New Proof of the Infinite Product Representation for Gamma Function and Pochhammer's Symbol and New Infinite Product Representation for Binomial Coefficient

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"It is the spirit that quickeneth; the flesh profiteth nothing: the words that I speak unto you, they are spirit, and they are life." - John 6:63.

Abstract. In this paper, we demonstrate some limit's formulae for gamma function and binomial coefficient among other things.

1. Introduction

Each mathematician looks at a function and sees in his own way. Leonhard Euler (1707-1783) contemplated the gamma function, and gave the infinite product expansion [1, p. 33]

$$\Gamma(z) = \frac{1}{z}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left(1+\frac{z}{j}\right)^{-1} \tag{1}$$

which is valid in $\mathbb{C}$, except for $z \in \{0, -1, -2, ...\}$.

Carl Friedrich Gauss (1777-1855) rewrote the Euler's product as

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z \cdot n!}{z(z+1)(z+2)\cdot \ldots \cdot (z+n)}, \tag{2}$$

see [2].

In 1854, Karl Weierstrass (1815-1897) gave the infinte product expansion for gamma function [1, p. 34-35]

$$\Gamma(z) = z e^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-z/j}, \tag{3}$$

which is valid for all $\mathbb{C}$.

Hence, the question: how do we see the gamma function? The answer: the wonderful limit's formula

$$\Gamma(n+1) = \lim_{k\to\infty}\left(\frac{k}{n+k}\right)^{n}\left(\frac{n+k}{k}\right)_{n}. \tag{4}$$

From this formula, we derive the a proof for the representation of infinite product of the gamma function and the binomial coefficient. In addition, we found the limit's formula for the coefficient binomial

$$\binom{z}{n} = \lim_{k\to\infty}\frac{\left(\frac{\ell}{k}+z\right)\left(\frac{\ell}{k}+z+1\right)_{n-1}}{\left(\frac{\ell}{k}+n\right)\left(\frac{\ell}{k}+n+1\right)_{n-1}},$$

among other things, such as the new infinite product representation for binomial coefficient, given by

$$\binom{z}{n} = \frac{z}{n}\prod_{j=1}^{\infty}\left(1+\frac{n-1}{j+n}\right)\left(1+\frac{n-1}{j+z}\right).$$

2. Some Lemmas

Lemma 1. *If n is an integer nonnegative, then*

$$\Gamma(n+1)=\lim_{k\to\infty}\left(\frac{k}{n+k}\right)^n\left(\frac{n+k}{k}\right)_n,$$

where $\Gamma(z)$ denotes the gamma function.

Proof. In elementary calculus, we well-know the identity

$$\lim_{k\to\infty}\frac{\ell+ak}{\ell+bk}=\frac{a}{b}, \tag{5}$$

The definition for gamma function [3], give us

$$n!=\prod_{r=1}^{n} r=\prod_{r=1}^{n}\frac{r}{1}. \tag{6}$$

Replaced a by r and b by 1 in (5)

$$\lim_{k\to\infty}\frac{\ell+rk}{\ell+k}=\frac{r}{1}, \tag{7}$$

Substitute the left hand side of (7) in the right hand side of (6)

$$\Gamma(n+1)=n!=\prod_{r=1}^{n}\lim_{k\to\infty}\frac{\ell+rk}{\ell+k}=\lim_{k\to\infty}\prod_{r=1}^{n}\frac{\ell+rk}{\ell+k}=\lim_{k\to\infty}\left(\frac{k}{n+k}\right)^n\left(\frac{n+k}{k}\right)_n,$$

which is the desired result. □

Lemma 2. *If n is an integer nonnegative, $z\in\mathbb{C}$ and ℓ is any number, then*

$$(z)_n=\lim_{k\to\infty}\left(\frac{k}{\ell+k}\right)^n\left(\frac{\ell}{k}+z\right)\left(\frac{\ell}{k}+z+1\right)_{n-1},$$

where $(z)_n$ denotes the Pochhammer symbol.

Proof. The definition for Pochhammer symbol [3], give us

$$(z)_n=\prod_{r=0}^{n-1}(z+r)=\prod_{r=0}^{n-1}\left(\frac{z+r}{1}\right). \tag{8}$$

Replaced a by $z+r$ and b by 1 in (5)

$$\lim_{k\to\infty}\frac{\ell+(z+r)\,k}{\ell+k}=\frac{z+r}{1}, \tag{9}$$

Substitute the left hand side of (9) in the right hand side of (8)

$$(z)_n=\prod_{r=0}^{n-1}\lim_{k\to\infty}\frac{\ell+(z+r)\,k}{\ell+k}=\lim_{k\to\infty}\prod_{r=0}^{n-1}\frac{\ell+(z+r)\,k}{\ell+k}=\lim_{k\to\infty}\left(\frac{k}{\ell+k}\right)^n\left(\frac{\ell}{k}+z\right)\left(\frac{\ell}{k}+z+1\right)_{n-1},$$

which is the desired result. □

Lemma 3. *If n is an integer nonnegative, $z\in\mathbb{C}$ and ℓ is any number, then*

$$\binom{z}{n}=\lim_{k\to\infty}\frac{\left(\frac{\ell}{k}+z\right)\left(\frac{\ell}{k}+z+1\right)_{n-1}}{\left(\frac{\ell}{k}+n\right)\left(\frac{\ell}{k}+n+1\right)_{n-1}},$$

where $\binom{z}{n}$ denotes the binomial coefficient.

Proof. The definition of the binomial coefficient [5], give us

$$\binom{z}{n}=\frac{(z)_n}{(n)_n}. \tag{10}$$

Usint the limit's formula of the Lemma 2 into (10), we obtain

$$\begin{aligned}\binom{z}{n}&=\frac{\lim\limits_{k\to\infty}\left(\frac{k}{\ell+k}\right)^n\left(\frac{\ell}{k}+z\right)\left(\frac{\ell}{k}+z+1\right)_{n-1}}{\lim\limits_{k\to\infty}\left(\frac{k}{\ell+k}\right)^n\left(\frac{\ell}{k}+n\right)\left(\frac{\ell}{k}+n+1\right)_{n-1}}\\&=\lim_{k\to\infty}\frac{\left(\frac{\ell}{k}+z\right)\left(\frac{\ell}{k}+z+1\right)_{n-1}}{\left(\frac{\ell}{k}+n\right)\left(\frac{\ell}{k}+n+1\right)_{n-1}},\end{aligned}$$

which is the desired result. □

3. Gamma Function: New Proof for the Infinite Product

3.1. Infinite Product Representation for Gamma Function.

Theorem 4. *(Euler, 1729) If $z\in\mathbb{C}-\{-1,-2,...\}$, then*

$$\Gamma(z)=\frac{1}{z}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^z\left(1+\frac{z}{j}\right)^{-1}$$

where $\Gamma(z)$ denotes the gamma function.

Proof. In [4], we have the infinite product for Pochhammer's symbol

$$(z)_n=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{n}{j+z-1}\right)^{-1}. \tag{11}$$

Replaced z by $(n+k)/k$ in (11)

$$\left(\frac{n+k}{k}\right)_n=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{nk}{jk+n+k-k}\right)^{-1}. \tag{12}$$

Substitute the right hand side of (12) in the right hand side of the Lemma 1 and encounter

$$\begin{aligned}\Gamma(n+1)&=\lim_{k\to\infty}\left(\frac{k}{n+k}\right)^n\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{nk}{jk+n}\right)^{-1}\\&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\lim_{k\to\infty}\left[\left(\frac{k}{n+k}\right)^n\left(1+\frac{nk}{jk+n}\right)^{-1}\right]\\&=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(\frac{j}{j+n}\right)=\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^n\left(1+\frac{n}{j}\right)^{-1},\end{aligned}$$

replaced n by z and use the identity $\Gamma(z+1)=z\Gamma(z)$, finding the desired result. □

4. Binomial Coefficient: New Infinite Product Representation

4.1. New Infinite Product Representation for Binomial Coefficient.

Theorem 5. *If $z\in\mathbb{C}-\{-1,-2,...\}$ and $n\in\mathbb{N}^+$, then*

$$\binom{z}{n}=\frac{z}{n}\prod_{j=1}^{\infty}\left(1+\frac{n-1}{j+n}\right)\left(1+\frac{n-1}{j+z}\right),$$

where $\binom{z}{n}$ denotes the binomial coefficient.

Proof. In [4], we have the infinite product for Pochhammer's symbol

$$(z)_n = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^n \left(1+\frac{n}{j+z-1}\right)^{-1}. \tag{13}$$

Replaced z by $\ell/k+z+1$ and n by $n-1$ in (13)

$$\left(\frac{\ell}{k}+z+1\right)_{n-1} = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{n-1} \left(1+\frac{(n-1)k}{jk+zk+\ell}\right)^{-1} \tag{14}$$

and replaced z by $\ell/k+n+1$ and n by $n-1$ in (13)

$$\left(\frac{\ell}{k}+n+1\right)_{n-1} = \prod_{j=1}^{\infty} \left(1+\frac{1}{j}\right)^{n-1} \left(1+\frac{(n-1)k}{jk+nk+\ell}\right)^{-1}. \tag{15}$$

Substitute the right hand side of (14) and (15) in the right hand side of the Lemma 3 and encounter

$$\begin{aligned}\binom{z}{n} &= \lim_{k\to\infty} \frac{\left(\frac{\ell}{k}+z\right)}{\left(\frac{\ell}{k}+n\right)} \prod_{j=1}^{\infty} \frac{\left(1+\frac{(n-1)k}{jk+zk+\ell}\right)^{-1}}{\left(1+\frac{(n-1)k}{jk+nk+\ell}\right)^{-1}} \\ &= \prod_{j=1}^{\infty} \lim_{k\to\infty} \left[\frac{\left(\frac{\ell}{k}+z\right)}{\left(\frac{\ell}{k}+n\right)} \cdot \frac{\left(1+\frac{(n-1)k}{jk+zk+\ell}\right)^{-1}}{\left(1+\frac{(n-1)k}{jk+nk+\ell}\right)^{-1}}\right] \\ &= \frac{z}{n} \prod_{j=1}^{\infty} \left(1+\frac{n-1}{j+n}\right)\left(1+\frac{n-1}{j+z}\right),\end{aligned}$$

which is the desired result. □

5. Gamma Function: Other Proof for the Infinite Product

5.1. Infinite Product Representation for Gamma Function.

Lemma 6. *If $a, b \in \mathbb{R}$ and $b \neq 0$, then*

$$\frac{a}{b} = \prod_{k=0}^{\infty} \frac{(k+2)(a+bk)}{(k+1)(a+b+bk)}.$$

Proof. In previous paper [6] the first author proved the integral representation for natural logarithm, for $\Re(z) > 0$,

$$\begin{aligned}\frac{\ln z}{z-1} &= \int_0^{\infty} \frac{\mathrm{d}x}{(z+x)(1+x)} = \sum_{k=0}^{\infty} \int_k^{k+1} \frac{\mathrm{d}x}{(z+x)(1+x)} \\ &= \frac{1}{z-1} \sum_{k=0}^{\infty} \ln \frac{(k+2)(k+z)}{(k+1)(k+z+1)} \\ &= \frac{1}{z-1} \ln \prod_{k=0}^{\infty} \frac{(k+2)(k+z)}{(k+1)(k+z+1)} \\ \Rightarrow \ln z &= \ln \prod_{k=0}^{\infty} \frac{(k+2)(k+z)}{(k+1)(k+z+1)}.\end{aligned} \tag{16}$$

The exponentiation of (16), give us

$$z = \prod_{k=0}^{\infty} \frac{(k+2)(k+z)}{(k+1)(k+z+1)}. \tag{17}$$

Replaced z by a/b in (17)

$$\frac{a}{b} = \prod_{k=0}^{\infty} \frac{(k+2)(a+bk)}{(k+1)(a+b+bk)},$$

which is the desired result. □

Theorem 7. *(Euler, 1729) If* $z \in \mathbb{C} - \{-1,-2,...\}$, *then*

$$\Gamma(z) = \frac{1}{z}\prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{z}\left(1+\frac{z}{j}\right)^{-1}$$

where $\Gamma(z)$ *denotes the gamma function.*

Proof. Replaced a by r and b by 1 in the Lemma 6

$$\frac{r}{1} = \prod_{k=0}^{\infty}\frac{(k+2)(r+k)}{(k+1)(r+k+1)}. \tag{18}$$

Substitute the right hand side of (18) into the right hand side of (6)

$$\begin{aligned} n! &= \prod_{r=1}^{n}\prod_{k=0}^{\infty}\frac{(k+2)(r+k)}{(k+1)(r+k+1)} = \prod_{k=0}^{\infty}\prod_{r=1}^{n}\frac{(k+2)(r+k)}{(k+1)(r+k+1)} \\ &= \prod_{k=0}^{\infty}\left(\frac{k+2}{k+1}\right)^{n}\left(\frac{1+k}{1+k+n}\right) = \prod_{k=1}^{\infty}\left(\frac{k+1}{k}\right)^{n}\left(\frac{k}{k+n}\right) \\ &= \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{n}\left(1+\frac{n}{k}\right)^{-1}, \end{aligned}$$

replaced n by z, k by j and use the identity $\Gamma(z+1) = z\Gamma(z)$, finding the desired result. □

6. Pochhammer Symbol: Other Proof for Infinite Product Representation

6.1. Other Proof for Infinite Product Representation for Pochhammer Symbol.

Theorem 8. *(Guedes, 2016 [4]) If* $z \in \mathbb{C} - \{-1,-2,...\}$ *and* $n \in \mathbb{N}^{+}$, *then*

$$(z)_n = \prod_{j=1}^{\infty}\left(1+\frac{1}{j}\right)^{n}\left(1+\frac{n}{j+z-1}\right)^{-1},$$

where $(z)_n$ *denotes the Pochhammer symbol.*

Proof. The definition for Pochhammer symbol [3], give us

$$(z)_n = \prod_{r=0}^{n-1}(z+r) = \prod_{r=0}^{n-1}\left(\frac{z+r}{1}\right). \tag{19}$$

Replaced a by $z+r$ and b by 1 in the Lemma 6

$$\frac{z+r}{1} = \prod_{k=0}^{\infty}\frac{(k+2)(z+r+k)}{(k+1)(z+r+k+1)} \tag{20}$$

Substitute the right hand side of (20) in the right hand side of the (19) and encounter

$$\begin{aligned} (z)_n &= \prod_{r=0}^{n-1}\prod_{k=0}^{\infty}\frac{(k+2)(z+r+k)}{(k+1)(z+r+k+1)} \\ &= \prod_{k=0}^{\infty}\prod_{r=0}^{n-1}\frac{(k+2)(z+r+k)}{(k+1)(z+r+k+1)} \\ &= \prod_{k=0}^{\infty}\left(\frac{k+2}{k+1}\right)^{n}\left(\frac{k+z}{k+n+z}\right) \\ &= \prod_{k=0}^{\infty}\left(1+\frac{1}{k+1}\right)^{n}\left(1+\frac{n}{k+z}\right)^{-1} \\ &= \prod_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{n}\left(1+\frac{n}{k+z-1}\right)^{-1}, \end{aligned}$$

replaced k by j, finding the desired result. □

References

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