New Infinite Product Representation for Cosine Function and Power Series Representation for Tangent Function

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"Enter into his gates with thanksgiving, and into his courts with praise: be thankful unto him, and bless his name." - Psalmos 100:4.

Abstract. In this paper, I demonstrate one new infinite product representation for cosine function, one new power series representation for tangent function and amazing identities involving radical.

1. Introduction

In this paper, I prove the new infinite product representation for cosine function given by

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left\{16 \cdot \frac{(2n-1)^4(n^4 - 5n^2z^2 + 4z^4)}{[(2n-1)^2 - z^2]^2(4n^2 - z^2)^2}\right\},$$

the power series representation for tangent function

$$\tan(\pi z) = \frac{2z}{\pi} \sum_{n=1}^{\infty} \left[\frac{64n^6 - 72n^5 + n^4(18 - 84z^2) + 108n^3z^2 - 3n^2z^2(5z^2 + 9) + 8z^6}{(4n^2 - z^2)(4n^2 - 4n - z^2 + 1)(n^4 - 5n^2z^2 + 4z^4)}\right]$$

and identity

$$\sqrt{5 + 2\sqrt{5}} = \sqrt{5 - 2\sqrt{5}} + 3\sqrt{1 - \frac{2}{\sqrt{5}}} + \sqrt{1 + \frac{2}{\sqrt{5}}}$$

among others.

2. Cosine Function: the Infinite Product

2.1. New Infinite Product Representation for Cosine Function.

Theorem 1. *If* $|z| \leqslant \frac{1}{2}$, *then*

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left\{16 \cdot \frac{(2n-1)^4(n^4 - 5n^2z^2 + 4z^4)}{[(2n-1)^2 - z^2]^2(4n^2 - z^2)^2}\right\},$$

where $\cos(z)$ *denotes the cosine function.*

Proof. In [1, p.12], I have the Euler's infinite product representation for sine function

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right), z \in \mathbb{C}. \tag{1}$$

In [1, p.13], again, I have the Euler's infinite product representation for cosine function

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{(2n-1)^2}\right), z \in \mathbb{C}. \tag{2}$$

I well know the trigonometric identity

$$\cos(\pi z) = \frac{1}{8} \frac{\sin(\pi z)\sin(2\pi z)}{\cos^2(\frac{\pi z}{2})\sin^2(\frac{\pi z}{2})}. \tag{3}$$

From (1), (2) and (3), it follows that

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left\{ 16 \cdot \frac{(2n-1)^4(n^4 - 5n^2z^2 + 4z^4)}{[(2n-1)^2 - z^2]^2(4n^2 - z^2)^2} \right\},$$

which is the desired result. □

Example 2. Using the above formula, I obtain the identities, for $z = 1/20, 9/20, 1/30$ and $3/40$, as follows,

$$\frac{1}{2}\sqrt{\frac{1}{2}\left(4 + \sqrt{2(5+\sqrt{5})}\right)} = \frac{\sqrt{5}-1}{\sqrt{2}+\sqrt{10-2\sqrt{5-\sqrt{5}}}},$$

$$\frac{1}{2}\sqrt{\frac{1}{2}\left(4 - \sqrt{2(5+\sqrt{5})}\right)} = \frac{\sqrt{5}-1}{\sqrt{2}+\sqrt{10+2\sqrt{5-\sqrt{5}}}},$$

$$\frac{1}{4}\sqrt{7+\sqrt{5}+\sqrt{6(5+\sqrt{5})}} = \frac{\sqrt{15}-\sqrt{3}-\sqrt{2(5+\sqrt{15})}}{2\left[1+\sqrt{5}-\sqrt{6(5-\sqrt{5})}\right]}$$

and

$$\frac{1}{2}\sqrt{\frac{1}{2}\left(4+\sqrt{2\left(4+\sqrt{2(5-\sqrt{5})}\right)}\right)} = \frac{\sqrt{5}-\sqrt{2(5+\sqrt{5})}-1}{\sqrt{4-2\sqrt{2}}+\sqrt{10(2-\sqrt{2})}-2\sqrt{(2+\sqrt{2})(5-\sqrt{5})}}.$$

3. TANGENT FUNCTION: THE POWER SERIES

3.1. New Power Series Representation for Tangent Function.

Theorem 3. *If $z \in \mathbb{C}$, then*

$$\tan(\pi z) = \frac{2z}{\pi}\sum_{n=1}^{\infty}\left[\frac{64n^6 - 72n^5 + n^4(18-84z^2) + 108n^3z^2 - 3n^2z^2(5z^2+9) + 8z^6}{(4n^2-z^2)(4n^2-4n-z^2+1)(n^4-5n^2z^2+4z^4)}\right],$$

where $\tan(z)$ *denotes the tangent function.*

Proof. Differentiating the equation of the Theorem 2 logarithmically with respect to z, I have the desired result. □

Example 4. Using the above formula, I obtain the identities, for $z = 1/5$ and $1/8$, as follows,

$$\sqrt{5+2\sqrt{5}} = \sqrt{5-2\sqrt{5}} + 3\sqrt{1-\frac{2}{\sqrt{5}}} + \sqrt{1+\frac{2}{\sqrt{5}}}$$

and

$$2 = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} - \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} - \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} - \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}.$$

4. INFINITE PRODUCTS FOR TRIGONOMETRIC AND HYPERBOLIC FUNCTIOSN

4.1. Infinite Products Representations for Cosine and Hyperbolic Cosine Functions.

I leave as easy exercises

Exercise 1. Prove that, for $z \in \mathbb{C}$,

$$\cos(\pi z) = \prod_{n=1}^{\infty} \frac{n^2 - 4z^2}{n^2 - z^2}, \tag{4}$$

$$\cos(\pi z)\sec\left(\frac{\pi z}{2}\right) = \prod_{n=1}^{\infty} \frac{(2n-1)^2 - 4z^2}{(2n-1)^2 - z^2}, \tag{5}$$

$$\frac{1 + 2\cos(2\pi z)}{3} = \prod_{n=1}^{\infty} \frac{n^2 - 9z^2}{n^2 - z^2}, \tag{6}$$

$$\frac{\sin(4\pi z)\csc(\pi z)}{4} = \prod_{n=1}^{\infty} \frac{n^2 - 16z^2}{n^2 - z^2}, \tag{7}$$

$$\cosh(\pi z) = \prod_{n=1}^{\infty} \frac{n^2 + 4z^2}{n^2 + z^2}, \tag{8}$$

$$\cosh(\pi z)\,\mathrm{sech}\left(\frac{\pi z}{2}\right) = \prod_{n=1}^{\infty} \frac{(2n-1)^2 + 4z^2}{(2n-1)^2 + z^2}, \tag{9}$$

$$\frac{1 + 2\cosh(2\pi z)}{3} = \prod_{n=1}^{\infty} \frac{n^2 + 9z^2}{n^2 + z^2}, \tag{10}$$

$$\frac{\sinh(4\pi z)\,\mathrm{csch}(\pi z)}{4} = \prod_{n=1}^{\infty} \frac{n^2 + 16z^2}{n^2 + z^2}, \tag{11}$$

$$\cos(\pi z) = \prod_{n=1}^{\infty} \left[\frac{n^2 - 16z^2}{n^2 - 4z^2} \cdot \frac{(2n-1)^2 - 4z^2}{(2n-1)^2 - 16z^2}\right] \tag{12}$$

and

$$\cosh(\pi z) = \prod_{n=1}^{\infty} \left[\frac{n^2 + 16z^2}{n^2 + 4z^2} \cdot \frac{(2n-1)^2 + 4z^2}{(2n-1)^2 + 16z^2}\right]. \tag{13}$$

Reference

[1] Remmert, Reinhold, *Classical Topics in Complex Function*, Graduate Texts in Mathematics, 172, Springer-Verlag, New York, 1998.