Ramanujan Trigonometric Formula

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abstract

This note presents formulas and fractals related with Ramanujan's trigonometric formula.

1. Introduction

Ramanujan's trigonometric formula:

$$\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{5-3\sqrt[3]{7}}{2}}=-\sqrt[3]{\frac{3\sqrt[3]{7}-5}{2}} \tag{1}$$

2. Related formulas

$$\sqrt[3]{\cos\frac{\pi}{7}}-\sqrt[3]{\cos\frac{2\pi}{7}}-\sqrt[3]{\cos\frac{4\pi}{7}}=\sqrt[3]{\frac{3\sqrt[3]{7}-5}{2}} \tag{2}$$

$$\sqrt[3]{\cos\frac{\pi}{7}}+\sqrt[6]{\frac{1}{2}-\frac{1}{2}\cos\frac{\pi}{7}}-\sqrt[6]{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}-\frac{1}{2}\cos\frac{\pi}{7}}}=\sqrt[3]{\frac{3\sqrt[3]{7}-5}{2}} \tag{3}$$

$$\sqrt[3]{\sin\frac{\pi}{14}}-\sqrt[3]{\sin\frac{3\pi}{14}}+\sqrt[3]{\sin\frac{5\pi}{14}}=\sqrt[3]{\frac{3\sqrt[3]{7}-5}{2}} \tag{4}$$

$$\frac{7\left(3\sqrt[3]{7}-5\right)}{\pi}=\left\{\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{14n}\right)^2\right)^{1/3}-\sqrt[3]{3}\prod_{n=1}^{\infty}\left(1-\left(\frac{3}{14n}\right)^2\right)^{1/3}+\sqrt[3]{5}\prod_{n=1}^{\infty}\left(1-\left(\frac{5}{14n}\right)^2\right)^{1/3}\right\}^3 \tag{5}$$

$$\frac{\pi\left(6\sqrt[3]{7}-8\right)}{14}=$$

$$=\left\{\prod_{n=1}^{\infty}\left(\frac{(14n)^2}{(14n)^2-1}\right)^{1/3}-\frac{1}{\sqrt[3]{3}}\prod_{n=1}^{\infty}\left(\frac{(14n)^2}{(14n)^2-9}\right)^{1/3}+\frac{1}{\sqrt[3]{5}}\prod_{n=1}^{\infty}\left(\frac{(14n)^2}{(14n)^2-25}\right)^{1/3}\right\}^3 \tag{6}$$

$$\cos\frac{2\pi}{7}=-\frac{1}{6}+\frac{1}{6}\sqrt[3]{7+21\sqrt[3]{7+21\sqrt[3]{7+\ldots}}} \tag{7}$$

$$\cos\frac{4\pi}{7}=-\frac{1}{6}-\frac{1}{2}\left(\frac{1}{9}+\frac{3}{7}\left(\frac{1}{9}+\frac{3}{7}\left(\frac{1}{9}+\ldots\right)^3\right)^3\right)^3 \tag{8}$$

$$\left(\cos\frac{4\pi}{7}\right)^{-1}=-\frac{4}{3}-\frac{1}{3}\sqrt[3]{56+84\sqrt[3]{56+84\sqrt[3]{56+\ldots}}} \tag{9}$$

$$\left(\cos\frac{8\pi}{7}\right)^{-1}=-\frac{4}{3}+\frac{1}{3}\left(\frac{2}{3}+\frac{1}{84}\left(\frac{2}{3}+\frac{1}{84}\left(\frac{2}{3}+\ldots\right)^3\right)^3\right)^3 \tag{10}$$

$$\cos\frac{\pi}{7}=\frac{1}{6}\sqrt{15+3\sqrt[3]{7+21\sqrt[3]{7+21\sqrt[3]{7+\ldots}}}} \tag{11}$$

$$\begin{cases} 2\cos\frac{\pi}{7}=\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2+\ldots}}}}}}} \\ x_{n+1}=\sqrt{2+\sqrt{2-\sqrt{2-x_n}}} \quad ,x_1=0\Rightarrow x_n\to 2\cos\frac{\pi}{7} \end{cases} \tag{12}$$

$$\pi=4\tan^{-1}\left(\cos\frac{2\pi}{7}\right)+4\tan^{-1}\left(\left(\tan\frac{\pi}{7}\right)^2\right) \tag{13}$$

$$\pi=4\tan^{-1}\left(\cos\frac{4\pi}{7}\right)+4\tan^{-1}\left(\left(\tan\frac{2\pi}{7}\right)^2\right) \tag{14}$$

$$\pi=4\tan^{-1}\left(\cos\frac{8\pi}{7}\right)+4\tan^{-1}\left(\left(\tan\frac{4\pi}{7}\right)^2\right) \tag{15}$$

$$\pi=2\sin^{-1}\sqrt{\left|2\cos\frac{4\pi}{7}\right|}+2\sin^{-1}\sqrt{\frac{1}{2}\sec\frac{\pi}{7}} \tag{16}$$

$$\sqrt[3]{\frac{7\left(3\sqrt[3]{7}-5\right)}{\pi}}=\prod_{n=1}^{\infty}e^{-\left(\zeta(2n)/3n\right)(1/14)^{2n}}-\sqrt[3]{3}\prod_{n=1}^{\infty}e^{-\left(\zeta(2n)/3n\right)(3/14)^{2n}}+\sqrt[3]{5}\prod_{n=1}^{\infty}e^{-\left(\zeta(2n)/3n\right)(5/14)^{2n}} \tag{17}$$

Remark. $\zeta\left(2n\right)=\sum_{k=1}^{\infty}k^{-2n} \quad ,n\in\mathbb{N}$,Riemann zeta function.

3. Related polynomials and fractals

Let

$$p(z) = 4z^9 + 30z^6 + 75z^3 - 32 \tag{18}$$

we get

$$p\left(\sqrt[3]{\frac{3\sqrt[3]{7}-5}{2}}\right) = 0 \tag{19}$$

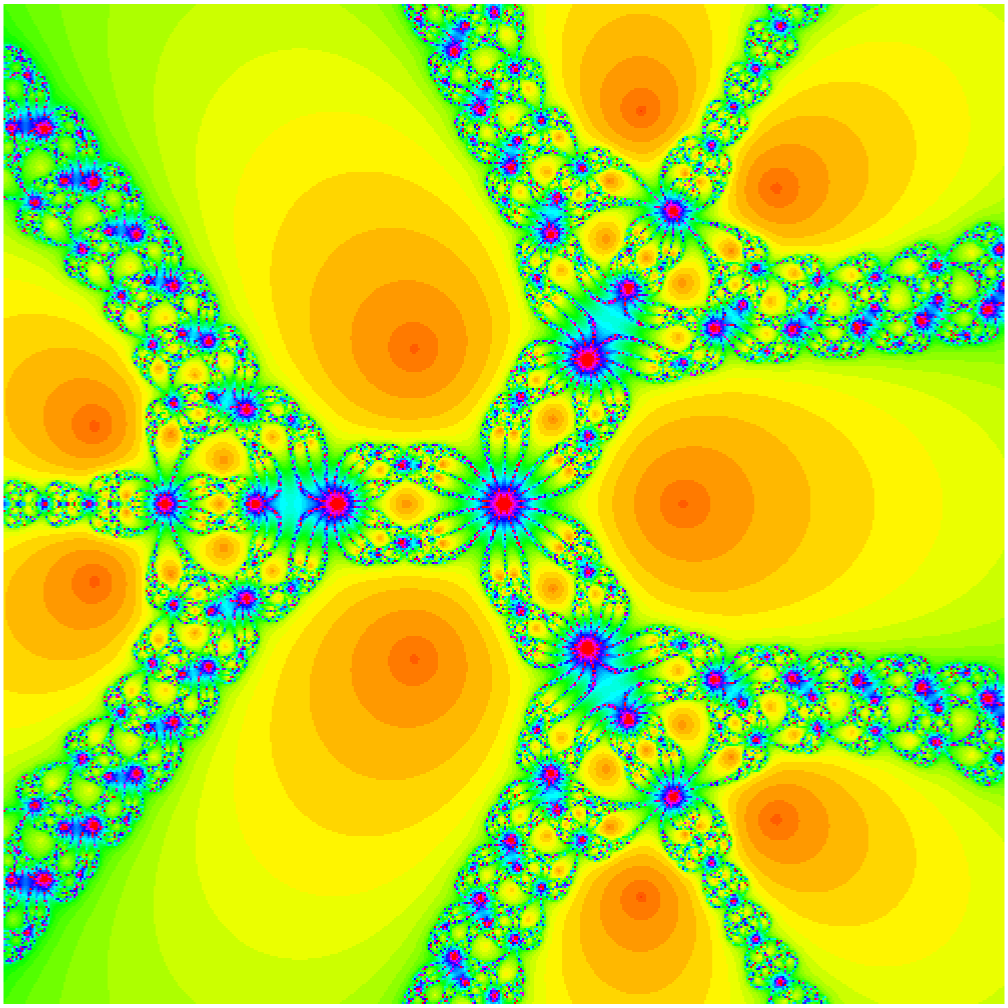

Fig. 1. Newton-Julia set for: $p(z)$.

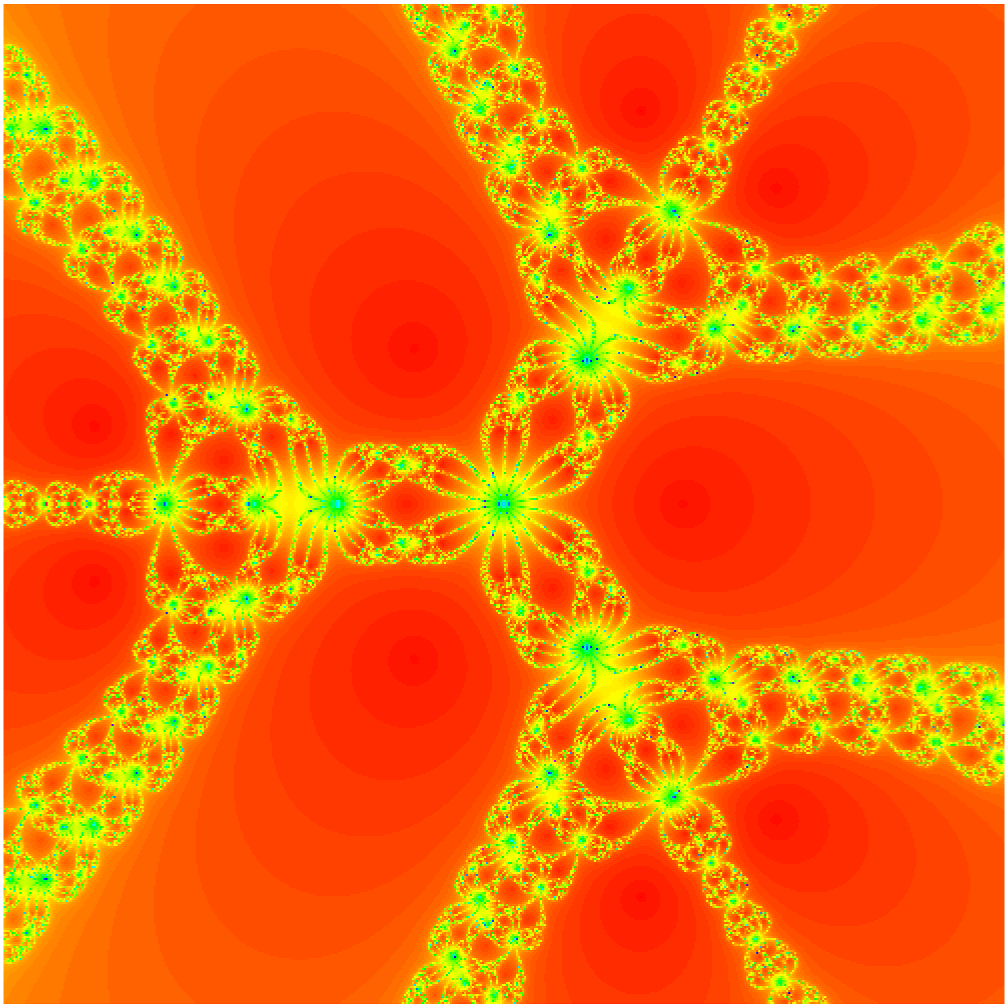

Fig. 2. Newton-Julia set for: $p(z)$.

Let

$$q(z) = z^3 + \frac{1}{2}z^2 - \frac{1}{2}z - \frac{1}{8} \tag{20}$$

we get

$$q\left(\cos\frac{2\pi}{7}\right) = q\left(\cos\frac{4\pi}{7}\right) = q\left(\cos\frac{8\pi}{7}\right) = 0 \tag{21}$$

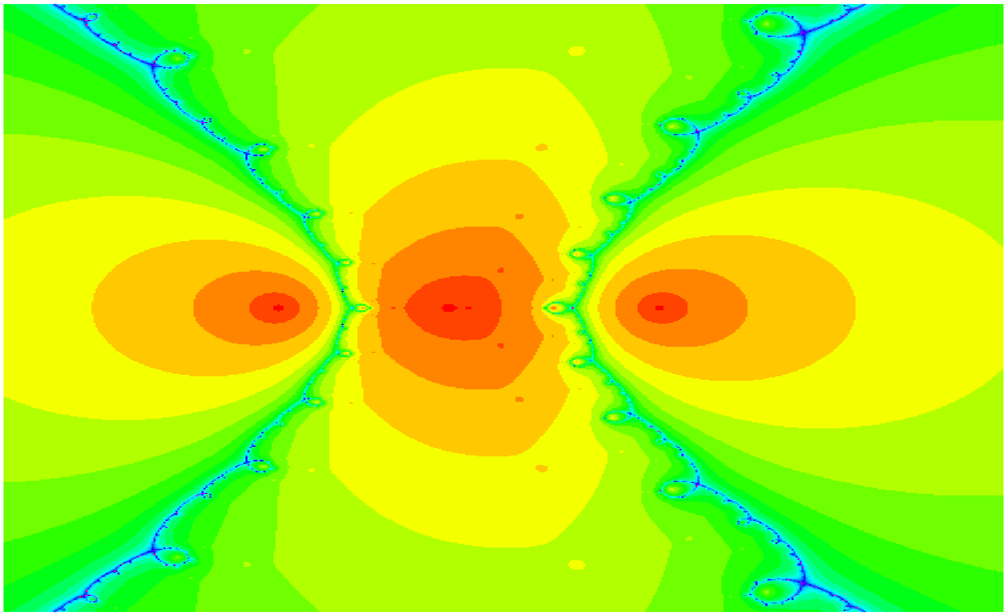

Fig. 3. Newton-Julia set for: $q(z)$.

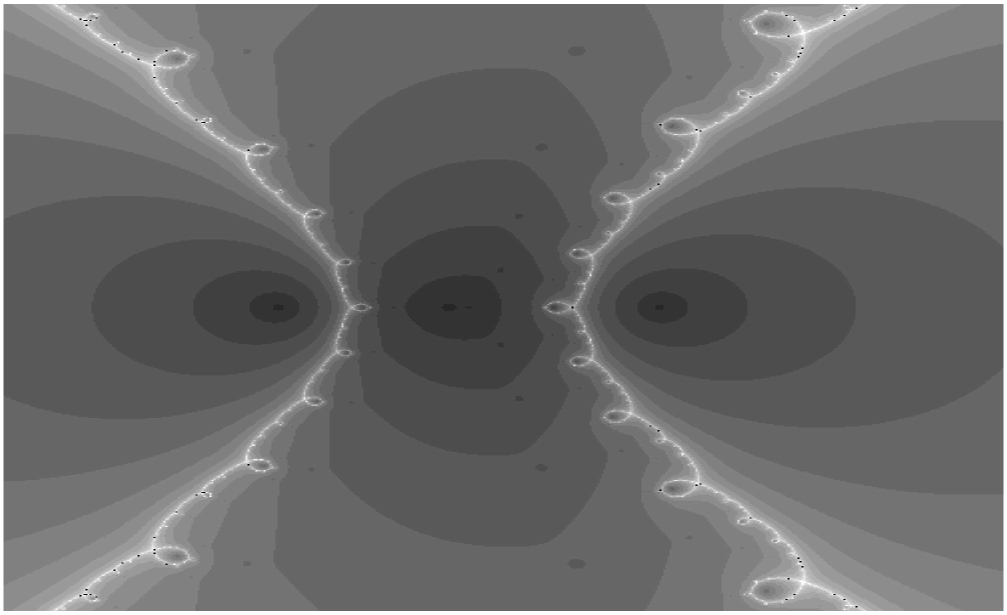

Fig. 4. Newton-Julia set for: $q(z)$.

4. Related functions,sequences and fractals

Let

$$f(z)=\sqrt{\frac{1+2z}{1+z}} \tag{22}$$

we get

$$z_{n+1}=f(z_n) \quad ,z_1=0 \Rightarrow z_n \to 2\cos\frac{2\pi}{7} \tag{23}$$

$$2\cos\frac{2\pi}{7}=\sqrt{\frac{1+2\sqrt{\frac{1+2\sqrt{\frac{1+2...}{1+1...}}}{1+1\sqrt{\frac{1+2...}{1+1...}}}}}{1+1\sqrt{\frac{1+2\sqrt{\frac{1+2...}{1+1...}}}{1+1\sqrt{\frac{1+2...}{1+1...}}}}}} \tag{24}$$

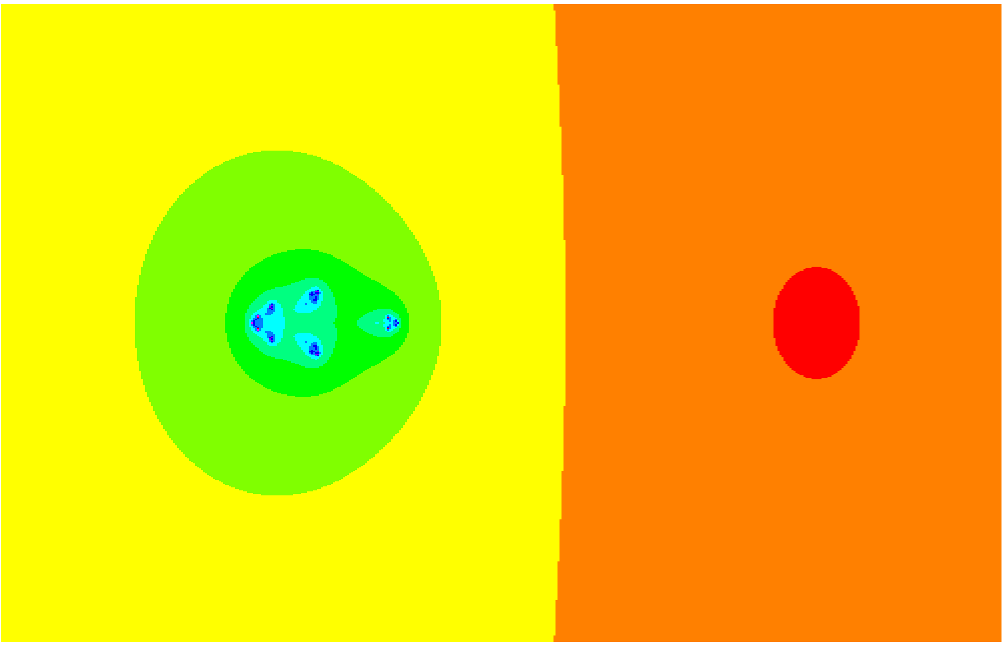

Fig. 5. Newton-Julia set for: $z-f(z)$.

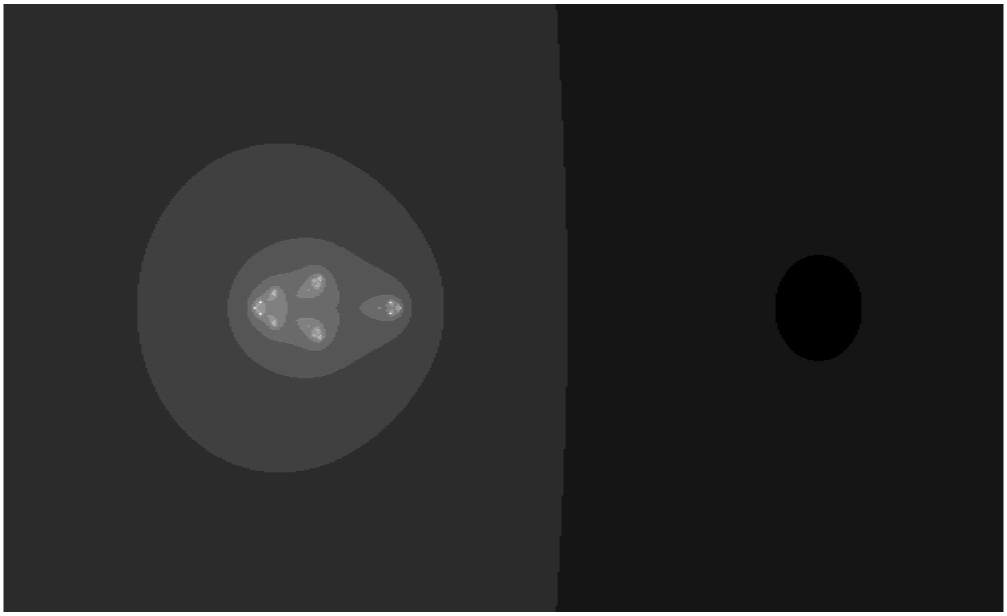

Fig. 6. Newton-Julia set for: $z - f(z)$.

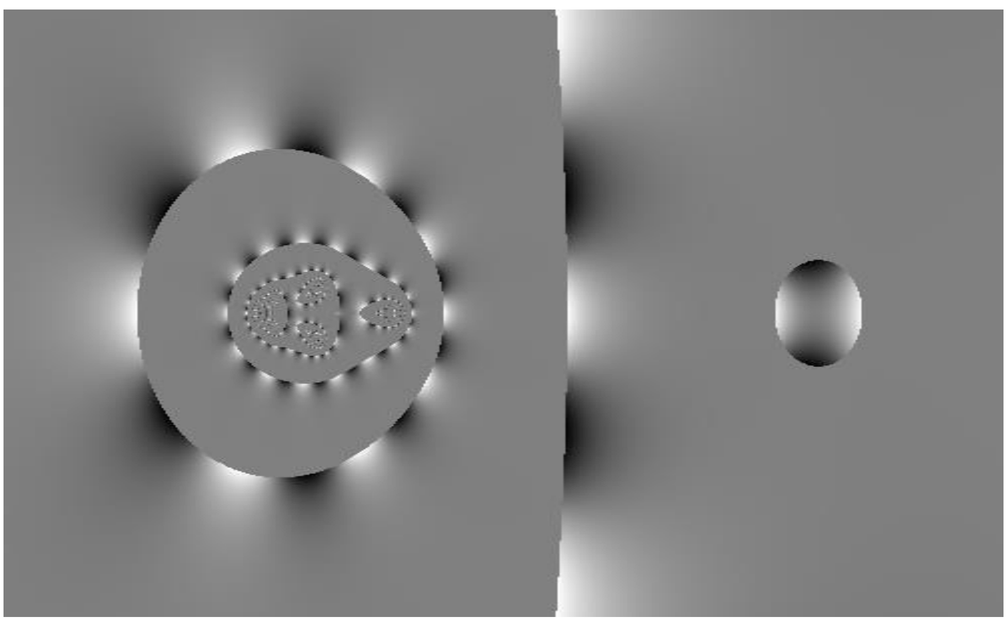

Fig. 7. Newton-Julia set for: $z - f(z)$.

Let

$$g(z)=\sqrt{\frac{8+4z}{4+z}} \tag{25}$$

we get

$$z_{n+1}=g(z_n) \quad ,z_1=0 \Rightarrow z_n \to \left(\cos\frac{2\pi}{7}\right)^{-1} \tag{26}$$

$$\left(\cos\frac{2\pi}{7}\right)^{-1}=\sqrt{\frac{8+4\sqrt{\frac{8+4\sqrt{\frac{8+4...}{4+1...}}}{4+1\sqrt{\frac{8+4...}{4+1...}}}}}{4+1\sqrt{\frac{8+4\sqrt{\frac{8+4...}{4+1...}}}{4+1\sqrt{\frac{8+4...}{4+1...}}}}}} \tag{27}$$

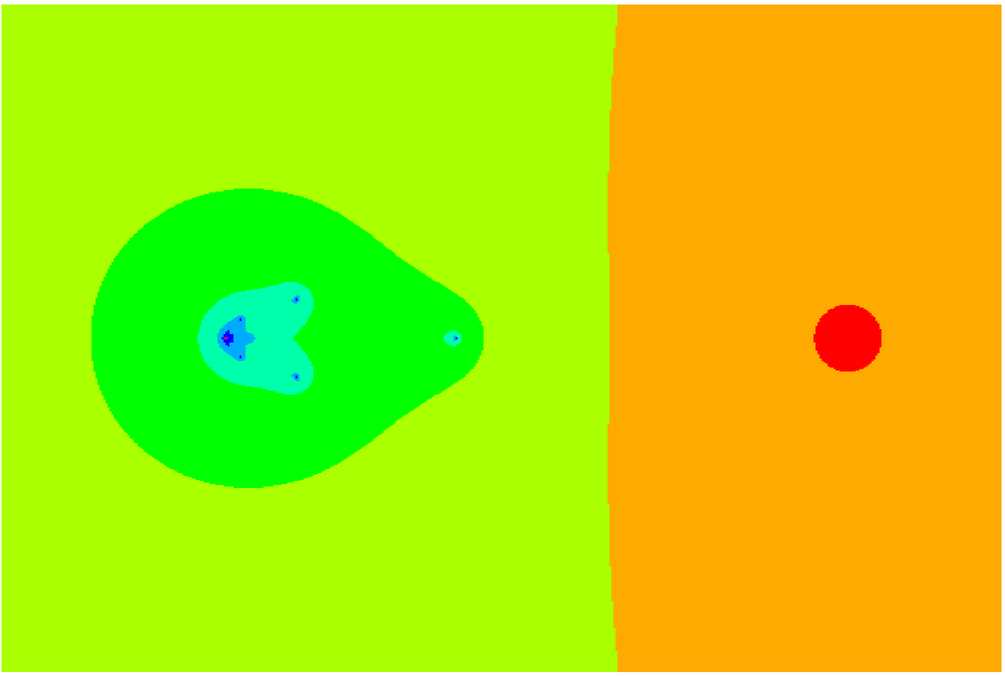

Fig. 8. Newton-Julia set for: $z-g(z)$.

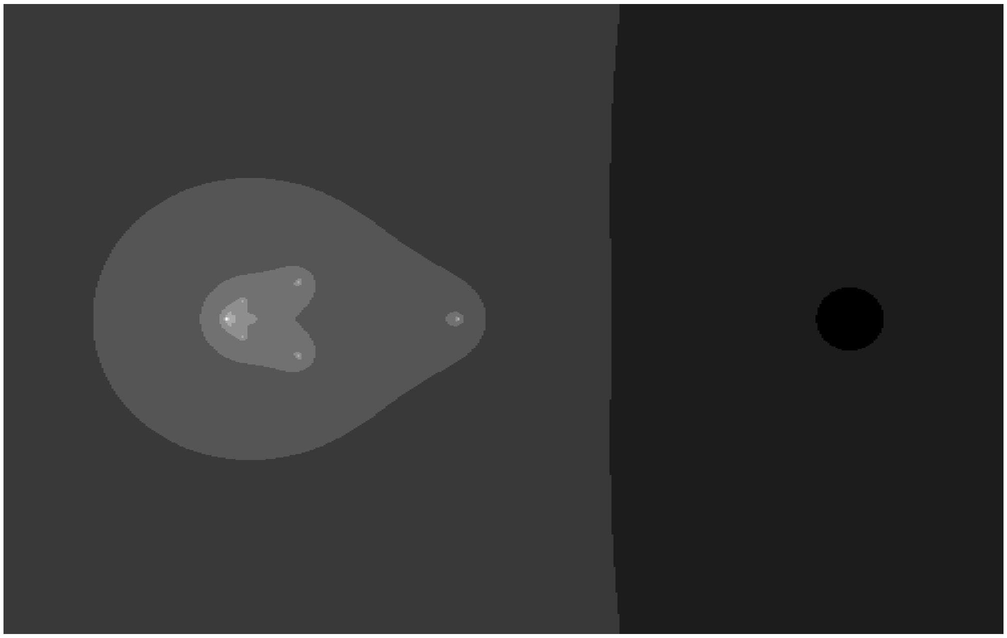

Fig. 9. Newton-Julia set for: $z - g(z)$.

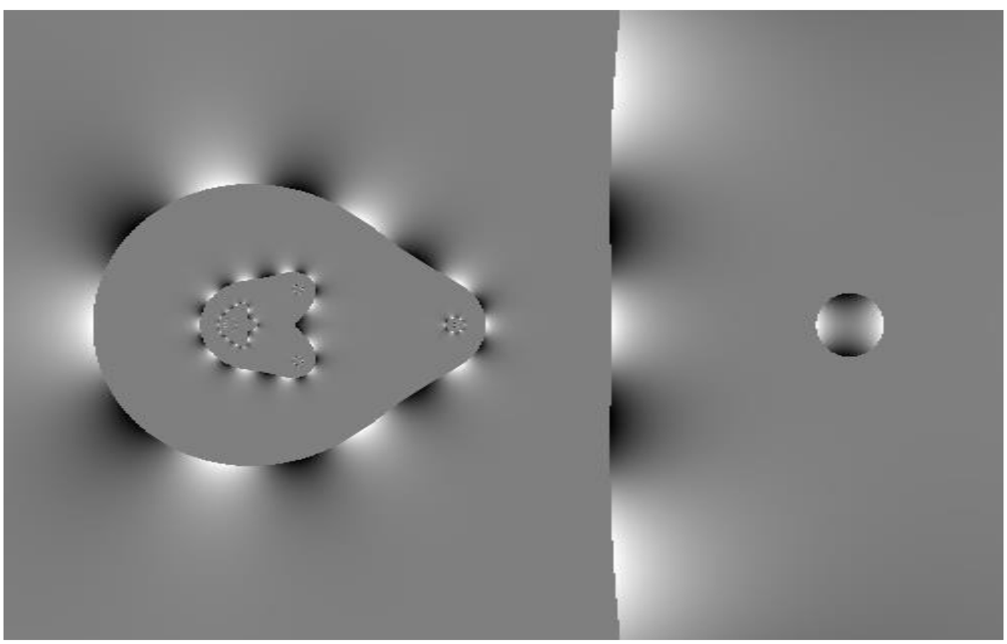

Fig. 10. Newton-Julia set for: $z - g(z)$.

Let

$$z_{n+1} = -1 + \sqrt{9 + \frac{8}{z_n}} \quad , z_1 = 0 \tag{28}$$

we get

$$z_n = \{0, \infty, 2, 2.605..., 2.474..., 2.497..., 2.493...., 2.494..., 2.493..., 2.493..., ...\} \tag{29}$$

$$z_n \to 4\cos\frac{2\pi}{7} \tag{30}$$

$$4\cos\frac{2\pi}{7} = -1 + \sqrt{9 - \frac{8}{1 - \sqrt{9 - \frac{8}{1 - \sqrt{9 - \frac{8}{1 - \sqrt{9 - ...}}}}}}} \tag{31}$$

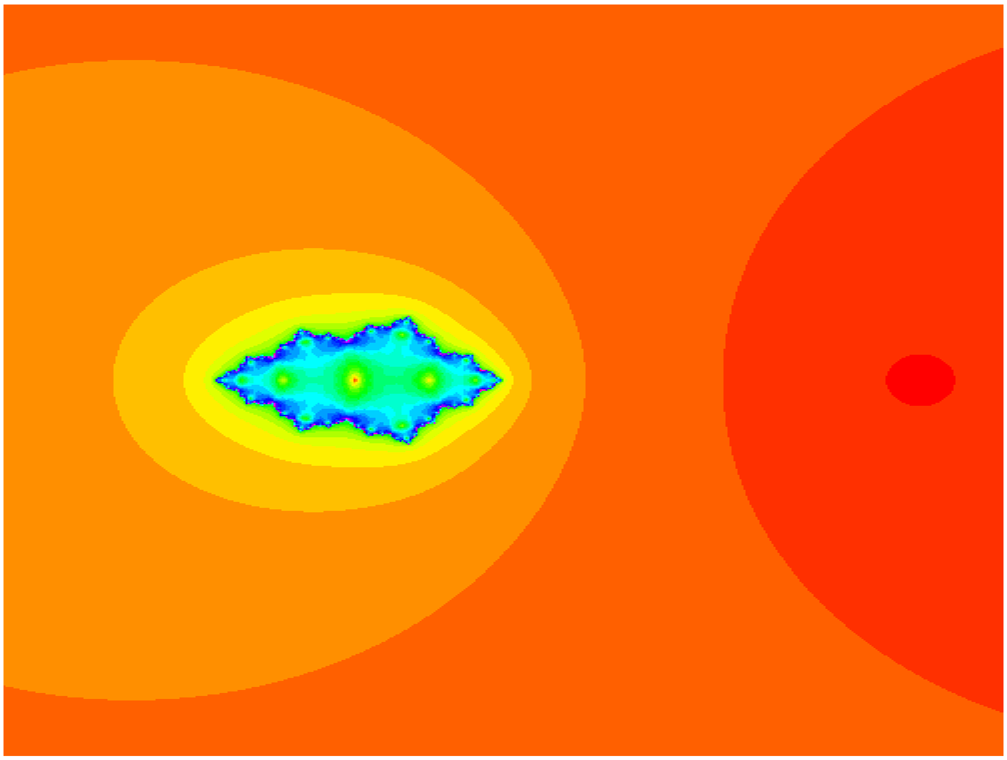

Fig. 11. Newton-Julia set for: $z + 1 - \sqrt{9 + (8/z)}$.

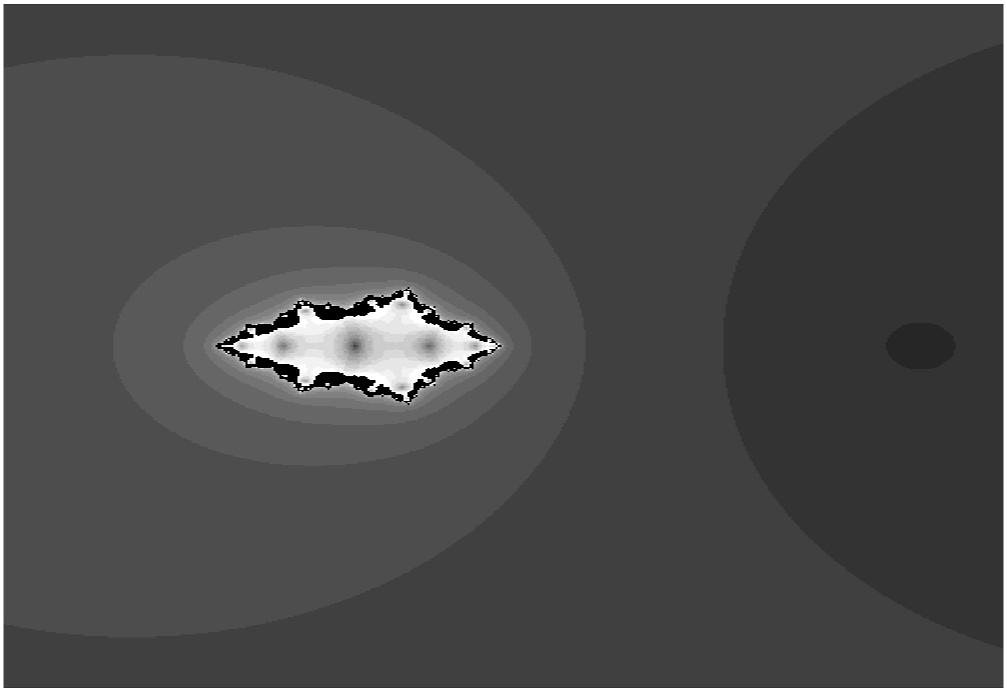

Fig. 12. Newton-Julia set for: $z+1-\sqrt{9+(8/z)}$.

Let

$$z_{n+1}=-1-\sqrt{9+\frac{8}{z_n}} \quad ,z_1=0 \tag{32}$$

we get

$$z_n=\{0,-\infty,-4,-3.645...,-3.608...,-3.604...,-3.603...,-3.603...,...\} \tag{33}$$

$$z_n \to 4\cos\frac{8\pi}{7} \tag{34}$$

$$4\cos\frac{8\pi}{7}=-1-\sqrt{9-\cfrac{8}{1+\sqrt{9-\cfrac{8}{1+\sqrt{9-\cfrac{8}{1+\sqrt{9-...}}}}}}} \tag{35}$$

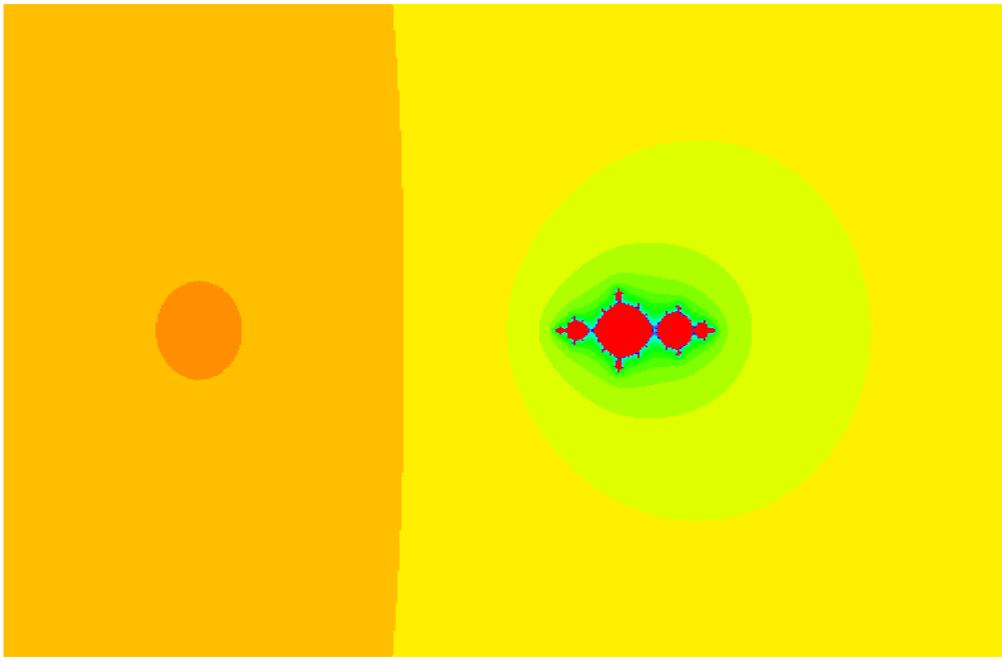

Fig. 13. Newton-Julia set for: $z+1+\sqrt{9+(8/z)}$.

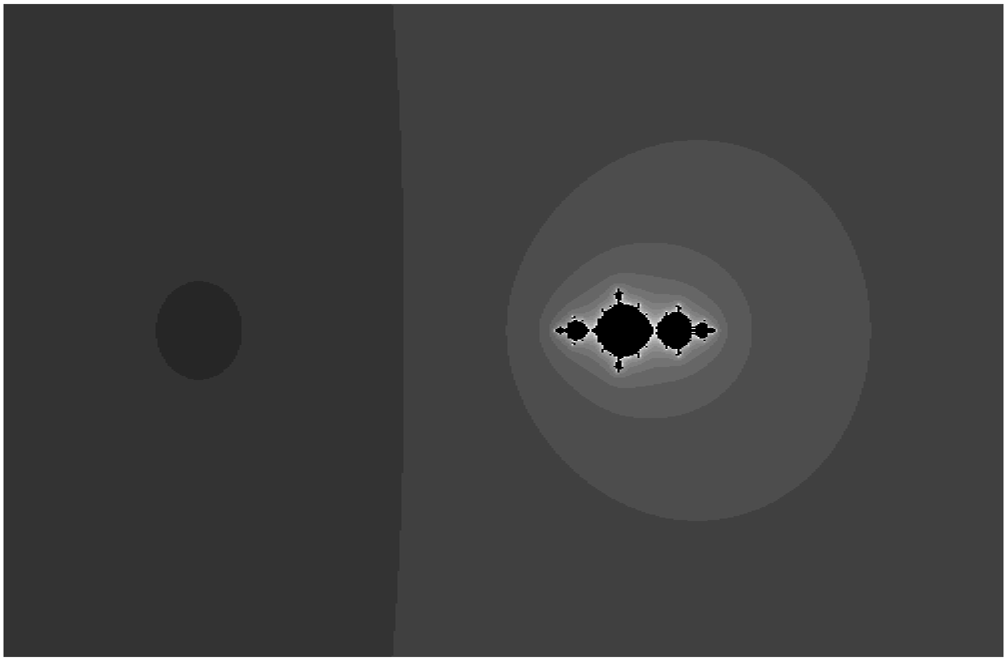

Fig. 14. Newton-Julia set for: $z+1+\sqrt{9+(8/z)}$.

Let

$$z_{n+1} = 1 + \sqrt{9 - \frac{8}{z_n}} \quad , z_1 = 0 \tag{36}$$

we get

$$z_n = \{0, 1 + i\infty, 4, 3.645..., 3.608..., 3.604..., 3.603..., 3.603..., ...\} \tag{37}$$

$$z_n \to 4\cos\frac{\pi}{7} \tag{38}$$

$$4\cos\frac{\pi}{7} = 1 + \sqrt{9 - \frac{8}{1 + \sqrt{9 - \frac{8}{1 + \sqrt{9 - \frac{8}{1 + \sqrt{9 - ...}}}}}}} \tag{39}$$

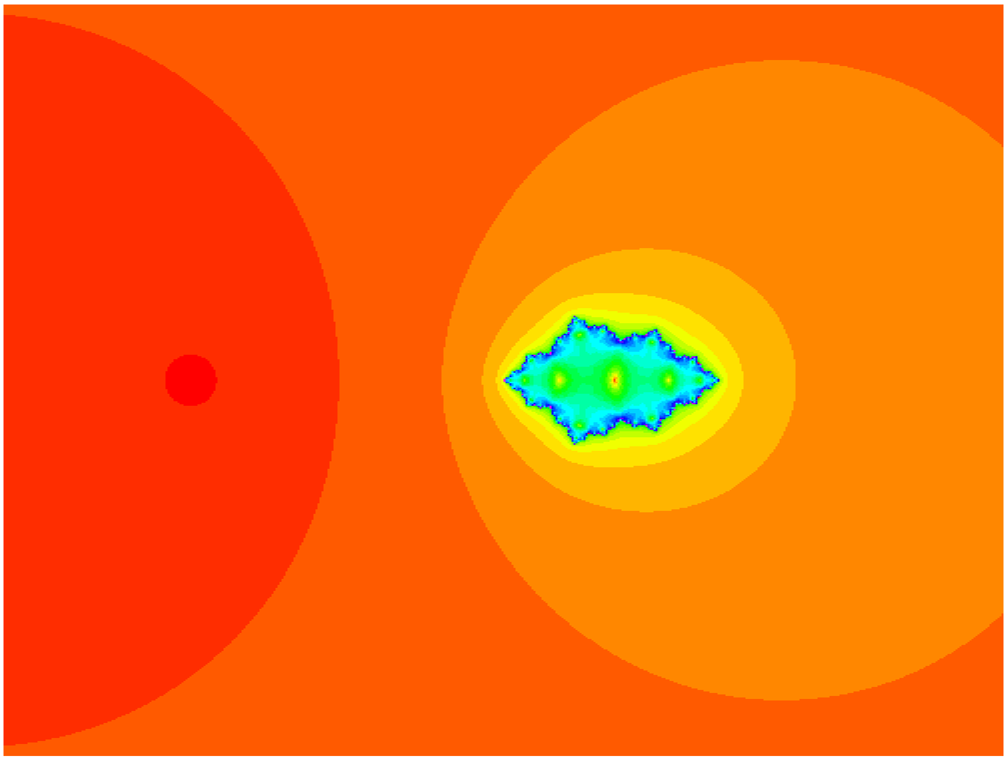

Fig. 15. Newton-Julia set for: $z - 1 - \sqrt{9 - (8/z)}$.

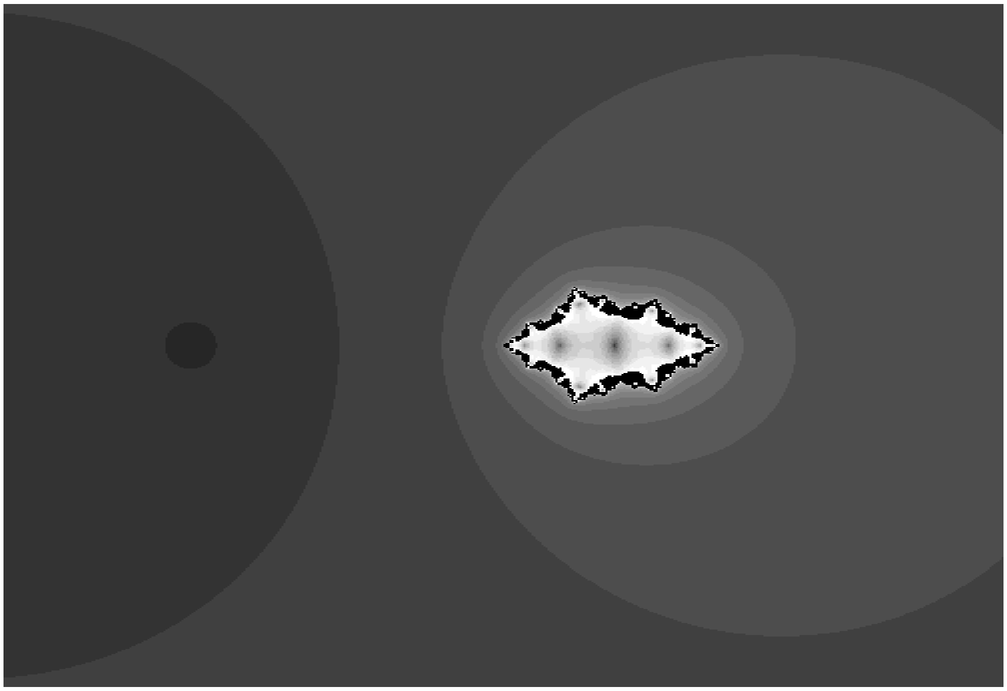

Fig. 16. Newton-Julia set for: $z - 1 - \sqrt{9 - (8/z)}$.

Let

$$z_{n+1} = 1 - \sqrt{9 - \frac{8}{z_n}} \quad , z_1 = 0 \tag{40}$$

we get

$$z_n = \{0, 1 - i\infty, -2, -2.605..., -2.474..., -2.497..., -2.493..., -2.944..., -2.493..., ...\} \tag{41}$$

$$z_n \to -4\cos\frac{2\pi}{7} \tag{42}$$

$$-4\cos\frac{2\pi}{7} = 1 - \sqrt{9 - \cfrac{8}{1 - \sqrt{9 - \cfrac{8}{1 - \sqrt{9 - \cfrac{8}{1 - \sqrt{9 - ...}}}}}}} \tag{43}$$

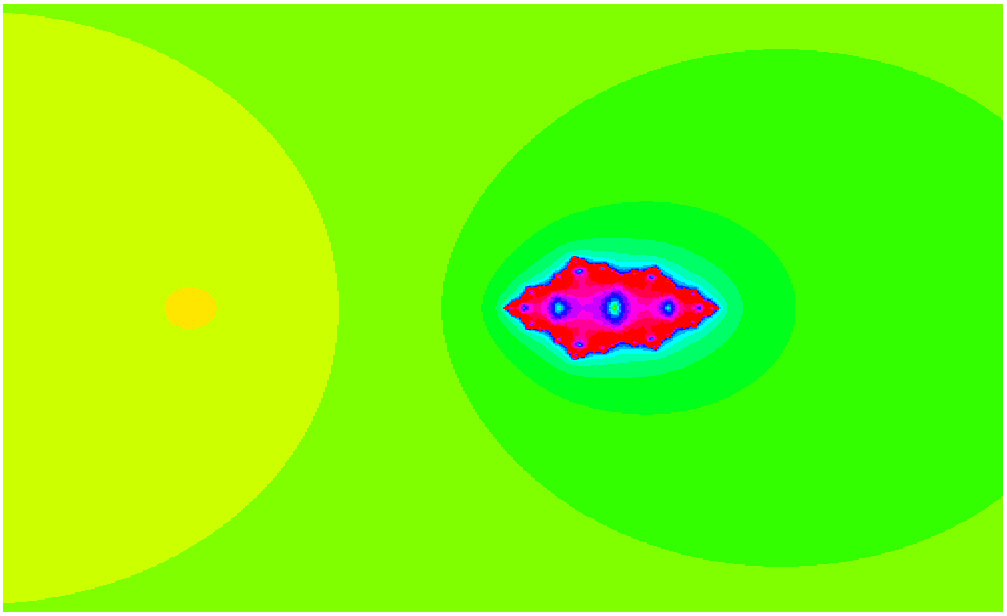

Fig. 17. Newton-Julia set for: $z - 1 + \sqrt{9 - (8/z)}$.

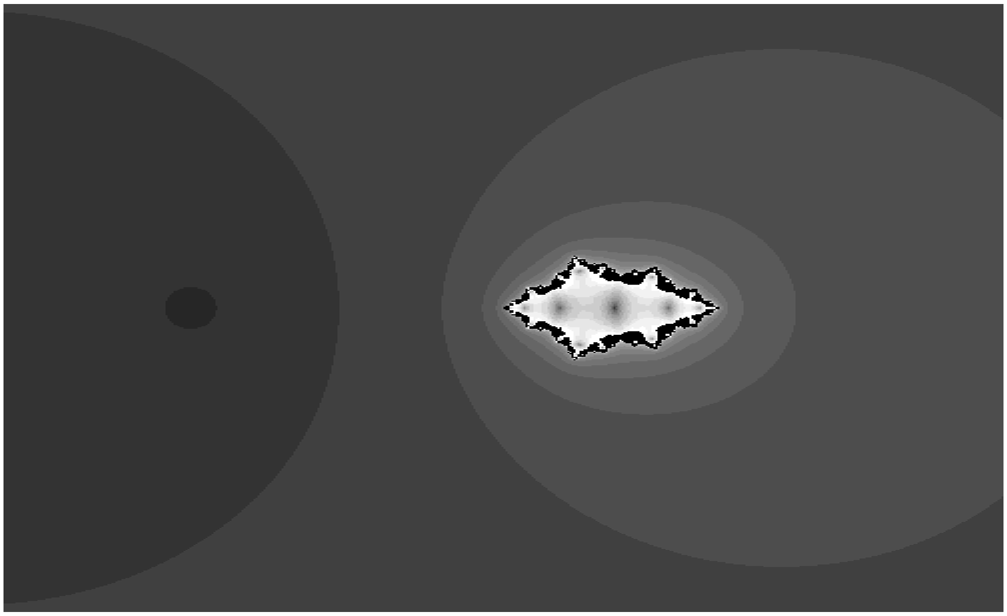

Fig. 18. Newton-Julia set for: $z - 1 + \sqrt{9 - (8/z)}$.

5. Final formulas

$$\pi = 2\sin^{-1}\left(\sqrt{\cos\frac{2\pi}{7}}\right) + 2\sin^{-1}\left(\sqrt{2}\sin\frac{\pi}{7}\right) \quad (44)$$

$$\pi = 2\sin^{-1}\left(\sqrt{2}\cos\frac{2\pi}{7}\right) + 2\sin^{-1}\left(\sqrt{-\cos\frac{4\pi}{7}}\right) \quad (45)$$

References

1. Witula, R.: Ramanujan Type Trigonometric Formulas: The General Form for the Argument $2\pi/7$. Journal of Integer Sequences,Vol.12 (2009),Article 09.8.5 .