PROJECTIVELY HERMITIAN YANG-MILLS METRICS ON HIGGS BUNDLES OVER ASYMPTOTICALLY CYLINDRICAL KÄHLER MANIFOLDS

PAN ZHANG

Abstract. Let V be an asymptotically cylindrical Kähler manifold with asymptotic cross-section $\mathfrak{D}$. Let $E_{\mathfrak{D}}$ be a stable Higgs bundle over $\mathfrak{D}$, and E a Higgs bundle over V which is asymptotic to $E_{\mathfrak{D}}$. In this paper, using the continuity method of Uhlenbeck and Yau, we prove that there exists an asymptotically translation-invariant Hermitian projectively Hermitian Yang-Mills metric on E.

1. Introduction

The Yang-Mills theory plays an important role for holomorphic vector bundles over a compact Kähler manifold. The relation between the existence of Hermitian Yang-Mills metrics and stable holomorphic vector bundles over compact Kähler manifolds is by now well understood, due to the work of Narasimhan-Seshadri [27], Donaldson [9], Siu [32], Uhlenbeck-Yau [31] and others. On the other hand, it was quite fruitful to consider the correspondences for vector bundles with some additional structures like Higgs field, which was initiated by Hitchin [13]. Such bundles have a rich structure and play an important role in many areas including gauge theory, Kähler and hyper-Kähler geometry, group representations, and nonabelian Hodge theory. Hitchin proved that a Higgs bundle on a compact Riemann surface admits a Hermitian Yang-Mills metric if and only if it is Higgs poly-stable. Later, Simpson [30] proved an analogue of the Donaldson-Uhlenbeck-Yau theorem for the Higgs bundle over higer dimensional Kähler manifolds, influenced by the work of Hitchin. In the compact case, the Higgs version of Donaldson-Uhlenbeck-Yau has been extensively studied, see references [1, 2, 3, 6, 7, 14, 18, 19, 24].

There are noncompact version of the theorem of Donaldson-Uhlenbeck-Yau, but it is not general enough to cover all cases, see references [4, 12, 16, 17, 25, 26, 28, 29, 30, 33]. In a very recent paper, Jacob and Walpuski [16] proved that if $\mathcal{E}$ is a reflexive sheaf over an asymptotically cylindrical Kähler manifold, which is asymptotic to a stable holomorphic vector bundle, then it admits an asymptotically translation-invariant projectively Hermitian Yang-Mills metric. Our aim is to generalize this result to Higgs bundles.

2010 Mathematics Subject Classification: 53C07; 14J60.

Key words and phrases: Higgs bundles; Kähler manifold; projectively Hermitian Yang-Mills Metrics.

Let (X,ω) be a compact Kähler manifold, and E a holomorphic vector bundle over X. The stability of holomorphic vector bundles, in the sense of Mumford-Takemoto, was a well established concept in algebraic geometry. A holomorphic vector bundle E is called stable (semi-stable), if for every coherent sub-sheaf $E' \hookrightarrow E$ of lower rank, one has

$$\mu(E') = \frac{\deg E'}{\text{rank } E'} < (\leq)\mu = \frac{\deg E}{\text{rank } E},$$

where $\mu(E')$ is called the slope of E'.

A holomorphic bundle $(E, \bar{\partial}_E)$ coupled with a Higgs field $\phi \in \Omega^{1,0}(\text{End}(E))$ which satisfies $\bar{\partial}_E \phi = 0$ and $\phi \wedge \phi = 0$ will be called by a Higgs bundle. A Higgs bundle $(E, \bar{\partial}_E, \phi)$ is called stable (semi-stable) if the usual stability condition $\mu(E') < \mu(E)(\leq)$holds for all proper ϕ-invariant sub-sheaves. A Hermitian metric H in Higgs bundle $(E, \bar{\partial}_E, \phi)$ is said to be projectively Hermitian Yang-Mills (PHYM) if the curvature $F_{H,\phi}$ of the Hitchin-Simpson connection $D_{H,\phi} = D_H + \phi + \phi^{*_H}$ satisfies

$$K_H := \sqrt{-1}\Lambda_w F_{H,\phi} - \frac{\text{tr}(\sqrt{-1}\Lambda_w F_{H,\phi})}{\text{rank } E}\text{id}_E = 0,$$

where D_H is the Chern connection induced by H, and ϕ^{*_H} is the adjoint of ϕ with respect to the metric H.

Theorem 1.1. *Let V be an asymptotically cylindrical Kähler manifold with asymptotic cross-section $\mathfrak{D}$. Let $E_{\mathfrak{D}}$ be a stable Higgs bundle over $\mathfrak{D}$, and E a Higgs bundle over V which is asymptotic to $E_{\mathfrak{D}}$. Then there exists an asymptotically translation-invariant projectively Hermitian Yang-Mills metric on E.*

Remark 1.2. *A PHYM metric H on E is Hermitian Yang-Mills (HYM) if and only if $\sqrt{-1}\Lambda_w F_{H,\phi}$ is constant. Every asymptotically translation-invariant Higgs line bundle over the asymptotically cylindrical Kähler manifold has an HYM metric; however, this metric will typically not be asymptotically translation invariant. This is a consequence of Proposition 2.6.*

2. Preliminaries

Definition 2.1. *Let $(\mathfrak{D}, g_{\mathfrak{D}}, J_{\mathfrak{D}})$ be a compact Kähler manifold. A Kähler manifold (V, g, J) is called asymptotically cylindrical with asymptotic cross-section $(\mathfrak{D}, g_{\mathfrak{D}}, J_{\mathfrak{D}})$ if there exists a constant $\delta_V > 0$, a compact subset $K \subset V$ and a diffeomorphism $\pi : V\backslash K \to (1,\infty) \times S^1 \times \mathfrak{D}$ such that*

$$\left|\nabla^k(\pi_* g - g_\infty)\right| + \left|\nabla^k(\pi_* J - J_\infty)\right| = O(e^{-\delta_V l}),$$

for all $k \in \mathbb{N}$, with

$$g_\infty := dl^2 \oplus d\theta^2 \oplus g_{\mathfrak{D}}, \quad J_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus J_{\mathfrak{D}}.$$

Here (l,θ) are the canonical coordinates on $(0,\infty)\times S^1$. Moreover, we assume that the map $V\backslash K\to(0,\infty)\times S^1$ is holomorphic.

In what follows, we suppose an asymptotically cylindrical Kähler manifold V with asymptotic cross-section $\mathfrak{D}$ has been fixed. By slight abuse of notation we denote by $l: V\to[0,\infty)$ a smooth extension of $l\circ\pi: V\backslash K\to(1,\infty)$ such that $l\leq 1$ on K. Given $L>1$, we define the truncated manifold

$$V_L := l^{-1}([0,L]).$$

Given $z=(L,\theta)\in(1,\infty)\times S^1$, we set

$$\mathfrak{D}_z := \pi^{-1}(\{L,\theta\}\times\mathfrak{D}).$$

Definition 2.2. *Let $(E_{\mathfrak{D}},\bar{\partial}_{\mathfrak{D}},\phi_{\mathfrak{D}})$ be a Higgs bundle over $\mathfrak{D}$ and $(E,\bar{\partial},\phi)$ a Higgs bundle over V. We say that E is asymptotic to $E_{\mathfrak{D}}$ if there exists a bundle isomorphism $\bar{\pi}: E\to E_\infty$ covering π and a constant $\delta_E>0$ such that*

$$\left|\nabla^k(\bar{\pi}_*\bar{\partial}-\bar{\partial}_\infty)\right| = O(e^{-\delta_E l}),\quad \left|\nabla^k(\bar{\pi}_*\phi-\phi_\infty)\right| = O(e^{-\delta_E l})$$

for all $k\in\mathbb{N}$. Here $(E_\infty,\bar{\partial}_\infty,\phi_\infty)$ is the pullback of $(E_{\mathfrak{D}},\bar{\partial}_{\mathfrak{D}},\phi_{\mathfrak{D}})$ to $(1,\infty)\times S^1\times D$; moreover, we have chosen an auxiliary Hermitian metric on $E_{\mathfrak{D}}$ and pulled it back to E_∞. We say that $(E,\bar{\partial},\phi)$ is asymptotically translation-invariant if it is asymptotic to some Higgs bundle over $\mathfrak{D}$.

Definition 2.3. *Let $(E,\bar{\partial},\phi)$ be a Higgs bundle over V asymptotic to $(E_{\mathfrak{D}},\bar{\partial}_{\mathfrak{D}},\phi_{\mathfrak{D}})$. Let $H_{\mathfrak{D}}$ be a Hermitian metric on $E_{\mathfrak{D}}$. We say that a Hermitian metric H on E is asymptotic to $H_{\mathfrak{D}}$ if there exist a constant $\delta_H>0$ such that*

$$\left|\nabla^k\log(H_\infty^{-1}\pi_* H)\right| = O(e^{-\delta_H l})$$

for all $k\in\mathbb{N}$. Here H_∞ is the pullback of $H_{\mathfrak{D}}$ to E_∞. We say that H is asymptotically translation-invariant if it is asymptotic to some Hermitian metric $H_{\mathfrak{D}}$.

Let $(E,\bar{\partial},\phi)$ be a Higgs bundle over V and H a Hermitian metric on E. Then we get an operator ∂_H so that $D_H=\bar{\partial}+\partial_H$ is the metric connection on E, and we can define ϕ^{*_H} by

$$\langle\phi u, v\rangle_H = \langle u, \phi^{*_H} v\rangle_H.$$

Set

$$D'_H = \partial_H+\phi^{*_H},\quad D''=\bar{\partial}+\phi,$$

and then

$$D_{H,\phi} = D'_H + D''.$$

We have the following Kähler identities [30]

$$\sqrt{-1}[\Lambda, D''] = (D'_H)^*,\quad \sqrt{-1}[\Lambda, D'_H] = -(D'')^*.$$

Then we have the Weitzenböck formulas

$$(D'_H)^* D'_H = \frac{1}{2}(D^*_{H,\phi}D_{H,\phi} + [K_H,\cdot]),$$

$$(D'')^*D'' = \frac{1}{2}(D^*_{H,\phi}D_{H,\phi} - [K_H, \cdot]).$$

Definition 2.4. *A differential operator $\mathcal{A} : \Gamma(E_1) \to \Gamma(E_2)$ on sections of tensor bundles on V is called asymptotically translation-invariant if there is a translation-invariant operator $\mathcal{A}_\infty$ on sections of the corresponding bundles on $D \times S^1 \times \mathbb{R}_l$ such that the difference between the coefficients of $\mathcal{A}$ and $\mathcal{A}_\infty$ goes to zero in C^∞ uniformly as t goes to infinity.*

Even if $\mathcal{A}$ is elliptic, we can not expect $\mathcal{A}$ to induce a Fredholm operator on ordinary Hölder or Sobolev spaces since V is noncompact. To fix this, it is helpful to introduce Hölder norms with exponential weights.

Definition 2.5. *For $k \in \mathbb{N}, \alpha \in (0, 1)$ and $\delta \in \mathbb{R}$, define*

$$C^{k,\alpha}_\delta := \{f \in C^{k,\alpha} : \| f \|_{C^{k,\alpha}_\delta} < \infty\},$$

with

$$\| \cdot \|_{C^{k,\alpha}_\delta} := \| e^{\delta l} \cdot \|_{C^{k,\alpha}},$$

and set

$$C^\infty_\delta(V) := \bigcap_{k\in\mathbb{N}} C^{k,\alpha}_\delta(V).$$

Similarly, we define $C^{k,\alpha}_\delta(V, \sqrt{-1}\mathfrak{su}(E, H))$ and $C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H))$. Here $\sqrt{-1}\mathfrak{su}(E, H)$ denotes the traceless endomorphisms of E and such endomorphisms are self-adjoint with respect to H.

Proposition 2.6 ([16, Proposition 2.7])**.** *For $0 < \delta \ll_{\mathfrak{D}} 1$, the linear map $C^{k+2,\alpha}_\delta(V) \oplus \mathbb{R} \to C^{k,\alpha}_\delta(V)$ defined by*

$$(f, A) \longmapsto \Delta f - A\Delta l^1$$

is an isomorphism.

Proposition 2.7. *If $(E_{\mathfrak{D}}, \bar\partial_{\mathfrak{D}}, \phi_{\mathfrak{D}})$ is stable and $\left|\delta\right| \ll_{\mathfrak{D}} 1$, the linear operator $D^*_{H_0,\phi}D_{H_0,\phi}$: $C^{k+2,\alpha}_\delta(V, \sqrt{-1}\mathfrak{su}(E, H)) \to C^{k,\alpha}_\delta(V, \sqrt{-1}\mathfrak{su}(E, H))$ is Fredholm of index zero.*

Proof. Since $(E_{\mathfrak{D}}, \bar\partial_{\mathfrak{D}}, \phi_{\mathfrak{D}})$ is stable, there is a PHYM metric $H_{\mathfrak{D}}$ on $E_{\mathfrak{D}}$. The linear operator $D^*_{H_0,\phi}D_{H_0,\phi}$ is asymptotic to the translation-invariant linear operator

$$-\partial_l^2 - \partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}$$

acting on the sections of $\sqrt{-1}\mathfrak{su}(E_\infty, H_\infty)$, where $D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}$ is the Hitchin-Simspon connection on $E_{\mathfrak{D}}$ associated to the metric $H_{\mathfrak{D}}$. Since $H_{\mathfrak{D}}$ is PHYM, the Weitzenböck formulas imply

$$\frac{1}{2}D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} = (D'_{H_{\mathfrak{D}}})^*D'_{H_{\mathfrak{D}}} = (D''_{E_{\mathfrak{D}}})^*D''_{E_{\mathfrak{D}}}.$$

If $0 \in \mathrm{Spec}\{D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}\}$, then there exists nonzero section s such that

$$D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}s = 0,$$

[1]Throughout this paper, we denote $\Delta = d^*d$.

equivalently,

$$D''_{E_{\mathfrak{D}}} s = 0,$$

which together with the stability of $E_{\mathfrak{D}}$ gives $s = 0$.

Consequently, the spectrum of $-\partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}$ is contained in $[\lambda_{\mathfrak{D}}, \infty)$, for some $\lambda_{\mathfrak{D}} > 0$.

We say that $\delta \in \mathbb{R}$ is a critical weight of $-\partial_l^2 - \partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}$ if there is a $\tau \in \mathbb{C}$ with

$$\operatorname{Im} \tau = \delta$$

and a non-zero section of E polynomial in l, say u, such that

$$(-\partial_l^2 - \partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}})(e^{\sqrt{-1}\tau l} u) = 0.$$

We can expand u in eigen-sections of $-\partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}$. Therefore if the above equation has a solution, we can find a non-zero real polynomial in l, say $p(l)$, and a non-zero section u with

$$(-\partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}) u = \lambda u$$

for some $\lambda \in [\lambda_{\mathfrak{D}}, \infty)$ such that

$$(-\partial_l^2 - \partial_\theta^2 + D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}})(e^{\sqrt{-1}\tau l} p(l) u) = 0.$$

Then

$$(\tau^2 + \lambda) p(l) - 2\sqrt{-1}\tau p'(l) + p''(l) = 0. \tag{2.1}$$

Considering the leading order term in l,

$$\tau^2 + \lambda = 0,$$

which means

$$\delta = \pm\sqrt{\lambda}.$$

This implies the Fredholm property for $\left|\delta\right| < \sqrt{\lambda_{\mathfrak{D}}}$ by [15, Proposition 2.4].

On the other hand, $D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}} D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}$ is formally self-adjoint and 0 is not a critical weight, then the index is zero [20, Theorem 7.4]. □

3. The Donaldson functional

Let (X, g, J) be a compact Kähler manifold and E a Higgs bundle over X. Given metric H_0 and $s \in C^\infty(X, \sqrt{-1}\mathfrak{su}(E, H_0))$, the value of Donaldson functional at $(H_0, H_0 e^s)$ is [30]

$$\mathcal{M}(H_0, H_0 e^s) := \int_X \operatorname{tr}(s\sqrt{-1}\Lambda F_{H_0,\phi}) + \langle \Psi(s)(D''s), D''s \rangle_{H_0},$$

where

$$\Psi(x, y) = \begin{cases} (x-y)^{-2}(e^{y-x} - (y-x) - 1) & x \neq y, \\ \frac{1}{2} & x = y. \end{cases}$$

The above integral is equivalent to

$$\mathcal{M}(H_0, H_0e^s) := \int_0^1 \int_X \langle s, \mathrm{Ad}(e^{\frac{us}{2}})K_{H_0e^{us}}\rangle \mathrm{dvol}_g \mathrm{d}u.$$

More details can be found in a recent paper [8].

Proposition 3.1 ([10, Lemma 24], [30, Proposition 5.3])**.** *If H_0 is PHYM, then*

$$\| s \|_{L^1} - 1 \lesssim \mathcal{M}(H_0, H_0e^s).$$

Proposition 3.2 ([30, Proposition 5.1])**.** *We have*

$$\mathcal{M}(H_0, H_2) = \mathcal{M}(H_0, H_1) + \mathcal{M}(H_1, H_2).$$

4. The Uhlenbeck-Yau continuity method

We will use the continuity method initiated by Uhlenbeck-Yau [31] (see also Lübke and Teleman's books [21, 22]).

At first, we fix some

$$0 < \delta < \min\{\delta_V, \delta_E, \sqrt{\lambda_{\mathfrak{D}}}\}$$

and will shortly construct a background Hermitian metric H_0 on E which is asymptotically translation-invariant and satisfies

$$K_{H_0} \in C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H_0)).$$

Given such an H_0, we define a map

$$\mathfrak{L} : C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H_0)) \times [0, 1] \to C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H_0))$$

by

$$\mathfrak{L}(s, t) := \mathrm{Ad}(e^{\frac{s}{2}})K_{H_0e^s} + ts.$$

Set

$$I := \{t \in [0, 1] : \mathfrak{L}(s, t) = 0 \ \text{ for some } \ s \in C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H_0))\}$$

By Simpson's theorem, there exists a PHYM metric $H_{\mathfrak{D}}$ on $E_{\mathfrak{D}}$. One can easily construct a Hermitian metric H_{-1} asymptotic to $H_{\mathfrak{D}}$ which satisfies

$$\kappa := K_{H_{-1}} \in C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H_{-1})).$$

The Hermitian metric $H_0 := H_{-1}e^\kappa$ is asymptotic to $H_{\mathfrak{D}}$ and

$$\mathfrak{L}(-\kappa, 1) = \mathrm{Ad}(e^{-\frac{\kappa}{2}})K_{H_{-1},\phi} - \kappa = 0.$$

Then $1 \in I$.

Then we need to show that $I \bigcap (0, 1]$ is open and I is closed; hence, $I = [0, 1]$. Since $\mathfrak{L}(s, 0) = 0$ precisely means that $H = H_0e^s$ satisfies the PHYM equation, this will prove Theorem 1.1.

To prove I is closed, the first step is to show that $\| s \|_{C^0}$ is bounded by a constant depending only on H_0. Then by an argument of Bando and Siu [5], $\| s \|_{C^k}$ is bounded by a constant depending only on H_0 and k. The second step is a decay estimate. And we will omit the second one since it is very similar to [16]. Then the closedness is an immediate consequence of Arzelà-Ascoli.

5. Linearising of the perturbed equation

One can extend $\mathfrak{L}(s,t)$ to a smooth map

$$\mathfrak{L} : C^{2,\alpha}_{\delta}(V, \sqrt{-1}\mathfrak{su}(E,H_0)) \times [0,1] \to C^{0,\alpha}_{\delta}(V, \sqrt{-1}\mathfrak{su}(E,H_0)).$$

The fact that $I \bigcap (0,1]$ is an immediate consequence of the following two propositions and the implicit function theorem for Banach spaces.

Proposition 5.1. *If $(s,t) \in C^{2,\alpha}_{\delta}(V, \sqrt{-1}\mathfrak{su}(E,H_0)) \times [0,1]$ is a solution of $\mathfrak{L}(s,t) = 0$, then $s \in C^{\infty}_{\delta}(V, \sqrt{-1}\mathfrak{su}(E,H_0))$.*

Proof. Fix a Hermitian metric H_0. Set

$$H := H_0 e^s \quad \text{and} \quad \widetilde{D}_s = e^{\frac{s}{2}}_{*} D_{H,\phi} = e^{\frac{s}{2}} \circ D_{H,\phi} \circ e^{-\frac{s}{2}}.$$

We set

$$\mathfrak{R}(s) := \mathrm{Ad}(e^{\frac{s}{2}}) K_{H_0 e^s}.$$

Since $D'_{H_0 e^s} = D'_{H_0} + e^{-s} D'_{H_0} e^s$, we have

$$\begin{aligned} \widetilde{D}'_s &:= e^{\frac{s}{2}} \circ D'_{H_0 e^s} \circ e^{-\frac{s}{2}} \\ &= e^{\frac{s}{2}} \circ (D'_{H_0} + e^{-s} D'_{H_0} e^s) \circ e^{-\frac{s}{2}} \\ &= D'_{H_0} + e^{-\frac{s}{2}} (D'_{H_0} e^{\frac{s}{2}}) \\ &= D'_{H_0} + \frac{1}{2} \Upsilon(-\frac{s}{2}) D'_{H_0} s, \end{aligned}$$

where $\Upsilon(s) \in \mathrm{End}(\mathfrak{gl}(E))$ is given by

$$\Upsilon(s) := \frac{e^{\mathrm{ad}_s} - \mathrm{id}}{\mathrm{ad}_s}.$$

On the other hand,

$$\widetilde{D}''_s := e^{\frac{s}{2}} \circ D'' \circ e^{-\frac{s}{2}} = D'' - \frac{1}{2} \Upsilon(\frac{s}{2}) D'' s,$$

which means

$$\widetilde{D}_s = \widetilde{D}_0 + \frac{1}{2} \Upsilon(-\frac{s}{2}) D'_{H_0} s - \frac{1}{2} \Upsilon(\frac{s}{2}) D'' s.$$

Then by the Kähler identities as well as the Weitzenböck formulas we have

$$\begin{aligned}\mathfrak{R}(s) &= (3-2\cosh(\mathrm{ad}_{\frac{s}{2}}))K_{H_0} + \frac{1}{4}(\Upsilon(\frac{s}{2})+\Upsilon(-\frac{s}{2}))D^*_{H_0,\phi}D_{H_0,\phi}\\ &+\frac{\sqrt{-1}}{2}\Lambda(D''\Upsilon(-\frac{s}{2})\wedge D'_{H_0}s)-\frac{\sqrt{-1}}{2}\Lambda(D'_{H_0}\Upsilon(\frac{s}{2})\wedge D''s)\\ &-\frac{\sqrt{-1}}{4}\Lambda(\Upsilon(-\frac{s}{2})D'_{H_0}s\wedge\Upsilon(\frac{s}{2})D''s+\Upsilon(\frac{s}{2})D''s\wedge\Upsilon(-\frac{s}{2})D'_{H_0}s),\end{aligned}$$

here we heavily used

$$\mathrm{tr}\sqrt{-1}\Lambda F_{\widetilde{D}_s} = \mathrm{tr}\sqrt{-1}\Lambda F_{H_0,\phi}.$$

Hence the equation $\mathfrak{L}(s,t)=0$ is equivalent to

$$(\frac{1}{2}D^*_{H_0,\phi}D_{H_0,\phi}+t)s + B(D_{H_0,\phi}s\otimes D_{H_0,\phi}s) = C(K_{H_0}),$$

where B and C are linear with coefficients depending on s, but not on its derivatives. The result now follows from a standard elliptic bootstrapping procedure.

□

Proposition 5.2. *If* $(s,t)\in C^{2,\alpha}_\delta(V,\sqrt{-1}\mathfrak{su}(E,H_0))\times(0,1]$ *is a solution of* $\mathfrak{L}(s,t)=0$, *then the linearisation*

$$L_{s,t} := \frac{d\mathfrak{L}}{ds}(s,t): C^{2,\alpha}_\delta(V,\sqrt{-1}\mathfrak{su}(E,H_0)) \to C^{0,\alpha}_\delta(V,\sqrt{-1}\mathfrak{su}(E,H_0))$$

is invertible.

Proof. If σ_t satisfies

$$e^{s+t\hat{s}} = e^s\mathrm{Ad}(e^{-\frac{s}{2}})e^{\sigma_t},$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\sigma_t = \mathrm{Ad}(e^{\frac{s}{2}})\Upsilon(-s)\hat{s},$$

here we used $\mathrm{d}_x\exp(y) = (\Upsilon(x)y)e^x = e^x(\Upsilon(-x)y)$.

Using the above fact, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widetilde{D}'_s &= \frac{1}{2}(e^{-\frac{s}{2}}D'_{H_0}(e^{\frac{s}{2}}\mathrm{Ad}(e^{-\frac{s}{2}})\Upsilon(\frac{s}{2})\hat{s}) - (\mathrm{Ad}(e^{-\frac{s}{2}})\Upsilon(\frac{s}{2})\hat{s})e^{-\frac{s}{2}}D'_{H_0}e^{\frac{s}{2}})\\ &= \frac{1}{2}(D'_{H_0}(\mathrm{Ad}(e^{-\frac{s}{2}})\Upsilon(\frac{s}{2})\hat{s}) + [e^{-\frac{s}{2}}D'_{H_0}e^{\frac{s}{2}}, \mathrm{Ad}(e^{-\frac{s}{2}})\Upsilon(\frac{s}{2})\hat{s}])\\ &= \frac{1}{2}\widetilde{D}'_s\mathrm{Ad}(e^{-\frac{s}{2}})\Upsilon(\frac{s}{2})\hat{s}.\end{aligned}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widetilde{D}''_s = -\frac{1}{2}\widetilde{D}''_s\Upsilon(\frac{s}{2})\hat{s}.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\sqrt{-1}\Lambda F_{\widetilde{D}_{s+t\hat{s}}} &= \sqrt{-1}\Lambda\widetilde{D}_s(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\widetilde{D}_s)\\ &= \frac{1}{4}\widetilde{D}_s^*\widetilde{D}_s(\mathrm{id}+\mathrm{Ad}(e^{-\frac{s}{2}}))\Upsilon(\frac{s}{2})\hat{s}\\ &\quad-\frac{1}{4}[\mathfrak{R}(s),(\mathrm{id}-\mathrm{Ad}(e^{-\frac{s}{2}}))\Upsilon(\frac{s}{2})\hat{s}].\end{aligned}$$

To see that $\hat{s}$ takes values in $\sqrt{-1}\mathfrak{su}(E,H_0)$, observe that one can identify the tangent space of $\sqrt{-1}\mathfrak{su}(E,H_0)$ as itself. Hence, one can easily verify that

$$\mathrm{tr}(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\sqrt{-1}\Lambda F_{\widetilde{D}_{s+t\hat{s}}})=0,$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathfrak{R}(s+t\hat{s})=\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\sqrt{-1}\Lambda F_{\widetilde{D}_{s+t\hat{s}}}.$$

Hence the linear operator $L_{s,t}$ is given by

$$L_{s,t}\hat{s}=\frac{1}{4}\widetilde{D}_s^*\widetilde{D}_s(\mathrm{id}+\mathrm{Ad}(e^{-\frac{s}{2}})\Upsilon(\frac{s}{2}))\hat{s}+t(\frac{\mathrm{ad}_s}{4}(1-\mathrm{Ad}(e^{-\frac{s}{2}}))\Upsilon(\frac{s}{2})+\mathrm{id})\hat{s}.$$

Since $s\in C^{2,\alpha}_\delta(V,\sqrt{-1}\mathfrak{su}(E,H_0))$, the linear operator $L_{s,t}$ can be joined to

$$\frac{1}{2}D^*_{H_0,\phi}D_{H_0,\phi}+t$$

by a path of bounded linear operators which are asymptotic to

$$\frac{1}{2}(-\partial_l^2-\partial_\theta^2+D^*_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}D_{H_{\mathfrak{D}},\phi_{\mathfrak{D}}}+2t).$$

The argument in the proof of Proposition 2.7 shows that this is a path of Fredholm operators. Therefore, the index of $L_{s,t}$ vanishes. To see that $L_{s,t}$ has trivial kernel and thus is invertible, observe that

$$\int_V\langle L_{s,t}\hat{s},(\mathrm{id}+\mathrm{Ad}(e^{-\frac{s}{2}}))\Upsilon(\frac{s}{2})\hat{s}\rangle\geq 2t\int_V\left|\hat{s}\right|^2.$$

More details about Fredholm theory can be found in the Appendix A in [23].

□

6. C^0-estimate and stability

Proposition 6.1. *If* $(s,t)\in C^\infty_\delta(V,\sqrt{-1}\mathfrak{su}(E,H_0))\times[0,1]$ *is a solution of* $\mathfrak{L}(s,t)=0$*, then*

$$\parallel s\parallel_{C^0}\leq c.$$

Proof. We denote by $c > 0$ a generic constant, which depends only on V, E, and the reference metric H_0 constructed in Section 4. We write $x \lesssim y$ for $x \leq cy$ and $x \asymp y$ for $c^{-1}y \leq x \leq cy$.

Fix $L_0 \gg 1$ and set

$$N := \| s \|_{L^\infty(V)} \quad \text{and} \quad M := \| s \|_{L^\infty(V \setminus V_{L_0})} .$$

Step 1. We have

$$N - M \lesssim L_0 + 1.$$

First we have

$$\begin{aligned}\langle \mathfrak{R}(s) - K_{H_0}, s\rangle &= \langle \sqrt{-1}\Lambda D''(e^{-s} D'_{H_0} e^s), s\rangle \\ &= \langle \sqrt{-1}\Lambda D''(\Upsilon(-s) D'_{H_0} s), s\rangle \\ &= \frac{1}{4}\Delta\big|s\big|^2 + \frac{1}{2}\big|\sqrt{\Upsilon(-s)} D_{H_0,\phi} s\big|^2,\end{aligned}$$

which together with $\mathfrak{L}(s,t) = 0$ gives

$$\Delta\big|s\big|^2 + 4t\big|s\big|^2 \leq -4\langle K_{H_0}, s\rangle. \tag{6.1}$$

Therefore,

$$\Delta\big|s\big|^2 \leq 4N\big|K_{H_0}\big|.$$

From Proposition 2.6, one can denote by $f \in C^{2,\alpha}_\delta(V)$ and $A > 0$ the unique solution to

$$\Delta(f - Al) = 4\big|K_{H_0}\big|.$$

We can assume that $\big|s\big|$ achieves its maximum at $x_0 \in V_{L_0}$. Applying the maximum principle to $\big|s\big|^2 - N(f - Al)$ on V_{L_0} we have the desired estimate.

Step 2. We have

$$M \lesssim \| K_{H_0 e^s|_{\mathfrak{D}_z}} \|_{L^2(V \setminus V_{L_0})},$$

where $z = (L, \theta) \in (L_0, \infty) \times S^1$.

Step 2.1. Suppose $x_0 \in \overline{V \setminus V_{L_0}}$ satisfies $\big|s\big|(x_0) = M$, then for all $L \geq l(x_0)$ we have

$$l(x_0) - L \lesssim \| s \|_{L^\infty(\partial V_L)} - \frac{1}{2}M.$$

By the maximum principle applied to $\big|s\big|^2 - N(f - Al)$ on V_L we have the desired estimate. Here we assume that $M \geq 8 \| f \|_{L^\infty(V \setminus V_{L_0})}$ and $N \leq 2M$ because otherwise we are already done.

Step 2.2. There are $L_0 \leq L_1 < L_2$ with $L_2 - L_1 \asymp M$ such that

$$M^2 \lesssim \int_{V_{L_2} \setminus V_{L_1}} \big|s\big|.$$

By Step 2.1 we have $M \lesssim \| s \|_{L^\infty(\partial V_L)}$ for $0 \leq L - l(x_0) \asymp M$. Having in mind $\Delta|s|^2 \leq 8M|K_{H_0}|$, then by the mean value inequality [11, Theorem 9.20] we have

$$M \lesssim \int_{V_{L+1}\setminus V_{L-1}} |s|.$$

Summing over $L - l(x_0) = 1, \cdots, k$ (with $k \asymp M$) yields the asserted inequality.

Step 2.3. We have

$$\| s \|_{L^1(\mathfrak{D}_z)} - \frac{1}{2} \lesssim M \| K_{H_0 e^s|_{\mathfrak{D}_z}} \|_{L^2(\mathfrak{D}_z)} .$$

Since $L_0 \gg 1$ and $E_{\mathfrak{D}}$ is stable, $E_{\mathfrak{D}_z}$ is stable as well. Denote by $H_{\mathfrak{D}_z}$ the PHYM metric on $E_{\mathfrak{D}_z}$ inducing the same metric on $\det(E_{\mathfrak{D}_z})$ as $H_0 |_{\mathfrak{D}_z}$. Then we can identify $H_{\mathfrak{D}_z}$ and $H_0 |_{\mathfrak{D}_z}$ when L_0 is sufficiently large, in other words,

$$\log(H_{\mathfrak{D}_z}^{-1} H_0 |_{\mathfrak{D}_z}) \in C^\infty_\delta(V, \sqrt{-1}\mathfrak{su}(E, H_0)).$$

And by the implicit function theorem, $H_{\mathfrak{D}_z}$ depends on z smoothly. Then from Proposition 3.1 and Proposition 3.2 we have

$$\begin{aligned}
\| s \|_{L^1(\mathfrak{D}_z)} - 1 &\lesssim \mathcal{M}(H_{\mathfrak{D}_z}, H_0 e^s |_{\mathfrak{D}_z}) \\
&= \mathcal{M}(H_0 |_{\mathfrak{D}_z}, H_0 e^s |_{\mathfrak{D}_z}) + \mathcal{M}(H_{\mathfrak{D}_z}, H_0 |_{\mathfrak{D}_z}) \\
&= \mathcal{M}(H_0 |_{\mathfrak{D}_z}, H_0 e^s |_{\mathfrak{D}_z}) + O(e^{-\delta L_0}) \\
&\lesssim \int_{\mathfrak{D}_z} |s||K_{H_0 e^s|_{\mathfrak{D}_z}}| + e^{-\delta L_0}.
\end{aligned}$$

This implies the asserted inequality.

Comparing the lower bounds from Step 2.2 with the upper bounds obtained by integrating Step 2.3 completes the proof of Step 2.

Step 3. We have

$$\| K_{H_0 e^s|_{\mathfrak{D}_z}} \|^2_{L^2(V\setminus V_{L_0})} \lesssim e^{-\delta L_0} + \| F^\perp_{H_0} \|^2_{L^2(V_{L_0})},$$

where $F^\perp_{H_0}$ denotes the trace-free part.

Step 3.1. We have

$$\lim_{L\to\infty} \| K_{H_0 e^s|_{\mathfrak{D}_z}} \|^2_{L^2(V_L\setminus V_{L_0})} \lesssim ce^{-\delta L_0} + \| F^\perp_{H_0} \|^2_{L^2(V_{L_0})} + \lim_{L\to\infty} \int_{V_L} |F^\perp_{H_0 e^s}|^2 - |F^\perp_{H_0}|^2.$$

If H is a Hermitian metric on a Higgs bundle E over an n-dimensional compact Kähler manifold X with Kähler form ω, then

$$\int_X q_4(H) \wedge \omega^{n-2} = \int_X c(|F^\perp_H|^2 - |K_H|^2)\mathrm{vol}$$

independent on the choice of the metric, where

$$q_4(H) := 2c_2(H) - \frac{r-1}{r} c_1(H)^2$$

with

$$c_1(H) := \frac{\sqrt{-1}}{2\pi}\mathrm{tr}F_{H,\phi}, \quad c_2(H) := -\frac{1}{8\pi^2}((\mathrm{tr}F_{H,\phi})^2 - \mathrm{tr}(F_{H,\phi}^2)).$$

Therefore

$$\begin{aligned}\int_{\mathfrak{D}_z} \left|K_{H_0e^s|_{\mathfrak{D}_z}}\right|^2 &= \int_{\mathfrak{D}_z} \left|K_{H_0|_{\mathfrak{D}_z}}\right|^2 + \int_{\mathfrak{D}_z} \left|F^{\perp}_{H_0e^s|_{\mathfrak{D}_z}}\right|^2 - \left|F^{\perp}_{H_0|_{\mathfrak{D}_z}}\right|^2 \\ &\lesssim \int_{\mathfrak{D}_z} \left|F^{\perp}_{H_0e^s}\right|^2 - \left|F^{\perp}_{H_0}\right|^2 + e^{-\delta L},\end{aligned}$$

here we used

$$| F_{H_0} - F_{H_0|_{D_z}} | \lesssim e^{-\delta L} \quad \text{and} \quad | K_{H_0|_{D_z}} | \lesssim e^{-\delta L}.$$

Step 3.2. We have

$$\lim_{L\to\infty} \int_{V_L} \left|F^{\perp}_{H_0e^s}\right|^2 - \left|F^{\perp}_{H_0}\right|^2 \leq 0.$$

Using (6.1), $\mathfrak{L}(s,t) = 0$ and

$$\lim_{L\to\infty} \int_{V_L} (q_4(H_0e^s) - q_4(H_0)) \wedge \omega^{n-2} = 0,$$

one can easily derive the inequality. □

References

[1] B. Anchouche, I. Biswas. Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold. Amer. J. Math., 2001, 207-228.

[2] I. Biswas. Stable Higgs bundles on compact Gauduchon manifolds. Comptes Rendus Math., 2011, 349, 71-74.

[3] I. Biswas, G. Schumacher. Yang-Mills equation for stable Higgs sheaves. Inter. J. Math., 2009, 20(05), 541-556.

[4] S. Bando. Einstein-Hermitian metrics on noncompact Kähler manifolds. Einstein metrics and Yang-Mills connections (Sanda, 1990), Vol. 145, Lecture Notes in Pure and Appl. Math. New York: Dekker, 1993, 27-33

[5] S. Bando, Y.T. Siu. Stable sheaves and Einstein-Hermitian metrics. Geometry and analysis on complex manifolds, 1994, 39, 39-50.

[6] U. Bruzzo, B.G. Otero. Metrics on semistable and numerically effective Higgs bundles. J. Reine Angew. Math., 2007, 612, 59-79.

[7] U. Bruzzo, B.G. Otero. Approximate Hermitian-Yang-Mills structures on semistable principal Higgs bundles. Ann. Global Anal. Geom., 2015, 47(1), 1-11.

[8] S.A.H. Cardona. Approximate Hermitian-Yang-Mills structures and semistability for Higgs bundles. I: generalities and the one-dimensional case. Ann. Global Anal. Geom., 2012, 42(3), 349-370.

[9] S.K. Donaldson. Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. Proc. London Math. Soc., 1985, 50(3), 1-26.

[10] S.K. Donaldson. Infinite determinants, stable bundles and curvature. Duke Math. J., 1987, 54, 231-247.

[11] D. Gilbarg and N.S. Trudinger. Elliptic partial differential equations of second order. Classics in Mathematics. Reprint of the 1998 edition. Berlin: Springer- Verlag, 2001.

[12] G.-Y. Guo. Yang-Mills fields on cylindrical manifolds and holomorphic bundles. I, II. Comm. Math. Phys., 1996, 179, 737-775, 777-788.
[13] N.J. Hitchin. The self-duality equations on a Riemann surface. Proc. London Math. Soc. 1987, 55(3), 59C126.
[14] M.C. Hong. Heat flow for the Yang-Mills-Higgs field and the Hermitian Yang-Mills-Higgs metric. Ann. Global Anal. Geom., 2001, 20(1), 23-46.
[15] M. Haskins, H.-J. Hein, and J. Nordstrom. Asymptotically cylindrical Calabi-Yau manifolds. J. Differ. Geom. 2015, 101(2), 213-265.
[16] A. Jacob, T. Walpuski. Hermitian Yang-Mills metrics on reflexive sheaves over asymptotically cylindrical Kähler manifolds. arXiv:1603.07702, 2016.
[17] T. Mochizuki. Kobayashi-Hitchin correspondence for tame harmonic bundles and an application. Astérisque 309, 117 (2006).
[18] J.Y. Li, X. Zhang. Existence of approximate Hermitian-Einstein structures on semi-stable Higgs bundles. Calc. Var., 2015, 52(3-4), 783-795.
[19] J.Y. Li, C. Zhang, X. Zhang. Semi-stable Higgs sheaves and Bogomolov type inequality. Calc. Var., 2017, 56, 1-33.
[20] R.B. Lockhart and R.C. McOwen. Elliptic differential operators on noncompact manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1985, 12, 409-447.
[21] M. Lübke and A. Teleman. The universal Kobayashi-Hitchin correspondence on Hermitian manifolds. Mem. Amer. Math. Soc. 183.863 (2006).
[22] M. Lübke and A. Teleman. The Kobayashi-Hitchin correspondence. World Scientific Publishing Co., Inc., River Edge, NJ, 1995, pp. x+254.
[23] D. McDuff and D. Salamon. J-holomorphic curves and symplectic topology. Second. Vol. 52. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2012.
[24] Y. Nie, X. Zhang. Semistable Higgs bundles over compact Gauduchon manifolds. J. Geom. Anal., 2017, DOI: 10.1007/s12220-017-9835-y.
[25] L. Ni. The Poisson equation and Hermitian-Einstein metrics on holomorphic vector bundles over complete noncompact Kähler manifolds. Indiana Univ. Math. J. 2002, 51, 679-704.
[26] L. Ni and H. Ren. Hermitian-Einstein metrics for vector bundles on complete Kähler manifolds. Trans. Amer. Math. Soc., 2001, 353, 441-456.
[27] M.S. Narasimhan and C.S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. Ann. Math., 1965, 82, 540-567.
[28] B. Owens. Instantons on cylindrical manifolds and stable bundles. Geom. Topol., 2001, 5, 761-797.
[29] H.N. Sá Earp. G2-instantons over asymptotically cylindrical manifolds. Geom. Topol., 2015, 19, 61-111.
[30] C.T. Simpson. Constructing variations of Hodge structure using Yang- Mills theory and applications to uniformization. J. Amer. Math. Soc., 1998, 1, 867-918.
[31] K.K. Uhlenbeck and S.-T. Yau. On the existence of Hermitian-Yang-Mills connections in stable vector bundles. Comm. Pure Appl. Math. 39.S, suppl. (1986). Frontiers of the mathematical sciences: 1985 (New York, 1985), S257-S293.
[32] Y.-T. Siu. Lectures on Hermitian-Einstein metrics for stable bundles and Kahler-Einstein metrics. DMV Seminar 8, Birkhauser Verlag, Basel, 1987.
[33] X. Zhang. Hermitian Yang-Mills-Higgs metrics on complete Kähler manifolds. Canad. J. Math., 2005, 57, 871-896.

P. Zhang
School of Mathematical Sciences, University of Science and Technology of China

Anhui 230026, P.R. China
Email: panzhang@mail.ustc.edu.cn