Veblen's Identities, Maxwell's Equations and Weyl's Unified Field Theory

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Abstract

Shortly after Einstein's announcement of general relativity in 1915, the German mathematical physicist Hermann Weyl proposed a non-Riemannian variant of the theory that appeared to successfully unify gravitation with electromagnetism. Several years later, the American mathematician Oswald Veblen derived a set of general identities for the Riemann-Christoffel curvature tensor that complemented the more familiar Bianchi identities. As we show in this brief paper, Veblen's identities allow an interesting and previously unexplored connection between Weyl's theory and the homogeneous set of Maxwell's equations. While neither Weyl nor Veblen seems to have pursued this connection, it appears to show that Weyl's failed 1918 theory might be deeper than previously thought.

Introduction

The Riemann-Christoffel curvature tensor is given by

$$R^{\lambda}{}_{\mu\alpha\beta} = \Gamma^{\lambda}_{\mu\alpha|\beta} - \Gamma^{\lambda}_{\mu\beta|\alpha} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\nu}_{\mu\beta}$$

where the single subscripted bar denotes ordinary partial differentiation with respect to the adjoining index. The quantity $\Gamma^{\lambda}_{\mu\alpha}$ is the *coefficient of affine connection* (which we do not immediately identify with the ordinary Levi-Civita connection). We will assume that the affine connection is symmetric with respect to its lower indices, but is otherwise completely arbitrary.

Given such a connection, the curvature tensor can be shown to obey the following familiar properties:

$$R^{\lambda}{}_{\mu\alpha\beta} = -R^{\lambda}{}_{\mu\beta\alpha} \tag{1}$$

$$R^{\lambda}{}_{\mu\lambda\beta} = -R^{\lambda}{}_{\mu\beta\lambda} = R_{\mu\beta} \tag{2}$$

$$R^{\mu}{}_{\mu\alpha\beta} = R_{\alpha\beta} - R_{\beta\alpha} \tag{3}$$

$$R^{\lambda}{}_{\mu\alpha\beta} + R^{\lambda}{}_{\beta\mu\alpha} + R^{\lambda}{}_{\alpha\beta\mu} = 0 \tag{4}$$

We also have the Bianchi identities, given by

$$R^{\lambda}{}_{\mu\alpha\beta||\nu} + R^{\lambda}{}_{\mu\nu\alpha||\beta} + R^{\lambda}{}_{\mu\beta\nu||\alpha} = 0 \tag{5}$$

where the double subscripted bar denotes covariant differentiation. These are the most general expressions possible for an arbitary affine connection, and we will assume no other properties for the curvature tensor.

Veblen's Identities

In 1922 Veblen derived (perhaps not independently) an additional set of equations similar to (5), which we reproduce here using the following (different) approach. If we add the cyclic set of the four Bianchi identities

$$R^{\lambda}{}_{\mu\nu\alpha||\beta} + R^{\lambda}{}_{\mu\beta\nu||\alpha} + R^{\lambda}{}_{\mu\alpha\beta||\nu} = 0$$

$$R^{\lambda}{}_{\beta\mu\nu||\alpha} + R^{\lambda}{}_{\beta\alpha\mu||\nu} + R^{\lambda}{}_{\beta\nu\alpha||\mu} = 0$$

$$R^{\lambda}{}_{\alpha\beta\mu||\nu} + R^{\lambda}{}_{\alpha\nu\beta||\mu} + R^{\lambda}{}_{\alpha\mu\nu||\beta} = 0$$

$$R^{\lambda}{}_{\nu\alpha\beta||\mu} + R^{\lambda}{}_{\nu\mu\alpha||\beta} + R^{\lambda}{}_{\nu\beta\mu||\alpha} = 0$$

we can write

$$\left(R^{\lambda}{}_{\mu\beta\nu}+R^{\lambda}{}_{\beta\mu\nu}+R^{\lambda}{}_{\nu\beta\mu}\right)_{||\alpha}+\left(R^{\lambda}{}_{\mu\nu\alpha}+R^{\lambda}{}_{\alpha\mu\nu}+R^{\lambda}{}_{\nu\mu\alpha}\right)_{||\beta}$$

$$+\left(R^{\lambda}{}_{\beta\nu\alpha}+R^{\lambda}{}_{\alpha\nu\beta}+R^{\lambda}{}_{\nu\alpha\beta}\right)_{||\mu}+\left(R^{\lambda}{}_{\mu\alpha\beta}+R^{\lambda}{}_{\beta\alpha\mu}+R^{\lambda}{}_{\alpha\beta\mu}\right)_{||\nu}=0 \tag{6}$$

Now, from (4), we have

$$R^{\lambda}{}_{\beta\mu\nu}=-R^{\lambda}{}_{\nu\beta\mu}-R^{\lambda}{}_{\mu\nu\beta}$$

$$R^{\lambda}{}_{\alpha\mu\nu}=-R^{\lambda}{}_{\nu\alpha\mu}-R^{\lambda}{}_{\mu\nu\alpha}$$

$$R^{\lambda}{}_{\beta\nu\alpha}=-R^{\lambda}{}_{\alpha\beta\nu}-R^{\lambda}{}_{\nu\alpha\beta}$$

$$R^{\lambda}{}_{\mu\alpha\beta}=-R^{\lambda}{}_{\beta\mu\alpha}-R^{\lambda}{}_{\alpha\beta\mu}$$

Plugging these identities into (6) we get, after some simplification,

$$R^{\lambda}{}_{\mu\beta\nu||\alpha}+R^{\lambda}{}_{\nu\mu\alpha||\beta}+R^{\lambda}{}_{\alpha\nu\beta||\mu}+R^{\lambda}{}_{\beta\alpha\mu||\nu}=0 \tag{7}$$

This is the set of identities Veblen derived in 1922. Although obviously different from the Bianchi identities, we maintain that (7) contains absolutely no additional geometric information. The two sets of identities are therefore consistent and equivalent, but with one major difference: Veblen's identities can be further simplified to give a new set of non-trivial identities when the covariant derivative of the metric tensor ($g_{\mu\nu||\alpha}$) does not vanish. These new identities are useful when considering a non-Riemannian geometry of the type Weyl considered, as we will see in the following.

Connection to Weyl's Theory

Let us contract the curvature tensor in (7) using $\lambda=\beta$ and the contraction property in (2). Then (7) reduces to

$$R_{\mu\nu||\alpha}+R^{\lambda}{}_{\nu\mu\alpha||\lambda}-R_{\alpha\nu||\mu}+R^{\lambda}{}_{\lambda\alpha\mu||\nu}=0 \tag{8}$$

Using (3), this simplifies to

$$R_{\mu\nu||\alpha}+R^{\lambda}{}_{\nu\mu\alpha||\lambda}-R_{\alpha\nu||\mu}+R_{\alpha\mu||\nu}-R_{\mu\alpha||\nu}=0$$

We can get rid of the curvature tensor term by an appeal to (5), which leads to

$$R_{\mu\nu||\alpha}-R^{\lambda}{}_{\nu\lambda\mu||\alpha}-R^{\lambda}{}_{\nu\alpha\lambda||\mu}-R_{\alpha\nu||\mu}+R_{\alpha\mu||\nu}=0$$

Using (2) once more and collecting terms, we have, finally,

$$\left(R_{\mu\nu}-R_{\nu\mu}\right)_{||\alpha}+\left(R_{\nu\alpha}-R_{\alpha\nu}\right)_{||\mu}+\left(R_{\alpha\mu}-R_{\mu\alpha}\right)_{||\nu}=0 \tag{9}$$

Because the affine connection is assumed to be symmetric in its lower indices, the affine connection terms associated with covariant differentiation cancel out, and (9) collapses to the equivalent ordinary partial differential expression

$$\left(R_{\mu\nu}-R_{\nu\mu}\right)_{|\alpha}+\left(R_{\nu\alpha}-R_{\alpha\nu}\right)_{|\mu}+\left(R_{\alpha\mu}-R_{\mu\alpha}\right)_{|\nu}=0 \tag{10}$$

The metric covariant derivative does not vanish in Weyl's 1918 theory, and as a consequence the quantity $R_{\mu\nu}-R_{\nu\mu}$ also does not vanish (as it does in Riemannian geometry). Conventional tensor calculus guarantees that any rank-two tensor with the cyclic property exhibited by (10) is derivable from the curl of some potential vector field ϕ_μ. Weyl associated this field with the four-potential of electromagnetism, along with

$$F_{\mu\nu}=R_{\mu\nu}-R_{\nu\mu}$$

where $F_{\mu\nu}=\phi_{\mu|\nu}-\phi_{\nu|\mu}$ is the antisymmetric electromagnetic tensor. Since the set of homogenous Maxwell's equations exhibit exactly the property in (10), it is indeed tempting to associate the quantity $R_{\mu\nu}-R_{\nu\mu}$ with the electromagnetic field, as did Weyl.

Comments

In his 1918 theory, Weyl developed a non-Riemannian geometry in which the metric covariant derivative is given by

$$g_{\mu\nu||\alpha} = 2\, g_{\mu\nu}\phi_\alpha$$

where ϕ_α is assumed to be proportional to the four-potential of the electromagnetic field. The non-vanishing of $g_{\mu\nu||\alpha}$ marks of abandonment of the notion of *metricity*, and so the Weyl geometry represents one example of a non-Riemannian space. It is for this reason that certain properties of the curvature tensor in Riemannian geometry no longer hold (such as $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$). Of course, non-metricity also means that the affine connection $\Gamma^\lambda_{\mu\nu}$ is no longer the familiar Levi-Civita connection given by

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \frac{1}{2} g^{\lambda\alpha}\left(g_{\mu\alpha|\nu} + g_{\alpha\nu|\mu} - g_{\mu\nu|\alpha}\right)$$

In Weyl's 1918 theory, this connection is appended by terms involving the vector field ϕ_μ. The student is encouraged to explore this highly interesting early effort by Weyl to unify gravitation and electromagnetism, the only forces of Nature known at the time.

Despite its stunning mathematical beauty and the apparent initial success of Weyl's unification scheme, Einstein subsequently showed the theory to be unphysical, and it was eventually discarded. Nevertheless, Weyl's non-Riemannian geometry continues to appear in various guises today, including cosmology and quantum physics. While the electromagnetic aspect of Weyl's theory is today considered a dead end, it is surprising how often quantities relating to electromagnetism seem to suddenly appear when exploring the theory, as the above argument demonstrates.

References

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