The n X n X n Points Problem optimal solution

Marco Ripà

Economics − Institutions and Finance, Roma Tre University, Rome, Italy.
e-mail: marcokrt1984@yahoo.it
YouTube Channel: https://www.youtube.com/user/marcokrt

Abstract: We provide an optimal strategy to solve the n X n X n points problem inside the box, considering only 90° turns, and at the same time a pattern able to drastically lower down the known upper bound. We use a very simple spiral frame, especially if compared to the previous *plane by plane* approach, that significantly reduces the number of straight lines connected at their end-points necessary to join all the n^3 dots. In the end, we combine the square spiral frame with the rectangular spiral pattern in the most profitable way, in order to minimize the difference $h_u(n) - h_l(n)$ between the upper and the lower bound, proving that it is $\leq 0.5 \cdot n \cdot (n + 3)$, for any $n > 1$.

Keywords: topology, inside the box, nine dots, straight line, outside the box, upper bound, graph theory, three-dimensional, segment, point.

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1. Introduction

As stated by the classic *nine dots problem* appeared in Samuel Loyd's *Cyclopedia of Puzzles* (Figure 1) [2], we have to "(…) draw a continuous line through the center of all the eggs so as to mark them off in the fewest number of strokes" [1]-[3]. However, this time we are considering n^3 points located in a three-dimensional space.

Thus, we will show how it is possible to join n X n X n points arranged in n equidistant grids, formed by n rows and n columns each, using at most $\frac{n^2+3\cdot n}{2}$ straight lines connected at their end-points, for any $n \in \mathbb{N}$.

Little Tommy Riddles calls attention to a couple of Christopher Columbus' famous egg tricks. In the first puzzle the famous trick-chicken, Americus Vespucius, after whom our great country was named, showed a clever puzzle wherein you are asked to lay nine eggs so as to form the greatest possible number of rows of three-in-line. King Puzzlepate has only succeeded in getting eight rows, as shown in the picture, but Tommy says a smart chicken can do better than that! The funny old King is now trying to work out a second puzzle, which is to draw a continuous line through the center of all of the eggs so as to mark them off in the fewest number of strokes. King Puzzlepate performs the feat in six strokes, but from Tommy's expression we take it to be a very stupid answer, so we expect our clever puzzlists to do better; it is a very ingenious trick, fully as good if not better than that of making an egg stand up on end, for the perpetration of which with an over ripe egg the great navigator was loaded with chains.

Figure 1. The original problem from Loyd's *Cyclopedia of Puzzles*, New York, 1914, p. 301.

2. The n X n X n problem bounds

The original *nine dots puzzle* can be naturally extended to an arbitrarily large number of distinct (zero-dimensional) points for each row / column [6]. This new problem asks to connect $n \text{X} n$ points, arranged in a grid formed by n rows and n columns, using the fewest straight lines connected at their end-points. Ripà and Remirez [4] showed that it is possible to do this for every $n \in \mathbb{N} - \{0, 1, 2\}$, using only $2 \cdot n - 2$ straight lines. For any $n \geq 5$, we can combine a given 8 line solution for the $5 \text{X} 5$ problem [11] and the square spiral frame [12].

In [5], Ripà further extended the $n \text{X} n$ result to a three-dimensional space [7] providing non-trivial bounds for this problem.

For $n > 3$, Ripà proved the lower bound [8]

$$h_l(n) = n^2 + \left\lceil \frac{3 \cdot n^2 - 4 \cdot n + 2}{2 \cdot (n-1)} \right\rceil \quad (1)$$

and the upper bound

$$h_u(n) = \begin{cases} \frac{2}{3} \cdot {i_{max}}^3 + 5 \cdot {i_{max}}^2 - 2 \cdot \left(n - \frac{14}{3}\right) \cdot i_{max} + 2 \cdot n^2 - 2 \cdot n + 3 \\ \qquad \boldsymbol{if} \quad n - {i_{max}}^2 - 5 \cdot i_{max} \leq 5 \\ \\ \frac{2}{3} \cdot {i_{max}}^3 + 6 \cdot {i_{max}}^2 - \left(2 \cdot n - \frac{43}{3}\right) \cdot i_{max} + 2 \cdot n^2 - 3 \cdot n + 8 \\ \qquad \boldsymbol{if} \quad n - {i_{max}}^2 - 5 \cdot i_{max} > 5 \end{cases} \tag{2}$$

Where i_{max} is the maximum value $i \in \mathbb{N}_0$ such that $n \geq i^2 + 5 \cdot i + 4 \mapsto \boldsymbol{i_{max}} = \left\lfloor \frac{1}{2} \cdot \left(\sqrt{\boldsymbol{4 \cdot n + 9}} - \boldsymbol{5}\right) \right\rfloor$.

Now, we will combine the square spiral pattern with the rectangular spiral one in order to drastically lower down the (2) following the *good average* approach: we have to put our effort on keeping high the average number of dots joined with two or more consecutive lines rather than focusing ourselves on a single one.

The method we are introducing lets us improve the (2) for any n and the best known upper bound for $n > 5$. On specifics, if $n = 5$ we match the best outcome of the [5].

This time it is not so important from which plane we start, anyway we know from the *pigeon problem* how to maximize the total number of dots connected skipping form a given plane to another. In fact, the pigeon problem asks: "Which is the maximal value of the sum of the lengths of $n - 1$ line segments (connected at their end-points) required to pass through n trail dots, with unit distance between adjacent points, visiting all of them without overlap two or more segments?", and its solution is given by the sequence A047838 of the OEIS [9].

Thus, let we start applying the square spiral frame to a central n X n grid, then we move to an external one, then we go through $\frac{n}{2}$ dots (if n is even) or $\frac{(n-1)}{2} + 1$ (if n is an odd number) and so on. We draw $2 \cdot n - 5$ lines for each frame and then we move to another grid, leaving 6 central dots for the ending, when we will use the rectangular spiral pattern. In this way we will be able to solve the puzzle without exiting from *the box*, the minimal cube that contains all the dots, not even once.

For every $n \geq 5$, the square spiral frame is as follows (Fiqure 2):

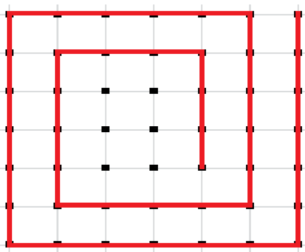

Figure 2. The square spiral frame using $2 \cdot n - 5$ lines (7 X 7 grid).

At the end of this process, we have used $2 \cdot n^2 - 4 \cdot n - 1$ lines and we need one more line in order to reach the starting position for the rectangular spiral pattern (Figure 3) [5].

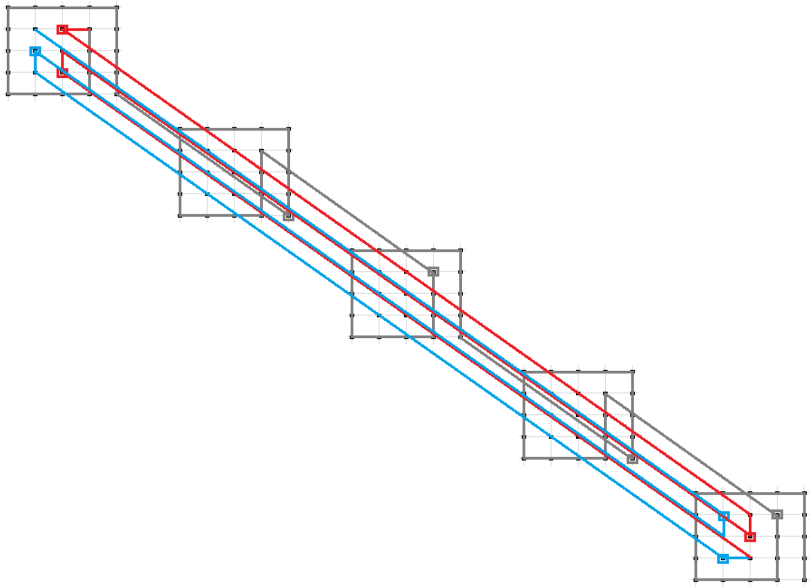

Figure 3. The rectangular spiral pattern for $n = 5$ (in red and light blue). This is the shortest path (length $= n^3 - 1$ units), no crossing lines, no points visited twice, $h_u(5) = 41$.

It is pretty easy to find out that the particular rectangular spiral arrangement we have chosen takes a total of 11 additional lines, for a new upper bound of $2 \cdot n^2 - 4 \cdot n + 11$.

$$h_u(n) = 2 \cdot n^2 - 4 \cdot n + 11 \tag{3}$$

Hence, by combining the (1) with the (3), for any $n \in \mathbb{N} - \{0, 1\}$, we can say that

$$n^2 + \left\lceil \frac{3 \cdot n^2 - 4 \cdot n + 2}{2 \cdot (n-1)} \right\rceil \leq h(n) \leq 2 \cdot n^2 - 4 \cdot n + 11 \tag{4}$$

3. The optimal upper bound

Looking carefully at the method described in Section 2, it is possible to discover how, as n grows, it would not be a good strategy to sacrifice the rectangular spiral waiting the last 11 moves of the game. There should be an optimal value to switch from the square spiral frame to the final stage, in order to get the best *inside the box* solution of the n X n X n dots puzzle. How many moves it is convenient to perform following the rectangular spiral path depends from n, keeping in mind that the best approach is to put the entire focus on maximizing the average number of dots connected with a large set of consecutive lines, according to the good average strategy that let us easily improve the previous upper bound for any $n > 6$.

Given that we are looking for the optimal $q \in \mathbb{N}$, the number of connected lines that belong to the square spiral frame, such that the (5) assumes the minimum value over the integers, we get two different cases depending on whether q is even or odd.

$$f(q,n) = \begin{cases} q \cdot n + n - 1 + 2 \cdot \left(n - \frac{q}{2}\right)^2 \\ \quad \textit{\textbf{if}} \quad q = 2 \cdot m, \forall m \in \mathbb{N} - \{0, 1, 2\} \\ \\ q \cdot n + n - 1 + 2 \cdot \left(n - \frac{q+1}{2}\right) \cdot \left(n - \frac{q-1}{2}\right) \\ \quad \textit{\textbf{if}} \quad q = 2 \cdot m - 1, \forall m \in \mathbb{N} - \{0, 1, 2\} \end{cases} \tag{5}$$

$$q_{opt}(n) \coloneqq min \ \{ \ a \mid \exists \ q \in \mathbb{N} - \{0, 1, 2, 3, 4\} : a = f(q,n) \ \}$$

Since the (5) implies that $q_{opt}(n) = n$ for any $n \geq 5$, the optimal solution requires n lines for each square spiral (Figure 4), n connecting lines and $\frac{n^2}{2} - 1$ or $\frac{n^2-1}{2} - 1$ lines (respectively if n is an even or an odd number) for the rectangular spiral pattern (see Table 1). Therefore, for a generic $m \in \mathbb{N} - \{0, 1, 2\}$, the number of lines necessary to visit all the n^3 dots is given by the (6)-(7):

$$h_u(n) = \begin{cases} n^2 + n + 2 \cdot \left(n - \frac{n}{2}\right) \cdot \left(n - \frac{n}{2}\right) - 1 \\ \quad \textit{\textbf{if}} \quad n = 2 \cdot m \\ \\ n^2 + n + 2 \cdot \left(n - \frac{n+1}{2}\right) \cdot \left(n - \frac{n-1}{2}\right) - 1 \\ \quad \textit{\textbf{if}} \quad n = 2 \cdot m - 1 \end{cases} = \begin{cases} \frac{3}{2} \cdot n^2 + n - 1 \\ \quad \textit{\textbf{if}} \quad n = 2 \cdot m \\ \\ \frac{3}{2} \cdot n^2 + n - \frac{3}{2} \\ \quad \textit{\textbf{if}} \quad n = 2 \cdot m - 1 \end{cases} \tag{6}$$

Hence,

$$\boldsymbol{h(n) \leq \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n - 1} \tag{7}$$

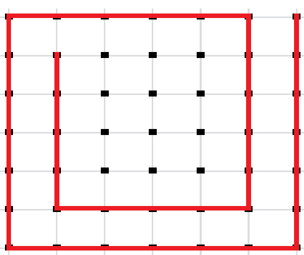

Figure 4. The *optimal* square spiral frame, where $q = n$, for a 7 X 7 grid.

Table 1: $n \mathrm{X} n \mathrm{X} n$ points puzzle upper bounds [7], following the *square* / *rectangular spiral pattern* by Figure 3 and Figure 4, for any $n \geq 5$.

n	**Best Upper Bound Currently Discovered**
1	/
2	**7**
3	**14**
4	**25**
5	**40**
6	**59**
7	**79**
8	**103**
9	**129**
10	**159**
11	**191**
12	**227**
13	**265**
14	**307**
15	**351**
16	**399**
17	**449**
18	**503**
19	**559**
20	**619**
21	**681**
22	**747**
23	**815**
24	**887**
25	**961**
26	**1039**
27	**1119**
28	**1203**
29	**1289**
30	**1379**
31	**1471**
32	**1567**
33	**1665**
34	**1767**
35	**1871**
36	**1979**
37	**2089**
38	**2203**
39	**2319**
40	**2439**
41	**2561**
42	**2687**
43	**2815**
44	**2947**
45	**3081**
46	**3219**
47	**3359**
48	**3503**
49	**3649**
50	**3799**
51	**3951**

Nota Bene. The upper bounds for $n = 3$, $n = 4$ and $n = 5$ are outside the box solutions: they are based on a combination of two-dimensional patterns as shown in Figure 5.

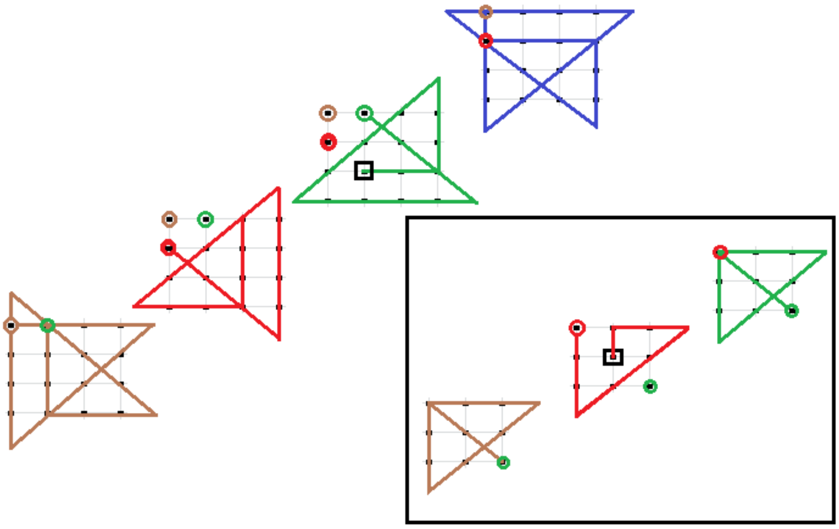

Figure 5. The outside the box solutions for $n = 3$ and $n = 4$
(14 and 25 lines respectively).

4. Conclusion

The research of the best solution to this problem could be the subject of another paper, following the same approach as above and trying to switch from the two main patterns more than once.

Simultaneously, we can improve the upper limit for the k-dimensions n X n X ... X n dots problem ($k > 3$) by simply defining $t := \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n - 1$.

Therefore, the current bounds are (Figure 6):

$$\left\lceil \frac{n^k + \left(\frac{k}{2} - 1\right) \cdot n^2 + (3 - 2 \cdot k) \cdot n + 2 \cdot k - 4}{n - 1} \right\rceil + 1 \leq h(n) \leq (t + 1) \cdot n^{k-3} - 1$$

$$\Rightarrow \left\lceil \frac{n^k + \frac{k}{2} \cdot (n-2)^2 - n^2 + 3 \cdot n - 4}{n - 1} \right\rceil + 1 \leq h(n) \leq \left(\left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n \right) \cdot n^{k-3} - 1 \qquad (8)$$

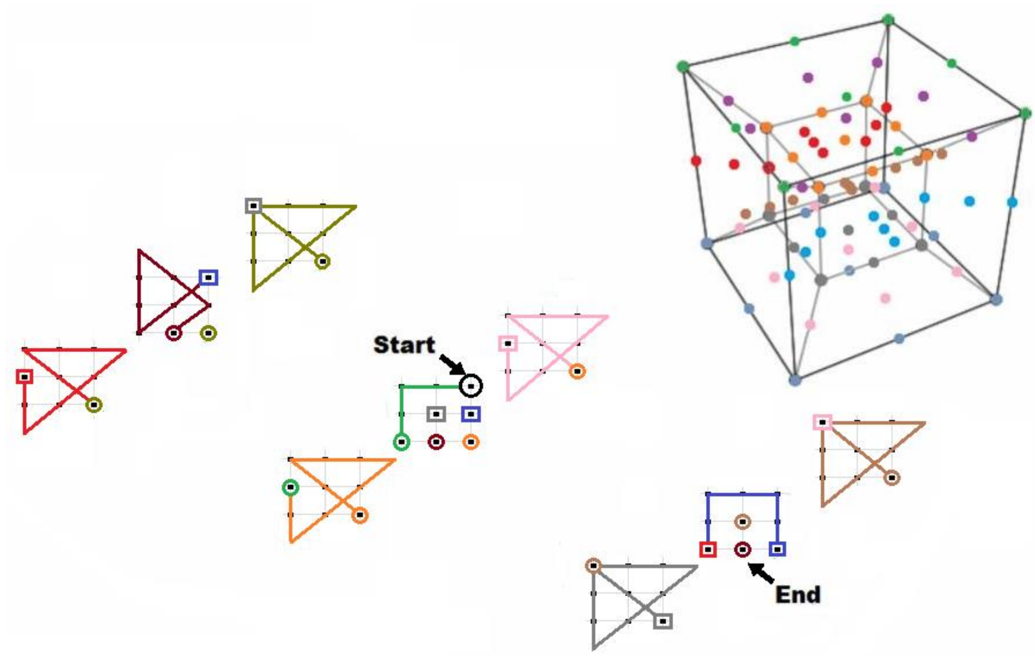

Figure 6. An extension of the puzzle in a four-dimensional space
($k = 4, n = 3; h_l = 41, h_u = 42$).

For $n > 5$, the new value of t is pretty much more accurate than the previous one (see [5]).

With specific regard to the $n \, \mathrm{X} \, n \, \mathrm{X} \, n$ dots problem, we have the following difference between the current upper bound and the lower one ($n \geq 5$):

$$h_u(n) - h_l(n) = \left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n - 1 - n^2 - \left\lceil \frac{3 \cdot n^2 - 4 \cdot n + 2}{2 \cdot (n-1)} \right\rceil \leq \left\lfloor \frac{\boldsymbol{n^2 \cdot (n-2)}}{\boldsymbol{2 \cdot (n-1)}} \right\rfloor$$

Thus,

$$h_u(n) - h_l(n) \leq \frac{\boldsymbol{n \cdot (n+3)}}{\boldsymbol{2}} \tag{9}$$

$h_u(n) - h_l(n) = \{\emptyset, 0, 0, 3\}$, for $n = \{1, 2, 3, 4\}$, while, for $n \geq 5$, the gaps are (at most) equal to the $a(n)$ belonging to the sequence A000096 of the OEIS [10].

Finally, using the method decribed in this paper, for any $n \geq 5$, it is possible to solve the puzzle inside the box without crossing two or more lines, with only $\left\lfloor \frac{3}{2} \cdot n^2 \right\rfloor + n - 1$ line segments connected at their end-points. In fact, we can apply the square spiral starting from an external grid, jumping to the next one after n lines and so on, with $n - 1$ connection lines of unitary length (let the distance between two adjacent points be a unit) for the square spiral frame. At this point we will spend the $(n^2 + n)$-th line to join the *central dots* from the opposit external grid (on the other side of the box) in order to finish with the classic "rectangular" pattern based on $\left(n - \frac{q}{2}\right)^2 - 1$ more connection lines of unitary length.

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