A new result about prime numbers: $\lim_{n\to+\infty} \frac{n}{p_n - n(ln\ \ n + ln\ \ ln\ \ n - 1)} = +\infty$

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Abstract

In this short paper we propose a new result about prime numbers:

$$\lim_{n\to+\infty} \frac{n}{p_n - n(ln\ n + ln\ ln\ n - 1)} = +\infty$$

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Keywords: Prime numbers, Dusart

Preliminaries.

We write $ln_2\ n$ instead of $ln\ ln\ n$.

Let p_n denote the n^{th} prime number.

In 1999 [1] Pierre Dusart showed that :

$$n(ln\ n + ln_2\ n - 1) < p_n < n(ln\ n + ln_2\ n - 0.9484) \quad \text{for n } \geq 39017$$

.

In [2] and [3] it is also proved that:

$$p_n \leq n(ln\ n + ln_2\ n - 1 + \frac{ln_2\ n - 2}{ln\ n}) \quad \text{for n } \geq 688383$$

Theorem.

$$\lim_{n \to +\infty} \frac{n}{p_n - n(ln\ n + ln_2\ n - 1)} = +\infty$$

Proof. Remember that (in [2]):

$$p_n \leq n(ln\ n + ln_2\ n - 1 + \frac{ln_2\ n - 2}{ln\ n}) \quad \text{for n } \geq 688383$$

We deduce:

$$n(ln\ n + ln_2\ n - 1 + \frac{ln_2\ n - 2}{ln\ n}) - n(ln\ n + ln_2\ n - 1) \geq p_n - n(ln\ n + ln_2\ n - 1) \quad \text{for n } \geq 688383$$

Hence:

$$\frac{n}{n(ln\ n + ln_2\ n - 1 + \frac{ln_2\ n - 2}{ln\ n}) - n(ln\ n + ln_2\ n - 1)} \leq \frac{n}{p_n - n(ln\ n + ln_2\ n - 1)} \quad \text{for n } \geq 688383$$

We have:

$$\frac{n}{n(ln\ n + ln_2\ n - 1 + \frac{ln_2\ n - 2}{ln\ n}) - n(ln\ n + ln_2\ n - 1)} = \frac{n}{\frac{n\ ln_2\ n - 2}{ln\ n})}$$

$$\frac{n}{\frac{n\ ln_2\ n - 2}{ln\ n})} = \frac{ln\ n}{ln_2\ n - 2}$$

And:

$$\lim_{n\to+\infty} \frac{ln\ n}{ln_2\ n-2} = +\infty$$

Because:

$$\frac{n}{n(ln\ n+ln_2\ n-1+\frac{ln_2\ n-2}{ln\ n})-n(ln\ n+ln_2\ n-1)} \leq \frac{n}{p_n-n(ln\ n+ln_2\ n-1)} \quad \text{for n } \geq 688383$$

We deduce:

$$\lim_{n\to+\infty} \frac{n}{p_n-n(ln\ n+ln_2\ n-1)} = +\infty$$

Consequence.

Let ϵ be a function defined by $\epsilon(n)$, n is a natural number and $n \geq 688383$

If:

$$\frac{n}{p_n-(n(ln\ n+ln_2\ n-1))} \geq \epsilon(n)$$

We have:

$$\frac{p_n-(n(ln\ n+ln_2\ n-1))}{n} \leq \frac{1}{\epsilon(n)}$$

And:

$$p_n-(n(ln\ n+ln_2\ n-1)) \leq \frac{n}{\epsilon(n)}$$

$$p_n-(n(ln\ n+ln_2\ n-1)) \leq \frac{n}{\epsilon(n)} = p_n \leq (n(ln\ n+ln_2\ n-1)) + \frac{n}{\epsilon(n)}$$

$$p_n \leq (n(ln\ n+ln_2\ n-1)) + \frac{n}{\epsilon(n)} = p_n \leq (n(ln\ n+ln_2\ n-1+\frac{1}{\epsilon(n)}))$$

Because:

$$\lim_{n\to+\infty} \frac{n}{p_n - n(ln\ n + ln_2\ n - 1)} = +\infty$$

The n^{th} prime number is smaller than $n(ln\ n + ln_2\ n - 1 + \frac{1}{\epsilon(n)})$ $\lim_{n\to+\infty} \epsilon(n) = +\infty$

Example. The function in [2] and previously described. We have $\epsilon(n) = \frac{ln\ n}{ln_2\ n-2}$ because $\frac{n}{p_n-(n(ln\ n+ln_2\ n-1))} \geq \frac{ln\ n}{ln_2\ n-2}$

Let ϕ be a function defined by $\phi(n)$, n is a natural number and $n \geq 688383$

If:

$$p_n \leq \phi(n)) \quad \text{for n } \geq 688383$$

We have:

$$\frac{n}{p_n - (n(ln\ n + ln_2\ n - 1))} \geq \frac{n}{\phi(n) - (n(ln\ n + ln_2\ n - 1))}$$

And:

$$\epsilon(n) = \frac{n}{\phi(n) - (n(ln\ n + ln_2\ n - 1))}$$

Consequently:

$$p_n \leq n(ln\, n + ln_2\, n - 1 + \frac{\phi(n) - (n(ln\ n + ln_2\ n - 1))}{n}) \qquad \lim_{n\to+\infty} \frac{\phi(n) - (n(ln\ n + ln_2\ n - 1))}{n} = 0$$

References

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