 = 1 - \frac{2m}{r_e}, \quad f_e = \pm 1, \quad d\bar{s}_e^2 = dr_e^2 + \left(1 - \frac{2m}{r_e}\right) r_e^2 d\Omega^2 \tag{10}$$

Gullstrand-Painlevé

$$A_g^2 = 1 - \frac{2m}{r_g}, \quad f_g = \sqrt{\frac{2m}{r_g}}, d\bar{s}_g^2 = dr_g^2 + \left(1 - \frac{2m}{r_g}\right) r_g^2 d\Omega^2 \tag{11}$$

but from now on I shall restrict my discussion to the models with $f = 0$. Including the models with $f \neq 0$ is not necessary to the purpose of this paper.

Excluding the Isotropic model the data for A and C can be parameterized as follows:

$$A = \left(\frac{r - \lambda m}{r + (2 - \lambda)m}\right)^{1/2}, \; B = 1, \quad C = \frac{(r + (2 - \lambda)m)(r - \lambda m)}{r^2} \tag{12}$$

$\lambda = 0$ corresponds to Brillouin's model, $\lambda = 1$ corresponds to Fock's model, and $\lambda = 2$ corresponds to the Droste-Hilbert model.

The inferior limit values that r can reach with $A^2 > 0$ are $r_b = 0$, $r_i = m/2$, $r_f = m$, $r_d = 2m$, $...r_\lambda = \lambda m$. They restrict the minimum values of the radius R that a spherical source could have to be matched to the corresponding exterior model.

The first order approximation of the line element corresponding to A, B, C above is:

$$ds^2 \simeq -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 + \frac{2m}{r}\right) dr^2 + r(r + 4m - 2\lambda m)(d\theta^2 + \sin^2\theta d\phi^2) \tag{13}$$

that it is as much as it is needed to derive the advance of the perihelium of Mercury and the deviation of light by the Sun.

2 Force

In Newton's theory of gravitation the basic and simple principle from which derives the magnificent description of the heavens that we know it is as simple as saying that a massive sphere of radius R and mass m exerts on a test mass located at a distance $r > R$ from its center a force F in the radial direction equal to $-m/r^2$ if units are used such that the gravitational constant is $G = 1$. And one would expects that any acceptable theory of gravitation should predict also a unique value to whatever correction it proposes.

In Einstein's theory of gravitation, because of the principle of General covariance and the concomitant principle according to which the coordinates do not have any physical or geometric meaning, the whole theory becomes meaningless beyond the first approximation. This can be seen very easily below calculating the corresponding relativistic force on a unit mass for each of the models that I have mentioned, up to the second order, according to the formula:

$$F_0^r = \left(\frac{d^2r}{dt^2}\right)_0 = -\Gamma^r_{tt} = -\frac{A^4}{B^2}\partial_r \ln A \tag{14}$$

where Γ^r_{tt} is the corresponding Christoffel connection symbol, and the subindex 0 means that the test particle is at rest with respect to the source of the gravitational field.

This is the result for the Isotropic model:

$$F_i = -\frac{64r^4(m-2r)m}{(2r+m)^7} \tag{15}$$

that can be approximated as follows;

$$F_i = -\frac{m}{r_i^2} + 4\frac{m^2}{r_i^3}, \quad r_i = \frac{1}{2} + \frac{1}{2}\sqrt{r_f^2 - m^2} \simeq r_f - \frac{1}{4}\frac{m^2}{r_f^2} \tag{16}$$

The results corresponding to the other models parameterized by λ are:

$$F = -\frac{(r-\lambda m)m}{(r+(2-\lambda)m)^3} \tag{17}$$

And can be approximated as follows:

Fock model

$$F_f \simeq -\frac{m}{r_f^2} + \frac{4m^2}{r_f^3} \tag{18}$$

Droste-Hilbert model

$$F_d \simeq -\frac{m}{r_d^2} + \frac{2m^2}{r_d^3}, \quad r_d = r_f + m \tag{19}$$

Brillouin model

$$F_b \simeq -\frac{m}{r_b^2} + \frac{6m^2}{r_b^3}, \quad r_b = r_f - m \tag{20}$$

Eddington-Finkelstein model

$$F_e \simeq -\frac{m}{r_e^2} + \frac{2m^2}{r_e^3}, \quad r_e = r_f + m \tag{21}$$

Gullstrand-Painlevé model

$$F_g \simeq -\frac{m}{r_g^2} + \frac{2m^2}{r_g^3}, \quad r_g = r_f + m \tag{22}$$

The force I am talking is supposed to be measured with gravimeters and produce a single result at each point, not many.

The models in the list above, while having all of them the correct Newtonian limit for m small enough, can be divided in three groups. A first group, the Fock and the Isotropic model predict the same second order deviation. A second group, the Droste-Hilbert, the Eddington-Finkelstein, and the Gullstrand-painlevéé models predict the same deviation although different from that of the preceding group. Finally the Brillouin model predicts a still different value for the deviation. Since only one or none of these predicted values can correspond to reality means that the Principle of general covariance should be questioned and some means of predicting a single result should be discovered a point of view that is already mentioned in the papers by Painlevé, Gullstram and Brillouin.

3 The coordinates have a meaning

At the beginning of this paper I said that the line-element of each of the models to be considered will be written using polar coordinates of space. Would this mean that every one of the coordinates $r_s, r_i, r_d, r_b, r_e, r_g, r_k, r_l$ have the same meaning or only one of them, or none, deserves to be called a radial polar coordinate?

Polar coordinates r, θ, ϕ are defined precisely only in the context of Riemannian metrics satisfying Helmholtz's free mobility postulate on which Metrology is based, the simplest being the Euclidean metric:

$$d\tilde{s}^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{23}$$

where the variable r is defined in the interval $r \geq 0$. Restricting this interval to $r \geq r_0 \neq 0$ would not change the metric properties but changes the topology of the domain where the line-element is defined. This is to keep in mind.

It is a careless decision, in a different context, to name three variables r, θ, ϕ polar coordinates because we are using the same letters without mentioning the relationship with polar coordinates of Euclidean space.

We got above three different results evaluating the correction to the Newtonian force value $-m/r^2$. The Isotropic model and the Fock model predict a correction of order $4m^2/r^3$, while the Droste-Hilbert model predicts $2m/r^3$ and the Brillouin model predicts $6m^2/rs$. This is unacceptable because only one or none of these estimations could be the result of a real experiment.

On the other hand this inconsistency comes as a natural consequence of the Principle of general covariance on which General relativity is founded but should not be oversimplified to the point of claiming that the coordinates do not have any meaning. It is obvious that if the three values above of the relativistic corrections are logical consequences of a correct theory then this theory must be completed differentiating the meaning of the coordinate r.

So, the question is: what is the connection between the coordinates above and the coordinates used in the line-elements $d\bar{s}^2$ used before? Considering the Isotropic model the connection is very simple because $d\bar{s}^2$ is conformal to $d\tilde{s}^2$ and therefore in this case it is legitimate to proceed assuming that the coordinates r, θ, ϕ have the same meaning in both cases. Incidently this is the case also with all cosmological Robertson-Walker models.

I have considered before in ([18]) and ([19]) this problem for the Fock model proving the following relations:

$$(\bar{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{jk})\bar{g}^{jk} = 0 \tag{24}$$

where $\bar{\Gamma}^i_{jk}$ and $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of the two metrics $d\bar{s}^2$ and $d\tilde{s}^2$. Notice however that this is true only for $r > m$ because this is the domain where the Fock model is defined.

They can be generalized to the one parameter λ family of models (12), where they become:

$$(\bar{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{jk})\bar{g}^{jk} = 2(\lambda - 1)\bar{g}^{is}\partial_s \ln A \tag{25}$$

That, somewhat more explicitly, can be read as a unique equation:

$$-\frac{dC}{dr}+\frac{2(B-C)}{r}=\frac{2mB^2C(\lambda-1)}{(\lambda m-2m-r)(\lambda m-r)} \tag{26}$$

Notice that (25) is a tensor relationship and therefore if they are verified for polar coordinates it will be verified also for any other system of coordinates of the Euclidean space. Whatever be the new meaning in Euclidean space this will be the same meaning in the relativistic model.

I think that it follows from this analysis that General covariance has to be broken. Other possibilities would be to consider General relativity as an envelope of several theories and wait until a very high precision observation discriminates one among the infinity of acceptable models. Or consider the diversity of acceptable predictions as bearing different information about the source of the field, as for example its radius or or its density layer structure.

If a choice has to be made, the harmonic condition (24) is the simplest way to do it. This should suffice to deal with interior models with a radius $R > m$ but not otherwise. Unfortunately it leads to the horizon problem.

To deal with sources of given mass m and arbitrary small radius R it will be necessary to deal with the M. Brillouin model. This is done in the last section.

4 The time-length of an optic fiber

Another of the benefits of the analysis of the preceding section is to give a meaning to the optical length of an arc of curve defined by parametric equations:

$$r=r(s),\ \theta=\theta(s)\ \phi=\phi(s) \tag{27}$$

of physical length:

$$L_{phys}=\int_{s_0}^{s_1} d\tilde{s} \tag{28}$$

by:

$$L_{opt}=\int_{s_0}^{s_1}\frac{1}{A^2}d\bar{s} \tag{29}$$

for any of the models corresponding to a particular value of the parameter λ introduced in (12).

Let us consider an ideal optic fiber with refraction index 1, and physical length L_{phys}. I consider here only two situations: i) the fiber is stretched in the vertical direction upwards from a particular position r. And ii) the fiber with the same origin follows a meridian arc.

Since light propagates along the fiber so that $ds^2 = 0$ using (34) I obtain that the time that the light takes to reach maximum height $r + L_{phys}$ where it is reflected towards the starting point will be:

$$\Delta T_{ver} = 2 \int_r^{r+L_{phy}} \frac{dr}{A(r,\lambda)^2} \tag{30}$$

While if light propagates in the fiber along a meridian angle $\Delta\theta$ from the origin the transit time would be:

$$\Delta T_{hor} = 2\frac{L}{A(R,\lambda)^2}\sqrt{C(R,\lambda)}L \tag{31}$$

A short calculation neglecting any powers of m, the mass of the Earth, and L, the physical length of the fibers, yields the following result:

$$\Delta T_{ver} - \Delta T_{hor} = \frac{2Lm}{R}(\lambda - 1) \tag{32}$$

With Fock's model ($\lambda = 1$) the result is zero at this order of approximation. With Brillouin's model ($\lambda = 0$) and with L of the order of one km the result is of the order of 10^{-14} s, and for Droste's model the result has the opposite value.

5 Spherically symmetric, static, interior solutions with $m < R$ or $m > R$

What it was known before as the the "The Schwarzschild singularity" and it is now known as its "Horizon" originates from the fact that each of the vacuum line-elements, but one, listed in the Introduction of this paper require that m be less than R. The Brillouin line-element (8) instead does not require this condition and this opens the possibility of considering ultra relativistic objects with densities never heard of. This is the subject of this section.

I consider the problem of matching, in the sense of Lichnerowicz, The Brillouin exterior model with mass m:

$$ds^2 = -A_b^2 dt^2 + A_b^{-2} d\bar{s}^2, \quad d\bar{s}^2 = dr^2 + C_b r^2 d\Omega^2 \tag{33}$$

to an interior static, spherically symmetric model with line-element:

$$ds^2 = -A^2 dt^2 + A^{-2} d\bar{s}^2, \quad d\bar{s}^2 = B^2 dr^2 + BCr^2 d\Omega^2 \tag{34}$$

across a surface of radius R in both cases $m < R$ and $m > R$ so that:

$$A = A_b,\ C = C_b,\ A' = A'_b,\ C' = C'_b,\ \text{across } r = R. \tag{35}$$

and $B = 1$ throughout so that r means everywhere the distance from the center.

Considering only perfect fluids only as sources, the equations to be solved with "G=c=1" are:

$$S^0_0 = 16\pi\rho \quad S^1_1 = S^2_2 = S^3_3 = P \tag{36}$$

S^α_β being the components of the Einstein tensor,ϱ a positive density, and$P$ a positive pressure, from where it follows , using the conservation the equation of hydrostatic equilibrium:

$$P' = -\frac{A'}{A}(\rho + P) \tag{37}$$

The preceeding equations lead to the following system of differential equations:

$$2AA''-3A'^2+\frac{2C'AA'}{C}-\frac{A^2C''}{C}+\frac{1}{4}\frac{C'^2A^2}{C^2}+\frac{4AA'}{r}-\frac{3C'A^2}{rC}-\frac{A^2}{r^2}+\frac{A^2}{C^2r^2} = 16\pi\rho \tag{38}$$

$$-2A'^2 - \frac{1}{2}\frac{A^2C''}{C} + \frac{1}{2}\frac{A^2C'^2}{C^2} + \frac{A^2}{r^2} - \frac{A^2}{Cr^2} = 0. \tag{39}$$

I consider two cases: i) the density ρ is constant and ii) $dP(\rho)/d\rho = 1$ meaning that the speed of sound be the speed of light. It follows then from (37) that:

$$\rho = \frac{1}{2}(2kA^{-2} - \rho(R)) \quad \text{and} \quad P = \frac{1}{2}(2kA^{-2} - 3\rho(R)) \tag{40}$$

k and $\rho(R)$ being two arbitrary constants.

Values of the parameters in case i) for which the numeric integration proceeds smoothly and yields plausible graphs includes the two cases:

$$m = 0.01,\ R = 1,\ \rho = 0.001 \quad \text{and} \quad m = 1,\ R = 0.01,\ \rho = 0.3 \tag{41}$$

Values that can be tested in case ii) yielding smooth graphs include:

$$m = 0.01,\ R = 1,\ k = 0.001 \ \text{ and } \ m = 1,\ R = 0.01,\ k = 0.001, \rho(R) = 0 \tag{42}$$

so that:

$$\rho = P = kA^{-2} \tag{43}$$

The graphs below correspond to the latest choice considered above. The blue graphics Ab, Bb and Cb are the graphics corresponding to the vacuum Brillouin model.

6 Concluding remarks

It is my opinion that summarizing the Principle of general covariance saying that "the coordinates mean nothing" is a very bad assertion. I believe, on the contrary that the meaning of the coordinates has to be made precise and this requires a Restricted principle of covariance.

It is possible to avoid the horizon problem matching the Brilloin model of Schwarzschild exterior solution to interior solutions with physically acceptable equations of state. The matching being made in the sense of Lichnerowicz, which is the most restrictive of those that have been proposed.

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References

[1] (a) k. Schwarzschild, (a) *Sitzungsber. Preuss. Akad. Wiss.*, Phys. Math. Kl., 189 (1916) (b) Reprinted as Golden Oldie *Gen. Rel. Grav.*, **35**, 951 (2003)

[2] Weyl H, , *Ann. Phys.,(Leipzig)*, 54, 117 (1917)

[3] (a) J. Droste, *Proc. K. Ned. Akad.Wet.*,Ser. A **19**, 197 (1917). (b) Reprinted as Golden Oldie *Gen. Rel. Grav.*, **34**, 1545 (2002)

[4] D. Hilbert, *Nachr. GesWiss. Götingen, Math. Phy. Kl.*, 53 (1917)

[5] P. Painlevé *C. R. Acad. Sc. (Paris)*, 173, 667 (1921)

[6] A. Gullstrand *Arkiv fr Matematik Astronomy och Fysik),* **16**, *(8), 1 (1922)*

[7] A. N. Whitehead (The Priciple of Relativity, University Press(1922)

[8] M. Brillouin, *Le Journal de Physique et Le Radium*, **23**, 43 (1923). English translation by S. Antoci: arXiv:Physics/0002009 v1

[9] L. Brillouin, *Relativity Reexamined*, Academic Press, (1970)

[10] A. S. Eddington *The Mathematical Theory of Relativity*(2nd ed.), Cambridge University Press, 93 (1924)

[11] V. Fock, *The theory of Space, time and Gravitation*, Chap. V, Sect. 56, Pergamon Press (1964)

[12] J. Ehlers and W. Kundt *GRAVITATION an introduction to current research*, Ed. L. Witten, John Wiley & Sons

[13] L. Bel, J. C. Escard *Acc. Nazionale dei Lincei*, Ser. VIII, **XVI**, fasc. 6 (1966)

[14] L. Bel, *J. Math. Phys.*, **10**, 1501 (1969)

[15] L. Bel, *Gen. Rel. Grav.*, **1**, 337 (1971)

[16] Ll. Bel, arXiv:0709.2257 [gr-qc] (2007)

[17] Ll. Bel, arXiv:1103.3767v2 [gr-qc] (2011)

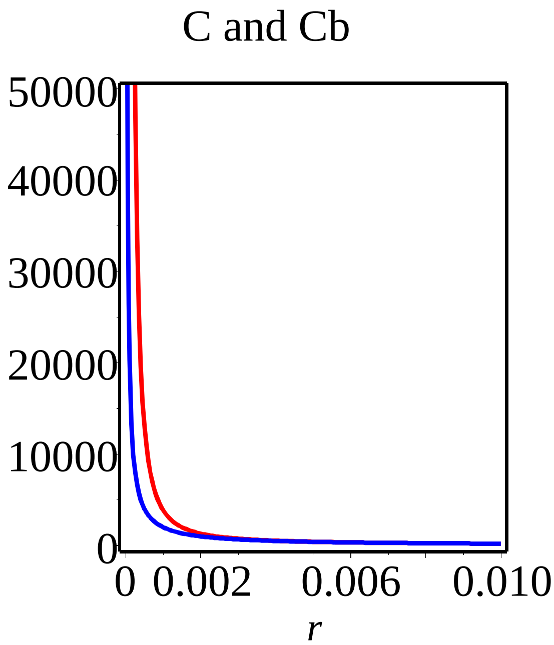
[18] Ll. Bel, arXiv:1111.4943v1 [gr-qc] (2011)

[19] Ll. Bel, arXiv:1205.6049v1 [gr-qc] (2012)

[20] R. Geroch *J. Math. Phys.*, **12**, 918 (1971)

[21] J. M. Senovilla, *Gen. Rel. Grav.*, **30**, 701, Sect. 3, Exemple 3.3 (1998)

[22] J. M. Senovilla, *Gen. Rel. Grav.*, **39**, 685 (2007)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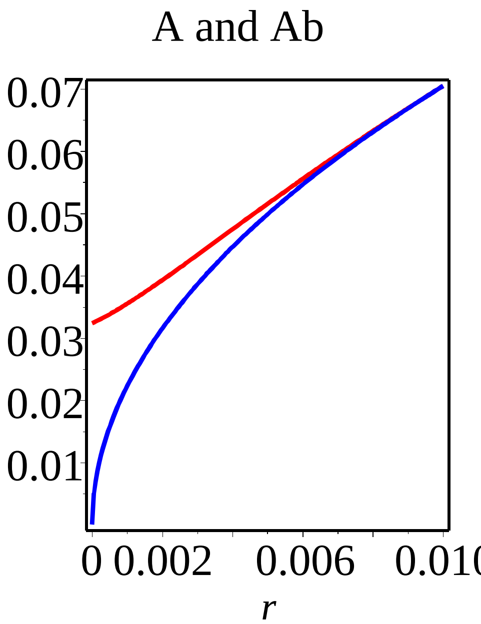
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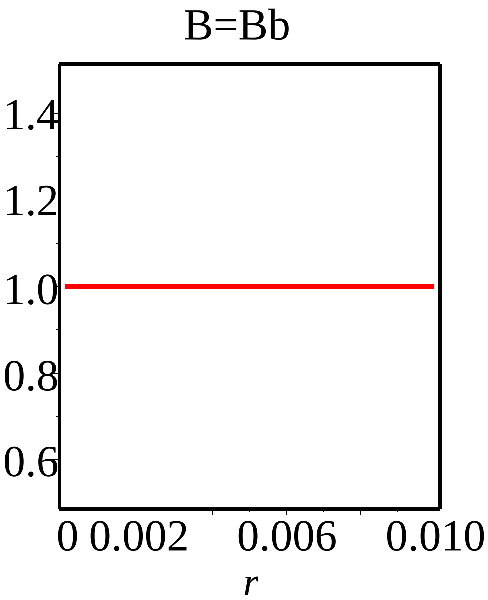
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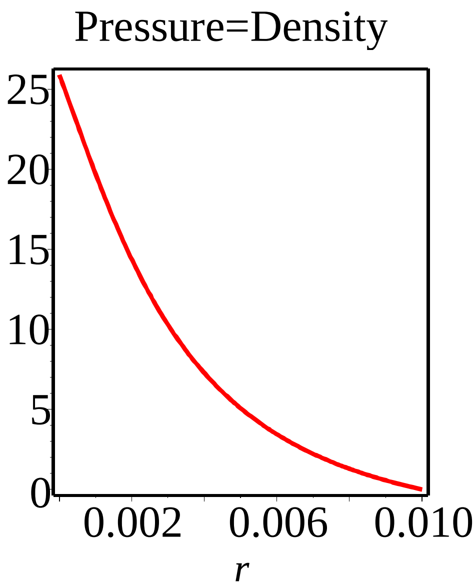
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