Upper bound of Prime gap - Legendre's conjecture was verified (I)

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Abstract

We have found the possible max- difference between two successive prime numbers, and by them, Legendre's conjecture is verified

A prime gap is the difference between two successive prime numbers.The n- th prime gap, denoted G_n is the difference between the (n+1)-th and the n- th prime numbers,i.e. [?]

$$Gn = P_{n+1} - P_n$$

1 The way for establishing Upper bound of Gn

1.1 The approach

A difference between two successive prime numbers is the length of sequence of composite numbers between them . To determine Upper bounds of Gn, we will determine the possible max-length of sequence of composite numbers next to P_n .
First, we consider a sequence of composite numbers below:
For a given natural number a!, there are two sequences of composite numbers before and after a! such as:
$a! - 2, a! - 3, a! - 4, a! - 5..., a! - a$ and $a! + 2, a! + 3, a! + 4, a! + 5.....a! + a$.
We have not determine the numbers: $a! \pm 1, a! \pm (a + 1)$, are composite or not. if they are composite numbers, they must a product of prime factors which are larger than a.
To connect two sequences above in such away that they become one sequence of composite numbers, $a! \pm 1$ must be a composite numbers. And we obtain a new sequence of composite numbers.
$a! - a, ..., a! - 5, a! - 4, a! - 3, a! - 2, a! - 1, a!, a! + 1, a! + 2, a! + 3, a! + 4, a! + 5, ..., a! + a$ (1).
In the sequence(1), if the numbers $a! \pm 1$, one is divisible by p_n, other one is divisible by p_{n+1}. (p_n and p_{n+1} are successive prime number), $p_n - 1 = a$, then the bounded numbers :$a! \pm (a + 1) = a! \pm p_n$ are not divisible by any numbers smaller than or equal to p_{n+1}.
So, we suppose that, the sequence (1) is the max-length of sequence of composite numbers, which have a divisor smaller than or equal to p_{n+1}.

1.2 The possible max -length of sequence of composite numbers between two successive prime numbers -prime gaps

If N is a composite with $P_n < N < P_{n+1}$, then N must have a divisor smaller than $\sqrt{P_{n+1}}$
Since the even number is composite number, so we consider only odd numbers, and its sequence, if the odd numbers of sequence are composite, they must have an odd divisor smaller than $\sqrt{P_{n+1}}$.
Assume p_{n+1} is the largest prime smaller than $\sqrt{P_{n+1}}$, p_n is the successive prime to p_{n+1}. We

arrange every odd divisors smaller and equal to $p_{(n+1)}$ next to P_n as follows (2):

P_n	N_1	N_2	...	...	...	$N_{(p_n-1)/2}$	$N_{(p_n+1)/2}$	...	...	...	$N_{(p_n-2)}$	$N_{(p_n-1)}$	N_{p_n}
Divisor	p_n-2	p_n-4	...	5	3	p_n	p_{n+1}	3	5	...	p_n-4	p_n-2	1

Above (2) is the possible max-length of sequence of composite numbers next to P_n.
For example: $p_{n+1}=17$, then $p_n=13$

P_n	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}	N_{13}
Dv	11	9	7	5	3	13(17)	17(13)	3	5	7	9	11	1

and the possible max-length = 26.

By the arrangement(2), N_p must be a prime, if N_p is a composite number, then it must have a odd divisor smaller or equal to p_{n+1}, this is impossible.
Difference between P_n and P_{n+1} is equal to $2p_n$. And this is such a possible max- Difference between P_n and P_{n+1} .
If this is not the possible max-Difference, then the number N_p is composite number.
If N_p is composite number, then it must have an odd divisor smaller or equal to p_{n+1}.
Hence, there will be other arrangement of divisors, and the length of sequence of composite numbers next to P_n will be shorter, so prime gap will be smaller than $2p_n$
In practice, there is only one case such that prime gap is equal $2p_n$ below:
$P_n=113, P_{n+1}=127, p_n=7, p_{n+1}=11$
And prime gap = 127 - 113 = 14 = $2p_n = 2.7$

Numbers	113	115	117	119	121	123	125	127
Divisor	1	5	3	7	11	3	5	1

For all other cases, prime gap is always smaller than $2p_n$, $Gn = P_{n+1} - P_n \leq 2p_n - 2$

Because, in the all other cases, arrangement of divisors is different from (2) (N_{p_n} is composite or not).
Write above arrangement as :

P_n	N_1	.	...	...	$N_{(p_n-1)/2}$	N_c	$N_{(p_n+1)/2}$	...	...	...	$N_{(p_n-2)}$
P_n	$N_c-(p_n-2)$	...	...	N_c-3	N_c-1	N_c	N_c+1	N_c+3	...	...	$N_c+(p_n-2)$
Dv	p_n-2	...	...	3	p_n		p_{n+1}	3	...	...	p_n-2

To obtain arrangement of divisor above, the below conditions must hold:

a. The middle even number N_c must be the form $N_c = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}...p_{n-1}^{\alpha_{n-1}}$.

b. $N_c \pm 1$, one is divisible by p_n, other one is divisible by p_{n+1}.

c. $N_c < p_{n+2}^2$

It is not difficult to show that only $N_c = 120 = 2^3.3.5$ holds.For large enough numbers, both of a and c could not be satisfied together .

We return to determine Gn:

Since $p_n \leq p_{n+1} - 2 < \sqrt{P_{n+1}} - 2 \leq \sqrt{P_n + 2p_n - 2} - 2$

so $p_n < \sqrt{P_n + 2p_n - 2} - 2$

$(p_n+2)^2 < P_n + 2p_n - 2$

$p_n^2 + 4p_n + 4 < P_n + 2p_n - 2$

$p_n^2 + 2p_n + 4 < P_n - 2$

$p_n^2 + 2p_n + 1 < P_n - 5$

$(p_n + 1)^2 < P_n - 5$

$p_n + 1 < \sqrt{P_n - 5}$

$p_n < \sqrt{P_n - 5} - 1$

$$\text{Hence: } Gn < 2(\sqrt{P_n - 5} - 1)^*, \text{ a short form: } G_n < 2\sqrt{P_n}$$

The formula * is true for all prime numbers $P_n \geq 17$.

2 Consequence

Legendre's conjecture was verified by prime gap above.
Legendre's conjecture, proposed by Adrien-Marie Legendre, states that there is a prime number between m^2 and$(m+1)^2$ for every positive integer m. The is one of Landau's problem (1912) on prime numbers; as of 2015, the conjecture has neither been proved nor disproved [**?**].
Difference between m^2 and $(m+1)^2$ is :$D = (m+1)^2 - m^2 = 2m+1$
Since $m+1 \geqslant p_{n+1}$, then $m > p_n$,$2m+1 > 2p_n$, hence $D > Gn$: There is a prime number between m^2 and$(m+1)^2$.
In fact, *there are at least two prime numbers between m^2 and $(m+1)^2$ for every positive integer m.*

References

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