# GENERAL EXACT TETRAHEDRON ARGUMENT FOR THE FUNDAMENTAL LAWS OF CONTINUUM MECHANICS

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ABSTRACT. In this article, we give a general exact mathematical framework that all the fundamental relations and conservation equations of continuum mechanics can be derived based on it. We consider a general integral equation contains the parameters that act on the volume and the surface of the integral's domain. The idea is to determine how many local relations can be derived from this general integral equation and what these local relations are. After obtaining the general Cauchy lemma, we derive two other local relations by a new general exact tetrahedron argument. So, there are three local relations that can be derived from the general integral equation. Then we show that all the fundamental laws of continuum mechanics, including the conservation of mass, linear momentum, angular momentum, energy, and the entropy law, can be considered in this general framework. Applying the general three local relations to the integral form of the fundamental laws of continuum mechanics in this new framework leads to exact derivation of the mass flow, continuity equation, Cauchy lemma for traction vectors, existence of stress tensor, general equation of motion, symmetry of stress tensor, existence of heat flux vector, differential energy equation, and differential form of the Clausius-Duhem inequality for entropy law.

The general exact tetrahedron argument is an exact proof that removes all the challenges on derivation of the fundamental relations of continuum mechanics. In this proof, there is no approximate or limited process and all the parameters are exact point-based functions. Also, it gives a new understanding and a deep insight into the origins and the physics and mathematics of the fundamental relations and conservation equations of continuum mechanics. This general mathematical framework can be used in many branches of continuum physics and the other sciences.

## 1. Introduction

Is there a general exact framework that all of the fundamental relations and conservation equations of continuum mechanics can be derived in it?

Continuum mechanics is a subject that is the base of a wide range of phenomena and physical behaviors of the nature and industry such as fluid mechanics, solid mechanics, continuum thermodynamics, heat transfer, etc. The birth of modern continuum mechanics is the introduction of the traction vector in 1822 by Cauchy that describes the nature of forces on the internal surfaces of the substance [6]. He gave a proof that is called Cauchy tetrahedron argument for the existence of stress tensor. The other important Cauchy's achievements in the foundations of continuum mechanics include the

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symmetry of stress tensor and the general equation of motion [6, 7, 12]. During about two centuries, the scientists and authors in continuum mechanics presented some different proofs and processes to derive the fundamental relations and conservation equations of continuum mechanics, more generally and precisely [11], [10], [13], [9], [1], [4], [8], [5].

We already gave two articles on this subject. In the first one (2017, [2]), we provided a comprehensive review of the different tetrahedron arguments and the proofs of the existence of stress tensor, and discussed the challenges and improvements of each one. In the second article (2017, [3]), for the first time, we presented and proved the exact tetrahedron argument that removes all of the challenges on the previous tetrahedron arguments and the proofs of the existence of stress tensor. Exact tetrahedron argument led to derivation of both the relations for the existence of stress tensor and general equation of motion, simultaneously. Also, we compared the exact tetrahedron argument with the previous proofs of the existence of stress tensor. Exact tetrahedron argument gave us a new understanding and a deep insight into the physics and mathematics of the stress tensor, general equation of motion, and their origins [3].

In this article, we generalize the exact tetrahedron argument for all of the fundamental relations and conservation laws of continuum mechanics. We prove a general exact mathematical framework and consider the different fundamental laws of continuum mechanics in this framework. Then, we will show that this leads to the exact derivation of the relations for the mass flux, existence of stress tensor, symmetry of stress tensor, surface heat flux, entropy flux, and the differential form of fundamental conservation laws of mass, linear momentum, angular momentum, energy, and the entropy law.

Here we consider an integral equation over the control volume $\mathcal{M}$ as the form:

$$\int_{\mathcal{M}} B\, dV = \int_{\partial\mathcal{M}} \phi\, dS \tag{1.1}$$

In general, $B = B(\boldsymbol{r}, t)$ is called body term and acts over the volume of $\mathcal{M}$, and $\phi = \phi(\boldsymbol{r}, t, \boldsymbol{n})$ is called surface term and acts on the surface of $\mathcal{M}$, i.e., $\partial\mathcal{M}$. Where $\boldsymbol{r}$ is the position vector, $t$ is time, and $\boldsymbol{n}$ is the outward unit normal vector on the surface of the control volume. If $B$ is scalar then $\phi$ must be scalar, and if $B$ is vector then $\phi$ must be vector. These two functions are continuous over their domains. In the later, we will show that the integral form of all the fundamental laws of continuum mechanics can be written in the form of the integral equation (1.1).

We want to find how many local relations can be derived from this integral equation and what they are.

We use the Eulerian approach in the entire of this article, where a control volume is utilized and the changes of quantities are recorded as the effective parameters on the surface or volume of the control volume and the fluxes that pass through control volume surface.

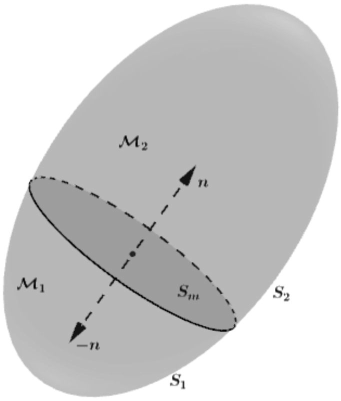


FIGURE 1. The control volumes $\mathcal{M}_1$ and $\mathcal{M}_2$, where $\partial\mathcal{M}_1 = S_1 \cup S_m$ and $\partial\mathcal{M}_2 = S_2 \cup S_m$, and the control volume $\mathcal{M}$ such that $V_{\mathcal{M}} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$ and $\partial\mathcal{M} = S_1 \cup S_2$.

## 2. General Cauchy lemma

Suppose the control volume $\mathcal{M}$ splits into $\mathcal{M}_1$ and $\mathcal{M}_2$ by the surface $S_m$. So, $V_{\mathcal{M}} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$, $\partial\mathcal{M}_1 = S_1 \cup S_m$, $\partial\mathcal{M}_2 = S_2 \cup S_m$, and $\partial\mathcal{M} = S_1 \cup S_2$, see Figure 1. If the integral equation (1.1) applies to $\mathcal{M}_1$ and $\mathcal{M}_2$, then the sum of these equations is:

$$\int_{\mathcal{M}_1} B_1\, dV + \int_{\mathcal{M}_2} B_2\, dV = \int_{\partial\mathcal{M}_1} \phi_1\, dS + \int_{\partial\mathcal{M}_2} \phi_2\, dS$$

By $V_{\mathcal{M}} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$, the sum of the body term integrals is equal to the integral of the body term on $\mathcal{M}$. In addition, by $\partial\mathcal{M}_1 = S_1 \cup S_m$ and $\partial\mathcal{M}_2 = S_2 \cup S_m$, the surface integrals split as:

$$\int_{\mathcal{M}} B\, dV = \int_{S_1} \phi_1\, dS + \int_{S_m} \phi_1\, dS + \int_{S_2} \phi_2\, dS + \int_{S_m} \phi_2\, dS$$

By $\partial\mathcal{M} = S_1 \cup S_2$, the sum of the surface integrals on $S_1$ and $S_2$ is equal to the surface integral of $\phi$ on $\partial\mathcal{M}$, so:

$$\int_{\mathcal{M}} B\, dV = \int_{\partial\mathcal{M}} \phi\, dS + \int_{S_m} \phi_1\, dS + \int_{S_m} \phi_2\, dS$$

Comparing this integral equation with the general integral equation (1.1), implies that:

$$\int_{S_m} \phi_1\, dS + \int_{S_m} \phi_2\, dS = 0$$

But $\phi_1$ on $S_m$ is $\phi(\boldsymbol{r}, t, \boldsymbol{n})$, and $\phi_2$ on $S_m$ is $\phi(\boldsymbol{r}, t, -\boldsymbol{n})$, so:

$$\int_{S_m} \big\{\phi(\boldsymbol{r}, t, \boldsymbol{n}) + \phi(\boldsymbol{r}, t, -\boldsymbol{n})\big\}\, dS = 0$$

therefore, we have

$$\phi(\boldsymbol{r}, t, \boldsymbol{n}) = -\phi(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{2.1}$$

This is the first local relation that is derived from the integral equation (1.1), and is called general Cauchy lemma. It states *"the surface terms acting on opposite sides of the same surface at a given point and time are equal in magnitude but opposite in sign"*.

It means that if we have the surface term on one side of a surface at a given point and time, then we can get the surface term on the other side of this surface at that point and time by the equation (2.1).

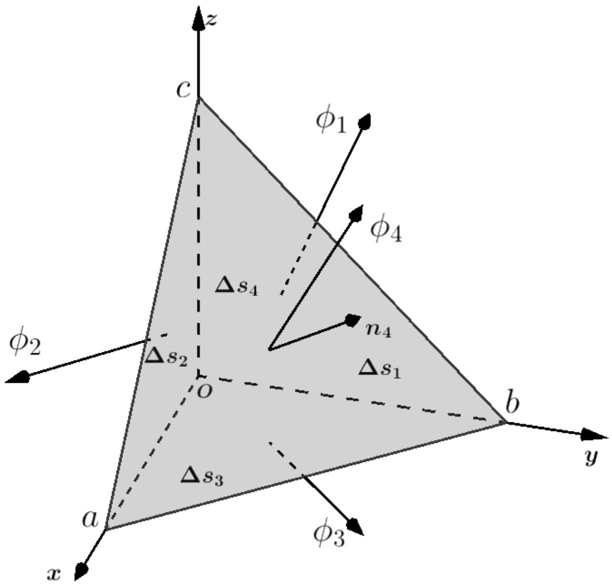


FIGURE 2. The geometry of tetrahedron control volume and the exact surface term vectors on the faces. Note that in this figure we suppose that the surface term is a vector, but in general it can be a scalar or a vector, depending on the type of the body term.

## 3. General exact tetrahedron argument

Consider a tetrahedron control volume in continuum media that its vortex is at the point $\boldsymbol{o}$ and its three orthogonal faces are parallel to the three orthogonal planes of the Cartesian coordinate system. The fourth surface of the tetrahedron, i.e., its base, has the outward unit normal vector $\boldsymbol{n}_4$. For simplicity, the vortex point is at the origin of the coordinate system. The geometrical parameters are shown in Figure 2. The vector $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ is the position vector from the origin of the coordinate system. Applying the general integral equation (1.1) to the tetrahedron control volume leads to:

$$\int_{\Delta s_4} \phi_4 \, dS + \int_{\Delta s_1} \phi_1 \, dS + \int_{\Delta s_2} \phi_2 \, dS + \int_{\Delta s_3} \phi_3 \, dS = \int_{\mathcal{M}} B \, dV \tag{3.1}$$

The key idea of this proof is to write the variables of this equation in terms of the exact Taylor series about a point in the domain. Here, we derive these series about the vortex point of tetrahedron (point $\boldsymbol{o}$), where the three orthogonal faces pass through it. Note that time $(t)$ is the same in the all terms, so it does not exist in the Taylor series. For $B(\boldsymbol{r}, t)$ at any point in the domain of the control volume, we have:

$$\begin{aligned} B &= B_o + \frac{\partial B_o}{\partial x}x + \frac{\partial B_o}{\partial y}y + \frac{\partial B_o}{\partial z}z \\ &+ \frac{1}{2!}\Big(\frac{\partial^2 B_o}{\partial x^2}x^2 + \frac{\partial^2 B_o}{\partial y^2}y^2 + \frac{\partial^2 B_o}{\partial z^2}z^2 + 2\frac{\partial^2 B_o}{\partial x \partial y}xy + 2\frac{\partial^2 B_o}{\partial x \partial z}xz + 2\frac{\partial^2 B_o}{\partial y \partial z}yz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!n!k!} \frac{\partial^{(m+n+k)} B}{\partial x^m \partial y^n \partial z^k}\Big|_o x^m y^n z^k \end{aligned} \tag{3.2}$$

Here $B_o$ and $\partial B_o/\partial x$ are the exact values of $B$ and $\partial B/\partial x$ at the point $\boldsymbol{o}$, respectively. Similarly, the other derivatives are the exact values of the corresponding derivatives of

$B$ at the point $\boldsymbol{o}$. On the surface $\Delta s_1$, $x = 0$ and $\boldsymbol{n}_1$ does not change, so:

$$\begin{aligned}\phi_1 = \phi_{1_o} + \frac{\partial \phi_{1_o}}{\partial y} y + \frac{\partial \phi_{1_o}}{\partial z} z + \frac{1}{2!}\Big(\frac{\partial^2 \phi_{1_o}}{\partial y^2} y^2 + \frac{\partial^2 \phi_{1_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \phi_{1_o}}{\partial y \partial z} yz\Big) \\ + \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)}\phi_1}{\partial y^m \partial z^k}\Big|_o y^m z^k\end{aligned} \tag{3.3}$$

where $\phi_{1_o}$ is the exact value of $\phi_1$ on $\Delta s_1$ at the point $\boldsymbol{o}$. On the surface $\Delta s_2$, $y = 0$ and $\boldsymbol{n}_2$ does not change, and on the surface $\Delta s_3$, $z = 0$ and $\boldsymbol{n}_3$ does not change, so:

$$\begin{aligned}\phi_2 = \phi_{2_o} + \frac{\partial \phi_{2_o}}{\partial x} x + \frac{\partial \phi_{2_o}}{\partial z} z + \frac{1}{2!}\Big(\frac{\partial^2 \phi_{2_o}}{\partial x^2} x^2 + \frac{\partial^2 \phi_{2_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \phi_{2_o}}{\partial x \partial z} xz\Big) \\ + \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)}\phi_2}{\partial x^m \partial z^k}\Big|_o x^m z^k\end{aligned} \tag{3.4}$$

$$\begin{aligned}\phi_3 = \phi_{3_o} + \frac{\partial \phi_{3_o}}{\partial x} x + \frac{\partial \phi_{3_o}}{\partial y} y + \frac{1}{2!}\Big(\frac{\partial^2 \phi_{3_o}}{\partial x^2} x^2 + \frac{\partial^2 \phi_{3_o}}{\partial y^2} y^2 + 2\frac{\partial^2 \phi_{3_o}}{\partial x \partial y} xy\Big) \\ + \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)}\phi_3}{\partial x^m \partial y^k}\Big|_o x^m y^k\end{aligned} \tag{3.5}$$

Similarly, $\phi_{2_o}$ and $\phi_{3_o}$ are the exact values of $\phi_2$ and $\phi_3$ at the point $\boldsymbol{o}$ on $\Delta s_2$ and $\Delta s_3$, respectively. For the surface term on $\Delta s_4$ a more explanation is needed. The surface term on $\Delta s_4$ expands based on the surface term on the inclined surface that is parallel to $\Delta s_4$ and passes through the vortex point of tetrahedron (point $\boldsymbol{o}$). Because the unit normal vectors of these two surfaces are the same, see Figure 3. Therefore:

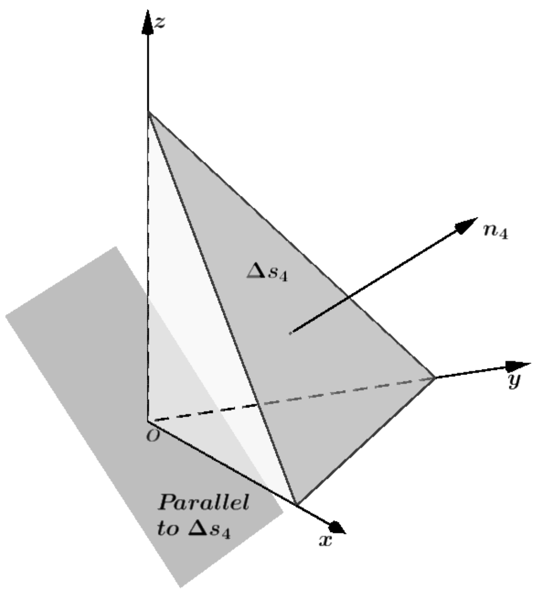


FIGURE 3. Inclined surface that is parallel to $\Delta s_4$ and passes through point $\boldsymbol{o}$.

$$
\begin{aligned}
\phi_4 &= \phi_{4_o} + \frac{\partial \phi_{4_o}}{\partial x}x + \frac{\partial \phi_{4_o}}{\partial y}y + \frac{\partial \phi_{4_o}}{\partial z}z \\
&+ \frac{1}{2!}\Big(\frac{\partial^2 \phi_{4_o}}{\partial x^2}x^2 + \frac{\partial^2 \phi_{4_o}}{\partial y^2}y^2 + \frac{\partial^2 \phi_{4_o}}{\partial z^2}z^2 + 2\frac{\partial^2 \phi_{4_o}}{\partial x \partial y}xy + 2\frac{\partial^2 \phi_{4_o}}{\partial x \partial z}xz + 2\frac{\partial^2 \phi_{4_o}}{\partial y \partial z}yz\Big) \\
&+ \ldots = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!n!k!}\frac{\partial^{(m+n+k)}\phi_4}{\partial x^m \partial y^n \partial z^k}\Big|_o x^m y^n z^k
\end{aligned} \tag{3.6}
$$

where $\phi_{4_o}$ is the exact surface term at the point $\boldsymbol{o}$ on the inclined surface with unit normal vector $\boldsymbol{n}_4$ that this surface passes exactly through point $\boldsymbol{o}$, the vertex point of tetrahedron control volume. Here $x$, $y$, and $z$ are the components of the position vector $\boldsymbol{r}$ on the surface $\Delta s_4$.

Note that $\phi_{1_o}$, $\phi_{2_o}$, $\phi_{3_o}$, and $\phi_{4_o}$ are the exact surface terms at the point $\boldsymbol{o}$ but on different surfaces with unit normal vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, and $\boldsymbol{n}_4$, respectively. The body term $B_o$ is exactly defined at the point $\boldsymbol{o}$. Therefore, all the surface terms and the body term with subscript $o$ and all their derivatives, such as $\partial^2\phi_{4_o}/\partial x \partial y$, are exactly defined at the point $\boldsymbol{o}$ and are bounded. As a result, for the convergence of the above Taylor series it is enough that we have $|\boldsymbol{r}| \leqslant 1$ in the domain of the control volume $\mathcal{M}$. But the scale of the coordinate system is arbitrary and we can define this scale such that the greatest distance in the domain of the control volume from the origin, is equal to one, i.e., $|\boldsymbol{r}|_{max} = 1$. By this scale, in the entire of the tetrahedron control volume we have $|\boldsymbol{r}| \leqslant 1$, that leads to the convergence condition for the above Taylor series.

Now all of the variables are prepared for integration in the integral equation (3.1). The integration of $B$ on the volume of $\mathcal{M}$:

$$
\begin{aligned}
\int_{\mathcal{M}} B\, dV &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} \Big\{ B_o + \frac{\partial B_o}{\partial x}x + \frac{\partial B_o}{\partial y}y + \frac{\partial B_o}{\partial z}z + \ldots \Big\}\, dx\, dy\, dz \\
&= \frac{1}{6}abc\Big\{B_o + \frac{1}{4}\Big(\frac{\partial B_o}{\partial x}a + \frac{\partial B_o}{\partial y}b + \frac{\partial B_o}{\partial z}c\Big) + \ldots\Big\}
\end{aligned} \tag{3.7}
$$

The integration of $\phi_4$ on $\Delta s_4$:

$$
\begin{aligned}
\int_{\Delta s_4} \phi_4\, dS &= \int_0^b \int_0^{a(1-\frac{y}{b})} \Big\{ \sqrt{\big(-\frac{c}{a}\big)^2 + \big(-\frac{c}{b}\big)^2 + 1}\Big( \phi_{4_o} + \frac{\partial \phi_{4_o}}{\partial x}x + \frac{\partial \phi_{4_o}}{\partial y}y \\
&+ \frac{\partial \phi_{4_o}}{\partial z}\big(c(1-\frac{x}{a}-\frac{y}{b})\big) + \frac{1}{2!}\Big(\frac{\partial^2 \phi_{4_o}}{\partial x^2}x^2 + \frac{\partial^2 \phi_{4_o}}{\partial y^2}y^2 + \frac{\partial^2 \phi_{4_o}}{\partial z^2}\big(c(1-\frac{x}{a}-\frac{y}{b})\big)^2 \\
&+ 2\frac{\partial^2 \phi_{4_o}}{\partial x \partial y}xy + 2\frac{\partial^2 \phi_{4_o}}{\partial x \partial z}x\big(c(1-\frac{x}{a}-\frac{y}{b})\big) + 2\frac{\partial^2 \phi_{4_o}}{\partial y \partial z}y\big(c(1-\frac{x}{a}-\frac{y}{b})\big)\Big) \\
&+ \ldots\Big)\Big\}\, dx\, dy \\
&= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}\Big\{\phi_{4_o} + \frac{1}{3}\Big(\frac{\partial \phi_{4_o}}{\partial x}a + \frac{\partial \phi_{4_o}}{\partial y}b + \frac{\partial \phi_{4_o}}{\partial z}c\Big) \\
&+ \frac{1}{12}\Big(\frac{\partial^2 \phi_{4_o}}{\partial x^2}a^2 + \frac{\partial^2 \phi_{4_o}}{\partial y^2}b^2 + \frac{\partial^2 \phi_{4_o}}{\partial z^2}c^2 + \frac{\partial^2 \phi_{4_o}}{\partial x \partial y}ab + \frac{\partial^2 \phi_{4_o}}{\partial x \partial z}ac + \frac{\partial^2 \phi_{4_o}}{\partial y \partial z}bc\Big) + \ldots\Big\}
\end{aligned} \tag{3.8}
$$

The integration of $\phi_1$ on $\Delta s_1$:

$$\begin{aligned}\int_{\Delta s_1} \phi_1\, dS &= \int_0^c \int_0^{b(1-\frac{z}{c})} \Big\{ \phi_{1_o} + \frac{\partial \phi_{1_o}}{\partial y} y + \frac{\partial \phi_{1_o}}{\partial z} z \\ &+ \frac{1}{2!}\Big( \frac{\partial^2 \phi_{1_o}}{\partial y^2} y^2 + \frac{\partial^2 \phi_{1_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \phi_{1_o}}{\partial y \partial z} yz \Big) + \dots \Big\}\, dy\, dz \\ &= \frac{1}{2} bc \Big\{ \phi_{1_o} + \frac{1}{3}\Big( \frac{\partial \phi_{1_o}}{\partial y} b + \frac{\partial \phi_{1_o}}{\partial z} c \Big) + \frac{1}{12}\Big( \frac{\partial^2 \phi_{1_o}}{\partial y^2} b^2 + \frac{\partial^2 \phi_{1_o}}{\partial z^2} c^2 + \frac{\partial^2 \phi_{1_o}}{\partial y \partial z} bc \Big) + \dots \Big\}\end{aligned} \tag{3.9}$$

The integration of $\phi_2$ on $\Delta s_2$ and $\phi_3$ on $\Delta s_3$ can be done, similarly. The geometrical relations for the area of faces and the volume of the tetrahedron are:

$$\begin{aligned} &\Delta s_1 = \frac{1}{2} bc, \qquad \Delta s_2 = \frac{1}{2} ac, \qquad \Delta s_3 = \frac{1}{2} ab \\ &\Delta s_4 = \frac{1}{2}\sqrt{a^2 b^2 + a^2 c^2 + b^2 c^2}, \qquad \Delta V = \frac{1}{6} abc \end{aligned} \tag{3.10}$$

By substituting the obtained relations for the surface terms and the body term into the equation (3.1) and using the above geometrical relations, we have:

$$\begin{aligned} &\Delta s_4 \Big\{ \phi_{4_o} + \frac{1}{3}\Big( \frac{\partial \phi_{4_o}}{\partial x} a + \frac{\partial \phi_{4_o}}{\partial y} b + \frac{\partial \phi_{4_o}}{\partial z} c \Big) \\ &\quad + \frac{1}{12}\Big( \frac{\partial^2 \phi_{4_o}}{\partial x^2} a^2 + \frac{\partial^2 \phi_{4_o}}{\partial y^2} b^2 + \frac{\partial^2 \phi_{4_o}}{\partial z^2} c^2 + \frac{\partial^2 \phi_{4_o}}{\partial x \partial y} ab + \frac{\partial^2 \phi_{4_o}}{\partial x \partial z} ac + \frac{\partial^2 \phi_{4_o}}{\partial y \partial z} bc \Big) + \dots \Big\} \\ &\quad + \Delta s_1 \Big\{ \phi_{1_o} + \frac{1}{3}\Big( \frac{\partial \phi_{1_o}}{\partial y} b + \frac{\partial \phi_{1_o}}{\partial z} c \Big) + \frac{1}{12}\Big( \frac{\partial^2 \phi_{1_o}}{\partial y^2} b^2 + \frac{\partial^2 \phi_{1_o}}{\partial z^2} c^2 + \frac{\partial^2 \phi_{1_o}}{\partial y \partial z} bc \Big) + \dots \Big\} \\ &\quad + \Delta s_2 \Big\{ \phi_{2_o} + \frac{1}{3}\Big( \frac{\partial \phi_{2_o}}{\partial x} a + \frac{\partial \phi_{2_o}}{\partial z} c \Big) + \frac{1}{12}\Big( \frac{\partial^2 \phi_{2_o}}{\partial x^2} a^2 + \frac{\partial^2 \phi_{2_o}}{\partial z^2} c^2 + \frac{\partial^2 \phi_{2_o}}{\partial x \partial z} ac \Big) + \dots \Big\} \\ &\quad + \Delta s_3 \Big\{ \phi_{3_o} + \frac{1}{3}\Big( \frac{\partial \phi_{3_o}}{\partial x} a + \frac{\partial \phi_{3_o}}{\partial y} b \Big) + \frac{1}{12}\Big( \frac{\partial^2 \phi_{3_o}}{\partial x^2} a^2 + \frac{\partial^2 \phi_{3_o}}{\partial y^2} b^2 + \frac{\partial^2 \phi_{3_o}}{\partial x \partial y} ab \Big) + \dots \Big\} \\ &\quad - \Delta V \Big\{ B_o + \frac{1}{4}\Big( \frac{\partial B_o}{\partial x} a + \frac{\partial B_o}{\partial y} b + \frac{\partial B_o}{\partial z} c \Big) + \dots \Big\} = 0 \end{aligned} \tag{3.11}$$

In the geometry of tetrahedron, $h$ is the height of the vertex $\boldsymbol{o}$ from the base face, i.e., $\Delta s_4$. So, we have the following geometrical relations for a tetrahedron with $\boldsymbol{n}_4 = n_x \boldsymbol{e}_x + n_y \boldsymbol{e}_y + n_z \boldsymbol{e}_z$, where $a$, $b$, and $c$ are greater than zero, see Figure 2.

$$\begin{aligned} &h = n_x a, \qquad h = n_y b, \qquad h = n_z c \\ &\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}, \qquad \Delta s_4 = \frac{abc}{2h} \\ &\Delta s_1 = n_x \Delta s_4, \qquad \Delta s_2 = n_y \Delta s_4, \qquad \Delta s_3 = n_z \Delta s_4 \\ &\Delta V = \frac{1}{6} abc = \frac{1}{3} h \Delta s_4 \end{aligned} \tag{3.12}$$

If we divide the equation (3.11) by $\Delta s_4$ and use the relations (3.12) for the faces and volume of the tetrahedron, then substitute the relations $a = h/n_x$, $b = h/n_y$, and

$c = h/n_z$ into the equation and rearrange it based on the powers of $h$, we have:

$$
\begin{aligned}
&\Big\{\phi_{4_o} + n_x\phi_{1_o} + n_y\phi_{2_o} + n_z\phi_{3_o}\Big\} \\
&\quad + \Big\{\Big(\frac{\partial\phi_{4_o}}{\partial x}\frac{1}{n_x} + \frac{\partial\phi_{4_o}}{\partial y}\frac{1}{n_y} + \frac{\partial\phi_{4_o}}{\partial z}\frac{1}{n_z}\Big) + n_x\Big(\frac{\partial\phi_{1_o}}{\partial y}\frac{1}{n_y} + \frac{\partial\phi_{1_o}}{\partial z}\frac{1}{n_z}\Big) \\
&\quad + n_y\Big(\frac{\partial\phi_{2_o}}{\partial x}\frac{1}{n_x} + \frac{\partial\phi_{2_o}}{\partial z}\frac{1}{n_z}\Big) + n_z\Big(\frac{\partial\phi_{3_o}}{\partial x}\frac{1}{n_x} + \frac{\partial\phi_{3_o}}{\partial y}\frac{1}{n_y}\Big) - B_o\Big\}\frac{1}{3}h \\
&\quad + \Big\{\Big(\frac{\partial^2\phi_{4_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2\phi_{4_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2\phi_{4_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2\phi_{4_o}}{\partial x\partial y}\frac{1}{n_x n_y} + \frac{\partial^2\phi_{4_o}}{\partial x\partial z}\frac{1}{n_x n_z} + \frac{\partial^2\phi_{4_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) \\
&\quad + n_x\Big(\frac{\partial^2\phi_{1_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2\phi_{1_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2\phi_{1_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) + n_y\Big(\frac{\partial^2\phi_{2_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2\phi_{2_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2\phi_{2_o}}{\partial x\partial z}\frac{1}{n_x n_z}\Big) \\
&\quad + n_z\Big(\frac{\partial^2\phi_{3_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2\phi_{3_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2\phi_{3_o}}{\partial x\partial y}\frac{1}{n_x n_y}\Big) - \Big(\frac{\partial B_o}{\partial x}\frac{1}{n_x} + \frac{\partial B_o}{\partial y}\frac{1}{n_y} + \frac{\partial B_o}{\partial z}\frac{1}{n_z}\Big)\Big\}\frac{1}{12}h^2 \\
&\quad + \ldots = 0
\end{aligned}
\tag{3.13}
$$

Note that by the coordinate system here and by $\Delta V \neq 0$, no one of $n_x$, $n_y$, and $n_z$ is exactly zero. So, all of the expressions in the braces {} of the equation (3.13) exist. We can rename the expressions in the braces and rewrite the equation as:

$$E_0 + E_1\frac{1}{3}h + E_2\frac{1}{12}h^2 + \ldots = 0 \tag{3.14}$$

If we continue to integrate the higher order derivatives of all terms based on their Taylor series, we have the following equation:

$$E_0 + E_1\frac{1}{3}h + E_2\frac{1}{12}h^2 + E_3\frac{1}{60}h^3 + \ldots + E_m\frac{2}{(m+2)!}h^m + \ldots = 0 \tag{3.15}$$

or

$$\sum_{m=0}^{\infty} E_m\frac{2}{(m+2)!}h^m = 0 \tag{3.16}$$

This is a great equation in the foundation of continuum mechanics. $E_0$, $E_1$, and $E_2$ are shown in the braces of the equation (3.13) and $E_3$ and other $E_m$'s will be presented. We now discuss some aspects of the equation (3.15):

- $E_m$'s are formed by the expressions of surface terms, body term and their derivatives, and the components of unit normal vector of the inclined surface.
- Each of the $E_m$'s exists, because the surface terms, body term and their derivatives are defined as continuous functions in continuum media and by the coordinate system here and by $\Delta V \neq 0$, no one of $n_x$, $n_y$, and $n_z$ is exactly zero.
- Each of the $E_m$'s depends on the variables at the point $\boldsymbol{o}$ and the components of unit normal vector of the inclined surface that is parallel to $\Delta s_4$ and passes through point $\boldsymbol{o}$. Because the surface terms, body term, and their derivatives are defined at the point $\boldsymbol{o}$.

- $E_m$'s do not depend on the volume of tetrahedron.
- $h$ is a geometrical variable and by the scale of the coordinate system on the tetrahedron control volume such that $|\boldsymbol{r}|_{max} \leqslant 1$, the altitude of the tetrahedron ($h$) is not greater than one.
- Note that $h = 0$ is not valid, because the general integral equation (1.1) is defined for the control volumes with nonzero volume.

By these properties, we return to the equation (3.15).

$$E_0 + E_1\frac{1}{3}h + E_2\frac{1}{12}h^2 + E_3\frac{1}{60}h^3 + \ldots + E_m\frac{2}{(m+2)!}h^m + \ldots = 0$$

We must find $E_m$'s. Since $E_m$'s are independent of $h$, the series on the left hand side is a power series. A power series is identically equal to zero if and only if all of its coefficients are equal to zero. Therefore:

$$E_m = 0, \qquad m = 0, 1, 2, \ldots, \infty \tag{3.17}$$

Note that these results are valid not only for $h \to 0$ but also for all values of $h$ in the domain. In other words, the results (3.17) are valid not only for an infinitesimal tetrahedron but also for any tetrahedron in the scaled coordinate system in continuum media. In addition, we have not done any approximate process during derivation of the equations (3.15) and (3.17). So, the results (3.17) hold exactly, not approximately.

Furthermore, the subscript $o$ in the expressions of $E_m$'s in the equation (3.13) indicates the vortex point of the tetrahedron. But any point in the domain in continuum media can be regarded as the vertex point of a tetrahedron and we could consider that tetrahedron. So, the point $\boldsymbol{o}$ can be any point in continuum media. We conclude that $E_m$'s are equal to zero at any point in continuum media. This implies that all their derivatives are equal to zero, as well. For example, we have for $E_0$:

$$\frac{\partial E_0}{\partial x} = \frac{\partial E_0}{\partial y} = \frac{\partial E_0}{\partial z} = 0 \tag{3.18}$$

and the other higher derivatives of $E_0$ are equal to zero. This trend holds for other $E_m$'s. But what are $E_m$'s?

For $E_0 = 0$, from the equation (3.13):

$$E_0 = \phi_{4_o} + n_x\phi_{1_o} + n_y\phi_{2_o} + n_z\phi_{3_o} = 0 \tag{3.19}$$

In this equation, the four surface terms are exactly defined at the point $\boldsymbol{o}$ on the surfaces that pass exactly through this point. The surface term $\phi_{1_o}$ is defined on the negative side of coordinate plane $yz$, i.e., $\boldsymbol{n}_1 = -1\boldsymbol{e}_x$, at the point $\boldsymbol{o}$. If $\phi_{x_o}$ is the surface term on the positive side of coordinate plane $yz$ at the point $\boldsymbol{o}$, then by the equation (2.1), i.e., $\phi(\boldsymbol{r}, t, \boldsymbol{n}) = -\phi(\boldsymbol{r}, t, -\boldsymbol{n})$, we have:

$$\phi_{1_o} = -\phi_{x_o} \tag{3.20}$$

Similarly, for $\phi_{2_o}$ and $\phi_{3_o}$:

$$\phi_{2_o} = -\phi_{y_o}, \qquad \phi_{3_o} = -\phi_{z_o} \tag{3.21}$$

By substituting these relations into (3.19) and rearranging it, we have:

$$\phi_{4_o} = n_{x4}\phi_{x_o} + n_{y4}\phi_{y_o} + n_{z4}\phi_{z_o} \tag{3.22}$$

where $n_{x4} = n_x$, $n_{y4} = n_y$, and $n_{z4} = n_z$. So, the surface term $\phi_{4_o}$ can be obtained by a linear relation between the surface terms on the three orthogonal planes and the components of its unit normal vector. But can we use the equation (3.22) for any unit normal vector rather than $\boldsymbol{n}_{4_o}$?

By considering the equations (3.11) and (3.13), we find that the equation (3.22) is really the following equation:

$$\phi_{4_o} = \frac{\Delta s_1}{\Delta s_4}\phi_{x_o} + \frac{\Delta s_2}{\Delta s_4}\phi_{y_o} + \frac{\Delta s_3}{\Delta s_4}\phi_{z_o} \tag{3.23}$$

and this equation is:

$$\phi_{4_o} = |n_{x4}|\phi_{x_o} + |n_{y4}|\phi_{y_o} + |n_{z4}|\phi_{z_o} \tag{3.24}$$

In Figure 2, by $a > 0$, $b > 0$, and $c > 0$, the components of unit normal vector on the inclined surface are greater than zero. So, the equation (3.22) is valid for these cases.

For the surfaces that their unit normal vector components are negative and are not zero, consider a tetrahedron control volume by the unit normal vector of its inclined surface (base face), $\boldsymbol{n}_{-4}$, that all of its components are negative. Therefore, we have $\boldsymbol{n}_{-4_o} = n_{x-4}\boldsymbol{e}_x + n_{y-4}\boldsymbol{e}_y + n_{z-4}\boldsymbol{e}_z = -n_x\boldsymbol{e}_x - n_y\boldsymbol{e}_y - n_z\boldsymbol{e}_z$, where $\boldsymbol{n}_{-4_o}$ is the outward unit normal vector of the surface that is parallel to the inclined surface and passes through the vortex point of this tetrahedron (point o), and $n_x$, $n_y$, and $n_z$ are positive values. Applying the process of exact tetrahedron argument to this new tetrahedron, leads to the following equation similar to the equation (3.19):

$$E_0 = \phi_{-4_o} + |n_{x-4}|\phi_{x_o} + |n_{y-4}|\phi_{y_o} + |n_{z-4}|\phi_{z_o} = 0 \tag{3.25}$$

As compared with the equation (3.19), in this equation we have $\phi_{x_o}$, $\phi_{y_o}$, and $\phi_{z_o}$ rather than $\phi_{1_o}$, $\phi_{2_o}$, and $\phi_{3_o}$, respectively. Because the outward sides of orthogonal faces of this new tetrahedron are in the positive directions of coordinate system. By the equation (3.25) and the components of $\boldsymbol{n}_{-4_o}$, we have:

$$\begin{aligned} \phi_{-4_o} &= -|n_{x-4}|\phi_{x_o} - |n_{y-4}|\phi_{y_o} - |n_{z-4}|\phi_{z_o} \\ &= -|-n_x|\phi_{x_o} - |-n_y|\phi_{y_o} - |-n_z|\phi_{z_o} \\ &= -n_x\phi_{x_o} - n_y\phi_{y_o} - n_z\phi_{z_o} \\ &= n_{x-4}\phi_{x_o} + n_{y-4}\phi_{y_o} + n_{z-4}\phi_{z_o} \end{aligned} \tag{3.26}$$

So, the surface term $\phi_{-4_o}$ can be obtained by a linear relation between the surface terms on the three orthogonal planes and the components of its unit normal vector. For the surfaces that one or two components of their unit normal vectors are negative but the other ones are not zero, the same process can be done.

For the other surfaces that one or two components of their unit normal vectors are equal to zero, the tetrahedron does not form, but due to the continuous property of the surface term on $\boldsymbol{n}$ and the arbitrary choosing for any orthogonal basis for the coordinate system, the surface terms on these surfaces can be described by the equation (3.22), as well. So, in general, the normal unit vector $\boldsymbol{n}_4$ can be related to any surface that passes through point $\boldsymbol{o}$ in three-dimensional continuum media. Thus, the subscript 4 removes from the equation (3.22) and we have for every $\boldsymbol{n} = n_x \boldsymbol{e}_x + n_y \boldsymbol{e}_y + n_z \boldsymbol{e}_z$:

$$\phi_o = n_x \phi_{x_o} + n_y \phi_{y_o} + n_z \phi_{z_o} \tag{3.27}$$

The subscript $o$ in this equation indicates the vortex point of the tetrahedron. But any point in the domain in continuum media can be the vertex point of a tetrahedron and we could consider this tetrahedron. So, the point $\boldsymbol{o}$ can be any point in continuum media and the subscript $o$ removes from the equation:

$$\phi = n_x \phi_x + n_y \phi_y + n_z \phi_z \tag{3.28}$$

or

$$\phi(\boldsymbol{r}, t, \boldsymbol{n}) = n_x \phi(\boldsymbol{r}, t, \boldsymbol{e}_x) + n_y \phi(\boldsymbol{r}, t, \boldsymbol{e}_y) + n_z \phi(\boldsymbol{r}, t, \boldsymbol{e}_z) \tag{3.29}$$

This is the second local relation that is derived from the general integral equation (1.1). It states that *"the surface term acting on any surface at a given point and time in the continuum domain can be obtained by a linear relation between the surface terms on the three orthogonal surfaces at that point and time and the components of the unit normal vector of the surface"*.

It means that if we have the surface terms on three orthogonal surfaces at a given point and time, then we can get the surface term on any surface that passes through that point at that time by using the unit normal vector of the surface and the linear relation (3.29).

In the next section, we will show that if $\phi(\boldsymbol{r}, t, \boldsymbol{n})$ is scalar then the equation (3.29) leads to the existence of a flux vector and if $\phi(\boldsymbol{r}, t, \boldsymbol{n})$ is vector then the equation (3.29) leads to the existence of a second order tensor.

Note that if we do not have the relation (2.1), i.e., the general Cauchy lemma, the equation (3.29) cannot be derived for every unit normal vector. Now the equation (3.29) contains the relation (2.1).

Let us see what $E_1 = 0$ tells.
From the equation (3.13):

$$\begin{aligned} E_1 = &\Big(\frac{\partial \phi_{4_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \phi_{4_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \phi_{4_o}}{\partial z}\frac{1}{n_z}\Big) + n_x\Big(\frac{\partial \phi_{1_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \phi_{1_o}}{\partial z}\frac{1}{n_z}\Big) \\ &+ n_y\Big(\frac{\partial \phi_{2_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \phi_{2_o}}{\partial z}\frac{1}{n_z}\Big) + n_z\Big(\frac{\partial \phi_{3_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \phi_{3_o}}{\partial y}\frac{1}{n_y}\Big) - B_o \end{aligned} \tag{3.30}$$

As previously stated, on the tetrahedron control volume with $\Delta V \neq 0$, no one of $n_x$, $n_y$, and $n_z$ is exactly zero. Therefore, $E_1$ exists. Furthermore, the unit normal vector $\boldsymbol{n}_4$ does not change on $\Delta s_4$, so:

$$\frac{\partial \boldsymbol{n}_4}{\partial x} = \frac{\partial \boldsymbol{n}_4}{\partial y} = \frac{\partial \boldsymbol{n}_4}{\partial z} = \boldsymbol{0} \tag{3.31}$$

Using the relations (3.31) and the equation (3.19), i.e., $\phi_{4_o} = E_0 - n_x\phi_{1_o} - n_y\phi_{2_o} - n_z\phi_{3_o}$, we have for (3.30):

$$E_1 = \frac{1}{n_x}\frac{\partial E_0}{\partial x} + \frac{1}{n_y}\frac{\partial E_0}{\partial y} + \frac{1}{n_z}\frac{\partial E_0}{\partial z} - \frac{\partial \phi_{1_o}}{\partial x} - \frac{\partial \phi_{2_o}}{\partial y} - \frac{\partial \phi_{3_o}}{\partial z} - B_o$$

If we define $E$ as:

$$E = -\frac{\partial \phi_{1_o}}{\partial x} - \frac{\partial \phi_{2_o}}{\partial y} - \frac{\partial \phi_{3_o}}{\partial z} - B_o \tag{3.32}$$

therefore, we have

$$E_1 = \frac{1}{n_x}\frac{\partial E_0}{\partial x} + \frac{1}{n_y}\frac{\partial E_0}{\partial y} + \frac{1}{n_z}\frac{\partial E_0}{\partial z} + E \tag{3.33}$$

But we saw in (3.18) that the derivatives of $E_0$ were equal to zero. So, from (3.33) and $E_1 = 0$, we have:

$$E_1 = E = 0 \tag{3.34}$$

By (3.32), $E$ is defined at the vertex point of tetrahedron. But as previously stated, the vertex point of the tetrahedron can be at any point in continuum media. Therefore, by (3.34), $E = 0$ at any point in continuum media. This implies that all derivatives of $E$ are equal to zero at any point in continuum media. So:

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial y} = \frac{\partial E}{\partial z} = 0 \tag{3.35}$$

By using the relations (3.20) and (3.21), i.e., $\phi_{1_o} = -\phi_{x_o}$, $\phi_{2_o} = -\phi_{y_o}$, and $\phi_{3_o} = -\phi_{z_o}$, the equation (3.32) becomes:

$$E = \frac{\partial \phi_{x_o}}{\partial x} + \frac{\partial \phi_{y_o}}{\partial y} + \frac{\partial \phi_{z_o}}{\partial z} - B_o \tag{3.36}$$

but $E = 0$, so

$$B_o = \frac{\partial \phi_{x_o}}{\partial x} + \frac{\partial \phi_{y_o}}{\partial y} + \frac{\partial \phi_{z_o}}{\partial z} \tag{3.37}$$

As explained earlier, we can remove the subscript $o$ from the equation and tell that this equation is valid at any point and at any time in the continuum domain. Therefore:

$$B = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z} \tag{3.38}$$

or

$$B(\boldsymbol{r},t) = \frac{\partial \phi(\boldsymbol{r},t,\boldsymbol{e}_x)}{\partial x} + \frac{\partial \phi(\boldsymbol{r},t,\boldsymbol{e}_y)}{\partial y} + \frac{\partial \phi(\boldsymbol{r},t,\boldsymbol{e}_z)}{\partial z} \tag{3.39}$$

This is the third local relation that is derived from the general integral equation (1.1). It is a partial differential equation and states that *"the body term at a given point and time in the continuum domain is equal to to the sum of the first order derivatives of the surface terms acting on the three orthogonal surfaces at that point and time"*.

It means that if we have the first derivatives of the surface terms on the three orthogonal surfaces at a given point and time, then we can get the body term at that point and time by using the equation (3.39).

Let us see what $E_2 = 0$ tells.
From the equation (3.13):

$$\begin{aligned} E_2 = &\Big(\frac{\partial^2\phi_{4_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2\phi_{4_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2\phi_{4_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2\phi_{4_o}}{\partial x\partial y}\frac{1}{n_x n_y} + \frac{\partial^2\phi_{4_o}}{\partial x\partial z}\frac{1}{n_x n_z} + \frac{\partial^2\phi_{4_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) \\ &+ n_x\Big(\frac{\partial^2\phi_{1_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2\phi_{1_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2\phi_{1_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) + n_y\Big(\frac{\partial^2\phi_{2_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2\phi_{2_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2\phi_{2_o}}{\partial x\partial z}\frac{1}{n_x n_z}\Big) \\ &+ n_z\Big(\frac{\partial^2\phi_{3_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2\phi_{3_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2\phi_{3_o}}{\partial x\partial y}\frac{1}{n_x n_y}\Big) - \Big(\frac{\partial B_o}{\partial x}\frac{1}{n_x} + \frac{\partial B_o}{\partial y}\frac{1}{n_y} + \frac{\partial B_o}{\partial z}\frac{1}{n_z}\Big) \end{aligned} \tag{3.40}$$

For $E_2$, similar to the process for $E_1 = 0$, we have:

$$\begin{aligned} E_2 = &\frac{1}{n_x^2}\frac{\partial^2 E_0}{\partial x^2} + \frac{1}{n_y^2}\frac{\partial^2 E_0}{\partial y^2} + \frac{1}{n_z^2}\frac{\partial^2 E_0}{\partial z^2} + \frac{1}{n_x n_y}\frac{\partial^2 E_0}{\partial x\partial y} + \frac{1}{n_x n_z}\frac{\partial^2 E_0}{\partial x\partial z} + \frac{1}{n_y n_z}\frac{\partial^2 E_0}{\partial y\partial z} \\ &+ \frac{1}{n_x}\frac{\partial E}{\partial x} + \frac{1}{n_y}\frac{\partial E}{\partial y} + \frac{1}{n_z}\frac{\partial E}{\partial z} \end{aligned} \tag{3.41}$$

By the previous explanations, all derivatives of $E_0$ and $E$ were equal to zero. Therefore, the equation (3.41) is a correct result of $E_2 = 0$.

Similar to the previous processes for $E_1$ and $E_2$, we have for $E_3 = 0$:

$$\begin{aligned} E_3 = &\frac{1}{n_x^3}\frac{\partial^3 E_0}{\partial x^3} + \frac{1}{n_y^3}\frac{\partial^3 E_0}{\partial y^3} + \frac{1}{n_z^3}\frac{\partial^3 E_0}{\partial z^3} + \frac{1}{n_x^2 n_y}\frac{\partial^3 E_0}{\partial x^2\partial y} + \frac{1}{n_x^2 n_z}\frac{\partial^3 E_0}{\partial x^2\partial z} + \frac{1}{n_y^2 n_z}\frac{\partial^3 E_0}{\partial y^2\partial z} \\ &+ \frac{1}{n_x n_y^2}\frac{\partial^3 E_0}{\partial x\partial y^2} + \frac{1}{n_x n_z^2}\frac{\partial^3 E_0}{\partial x\partial z^2} + \frac{1}{n_y n_z^2}\frac{\partial^3 E_0}{\partial y\partial z^2} + \frac{1}{n_x n_y n_z}\frac{\partial^3 E_0}{\partial x\partial y\partial z} \\ &+ \frac{1}{n_x^2}\frac{\partial^2 E}{\partial x^2} + \frac{1}{n_y^2}\frac{\partial^2 E}{\partial y^2} + \frac{1}{n_z^2}\frac{\partial^2 E}{\partial z^2} + \frac{1}{n_x n_y}\frac{\partial^2 E}{\partial x\partial y} + \frac{1}{n_x n_z}\frac{\partial^2 E}{\partial x\partial z} + \frac{1}{n_y n_z}\frac{\partial^2 E}{\partial y\partial z} \end{aligned} \tag{3.42}$$

We saw that all derivatives of $E_0$ and $E$ were equal to zero. So, the equation (3.42) is a correct result of $E_3 = 0$. This process for other $E_m$'s, leads to the expressions that contain the higher derivatives of $E_0$ and $E$ and the higher powers of the components of the unit normal vector and the results are equal to zero.

Therefore, the general integral equation (1.1) leads to the three important local relations (2.1), (3.29), and (3.39).

## 4. Fundamental laws of continuum mechanics, integral forms, basic local relations, and differential forms

In this section, we show that each of the fundamental laws of continuum mechanics can be written in the form of the general integral equation (1.1) on control volume $\mathcal{M}$, i.e.:

$$\int_{\mathcal{M}} B\, dV = \int_{\partial\mathcal{M}} \phi\, dS \tag{4.1}$$

In this equation $B = B(\boldsymbol{r}, t)$ and $\phi = \phi(\boldsymbol{r}, t, \boldsymbol{n})$ are continuous over the volume and the surface of $\mathcal{M}$, respectively. Where $\boldsymbol{r}$ is the position vector, $t$ is time, and $\boldsymbol{n}$ is the outward unit normal vector on the surface of the control volume. Here if $B$ is scalar then $\phi$ must be scalar, and if $B$ is vector then $\phi$ must be vector. In the previous sections, by using the Eulerian approach, we showed that this integral equation leads to the three local equations, as:
the first

$$\phi(\boldsymbol{r}, t, \boldsymbol{n}) = -\phi(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{4.2}$$

second

$$\phi(\boldsymbol{r}, t, \boldsymbol{n}) = n_x\phi(\boldsymbol{r}, t, \boldsymbol{e}_x) + n_y\phi(\boldsymbol{r}, t, \boldsymbol{e}_y) + n_z\phi(\boldsymbol{r}, t, \boldsymbol{e}_z) \tag{4.3}$$

and third

$$B(\boldsymbol{r}, t) = \frac{\partial\phi(\boldsymbol{r}, t, \boldsymbol{e}_x)}{\partial x} + \frac{\partial\phi(\boldsymbol{r}, t, \boldsymbol{e}_y)}{\partial y} + \frac{\partial\phi(\boldsymbol{r}, t, \boldsymbol{e}_z)}{\partial z} \tag{4.4}$$

In the following, we present some properties of a general integral equation in continuum media. If we have the following relation:

$$M_{t_0+\Delta t} - M_{t_0} = \int_{t_0}^{t_0+\Delta t} \psi \, d\tau \tag{4.5}$$

then by the definition of integrals it can be written as:

$$\int_{t_0}^{t_0+\Delta t} \frac{\partial M}{\partial t} \, d\tau = \int_{t_0}^{t_0+\Delta t} \psi \, d\tau$$

Note that we use the Eulerian approach. This implies:

$$\left(\frac{\partial M}{\partial t} - \psi\right) = 0, \qquad\qquad t_0 \leqslant t \leqslant (t_0 + \Delta t)$$

If $t_0$ and $\Delta t$ are any time and time interval in the time domain, then the general equation (4.5) leads to below equation that holds for any time:

$$\frac{\partial M}{\partial t} = \psi \tag{4.6}$$

In addition, the following integral equation holds for the control volume $\mathcal{M}$ in the Eulerian approach:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}} Q \, dv = \int_{\mathcal{M}} \frac{\partial Q}{\partial t} \, dv \tag{4.7}$$

Before considering the fundamental laws of continuum mechanics, let us discuss the flow of a physical quantity into a surface in continuum media. If we have a physical quantity such as $U = U(\boldsymbol{r}, t)$ that transfers by the velocity of the substance in continuum media, and $u = u(\boldsymbol{r}, t)$ is $U$ per unit volume, then the flow of this quantity into a surface with outward unit normal vector $\boldsymbol{n}$ is in the form:

$$\phi_U = -u \, \boldsymbol{v}.\boldsymbol{n} \tag{4.8}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ is the velocity vector of the substance. So, $\phi_U = \phi_U(\boldsymbol{r}, t, \boldsymbol{n})$ and it has the dimension of $[U]/(m^2.s)$. The negative sign is used because $\boldsymbol{n}$ is the outward unit normal vector of the surface. Here we suppose the fixed control volumes and for the moving control volumes the relative velocity must be used. Note that by the equation (4.8), $\phi_U$ satisfies the first and second local relations (4.2) and (4.3), as below:

$$-u \, \boldsymbol{v}.\boldsymbol{n} = -(-u \, \boldsymbol{v}.(-\boldsymbol{n}))$$

therefore

$$\phi_U(\boldsymbol{r},t,\boldsymbol{n}) = -\phi_U(\boldsymbol{r},t,-\boldsymbol{n}) \tag{4.9}$$

and

$$-u\,\boldsymbol{v}.\boldsymbol{n} = -u\,\boldsymbol{v}.(n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z) = n_x(-u\,\boldsymbol{v}.\boldsymbol{e}_x) + n_y(-u\,\boldsymbol{v}.\boldsymbol{e}_y) + n_z(-u\,\boldsymbol{v}.\boldsymbol{e}_z)$$

so

$$\phi_U(\boldsymbol{r},t,\boldsymbol{n}) = n_x\phi_U(\boldsymbol{r},t,\boldsymbol{e}_x) + n_y\phi_U(\boldsymbol{r},t,\boldsymbol{e}_y) + n_z\phi_U(\boldsymbol{r},t,\boldsymbol{e}_z) \tag{4.10}$$

By these general relations, we will consider the fundamental laws of continuum mechanics in the next subsections.

### 4.1. **Conservation of mass.**

The basic law of conservation of mass of a control volume $\mathcal{M}$ says:
*The total mass over the control volume $\mathcal{M}$ at time $(t_0+\Delta t)$ equals the total mass over $\mathcal{M}$ at time $t_0$ plus the net of mass flow into $\mathcal{M}$ from $t_0$ to $(t_0+\Delta t)$.* So:

$$\left\{\int_{\mathcal{M}} \rho\, dV\right\}_{t_0+\Delta t} = \left\{\int_{\mathcal{M}} \rho\, dV\right\}_{t_0} + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \phi_m\, dS\right\} d\tau \tag{4.11}$$

where $\rho = \rho(\boldsymbol{r},t)$ is the density (mass per unit volume) and $\phi_m = \phi_m(\boldsymbol{r},t,\boldsymbol{n})$ is the mass flow into the surface that it acts. Using the equation (4.8), we have $\phi_m = -\rho\,\boldsymbol{v}.\boldsymbol{n}$. By rearranging the equation:

$$\left\{\int_{\mathcal{M}} \rho\, dV\right\}_{t_0+\Delta t} - \left\{\int_{\mathcal{M}} \rho\, dV\right\}_{t_0} = \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} -\rho\,\boldsymbol{v}.\boldsymbol{n}\, dS\right\} d\tau \tag{4.12}$$

This is similar to the general equation (4.5), using (4.6) it becomes:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}} \rho\, dV = \int_{\partial\mathcal{M}} -\rho\,\boldsymbol{v}.\boldsymbol{n}\, dS \tag{4.13}$$

This is the integral equation of mass conservation law in continuum mechanics. By using (4.7) we have:

$$\int_{\mathcal{M}} \frac{\partial\rho}{\partial t}\, dV = \int_{\partial\mathcal{M}} -\rho\,\boldsymbol{v}.\boldsymbol{n}\, dS \tag{4.14}$$

This equation is similar to the general integral equation (4.1), where $B = \partial\rho/\partial t$ and $\phi = \phi_m(\boldsymbol{r},t,\boldsymbol{n}) = -\rho\,\boldsymbol{v}.\boldsymbol{n}$, so the three general local relations (4.2), (4.3), and (4.4) hold for it. The first and second local relations (4.2) and (4.3) lead to:

$$\phi_m(\boldsymbol{r},t,\boldsymbol{n}) = -\phi_m(\boldsymbol{r},t,-\boldsymbol{n}) \tag{4.15}$$

and

$$\phi_m(\boldsymbol{r},t,\boldsymbol{n}) = n_x\phi_m(\boldsymbol{r},t,\boldsymbol{e}_x) + n_y\phi_m(\boldsymbol{r},t,\boldsymbol{e}_y) + n_z\phi_m(\boldsymbol{r},t,\boldsymbol{e}_z) \tag{4.16}$$

But as we showed in (4.9) and (4.10), the mass flow $\phi_m(\boldsymbol{r},t,\boldsymbol{n}) = -\rho\,\boldsymbol{v}.\boldsymbol{n}$ satisfies the two local relations (4.2) and (4.3), and their meanings. So, the above two relations do not give us new results. The third local relation (4.4) for $B = \partial\rho/\partial t$ and $\phi = \phi_m(\boldsymbol{r},t,\boldsymbol{n}) = -\rho\,\boldsymbol{v}.\boldsymbol{n}$ leads to:

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}(-\rho\,\boldsymbol{v}.\boldsymbol{e}_x) + \frac{\partial}{\partial y}(-\rho\,\boldsymbol{v}.\boldsymbol{e}_y) + \frac{\partial}{\partial z}(-\rho\,\boldsymbol{v}.\boldsymbol{e}_z)$$

for $\boldsymbol{v} = v_x\boldsymbol{e}_x + v_y\boldsymbol{e}_y + v_z\boldsymbol{e}_z$, we have $\boldsymbol{v}.\boldsymbol{e}_x = v_x$, $\boldsymbol{v}.\boldsymbol{e}_y = v_y$, and $\boldsymbol{v}.\boldsymbol{e}_z = v_z$. Substituting these relations into the equation and rearranging it, yields:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{4.17}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\boldsymbol{v}) = 0 \tag{4.18}$$

This is the differential equation of mass conservation law in continuum mechanics that is called the continuity equation.

4.2. **Conservation of linear momentum.**
The basic law of conservation of linear momentum of a control volume $\mathcal{M}$ says:
*The total linear momentum over the control volume $\mathcal{M}$ at time $(t_0+\Delta t)$ equals the total linear momentum over $\mathcal{M}$ at time $t_0$ plus the net of linear momentum flow into $\mathcal{M}$ from $t_0$ to $(t_0+\Delta t)$ plus the total surface and body forces over $\mathcal{M}$ from $t_0$ to $(t_0+\Delta t)$.* So:

$$\begin{aligned}\left\{\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV\right\}_{t_0+\Delta t} &= \left\{\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV\right\}_{t_0} + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \boldsymbol{\phi}_{lm}\, dS\right\}d\tau \\ &+ \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \boldsymbol{t}\, dS\right\}d\tau + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\mathcal{M}} \rho\boldsymbol{b}\, dV\right\}d\tau\end{aligned} \tag{4.19}$$

where $\rho\boldsymbol{v}$ is the linear momentum per unit volume, $\boldsymbol{\phi}_{lm} = \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{n})$ is the linear momentum flow into the surface that it acts, $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ is the surface force per unit area that is called traction vector, and $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{r}, t)$ is the body force per unit mass. By using the general equation (4.8) for the flow of linear momentum, we have $\boldsymbol{\phi}_{lm} = -(\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}$, and rearranging the equation yields:

$$\left\{\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV\right\}_{t_0+\Delta t} - \left\{\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV\right\}_{t_0} = \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \{\boldsymbol{t} - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\}\, dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\, dV\right\}d\tau \tag{4.20}$$

This is similar to the general equation (4.5). So, by using (4.6) it becomes:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}} \rho\boldsymbol{v}\, dV = \int_{\partial\mathcal{M}} \{\boldsymbol{t} - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\}\, dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\, dV \tag{4.21}$$

This is the integral equation of linear momentum conservation law in continuum mechanics. Using (4.7) and rearranging the equation:

$$\int_{\mathcal{M}} \left\{\frac{\partial(\rho\boldsymbol{v})}{\partial t} - \rho\boldsymbol{b}\right\} dV = \int_{\partial\mathcal{M}} \{\boldsymbol{t} - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\}\, dS \tag{4.22}$$

This is similar to the general integral equation (4.1) by the vector forms of $B$ and $\phi$, where $B = \partial(\rho\boldsymbol{v})/\partial t - \rho\boldsymbol{b}$ and $\phi = \boldsymbol{t} - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n} = \boldsymbol{t} + \boldsymbol{\phi}_{lm}$. So, the three general local relations (4.2), (4.3), and (4.4) hold for it. The first and second local relations (4.2) and (4.3) lead to:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n}) - \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{4.23}$$

and

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n})+\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{n})=n_x\big\{\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x)+\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{e}_x)\big\}+n_y\big\{\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y)+\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{e}_y)\big\}$$
$$+\,n_z\big\{\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z)+\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{e}_{\boldsymbol{z}})\big\} \tag{4.24}$$

But as we showed in (4.9) and (4.10), the linear momentum flow $\boldsymbol{\phi}_{lm}=-(\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}$ satisfies the two local relations (4.2) and (4.3), and their meanings. i.e.:

$$\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{n})=-\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,-\boldsymbol{n}) \tag{4.25}$$

and

$$\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{n})=n_x\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{e}_x)+n_y\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{e}_y)+n_z\boldsymbol{\phi}_{lm}(\boldsymbol{r},t,\boldsymbol{e}_{\boldsymbol{z}}) \tag{4.26}$$

So, these terms remove from the equations (4.23) and (4.24). Thus, we have from (4.23):

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n})=-\boldsymbol{t}(\boldsymbol{r},t,-\boldsymbol{n}) \tag{4.27}$$

This is the Cauchy lemma for traction vectors and states that "the traction vectors acting on opposite sides of the same surface at a given point and time are equal in magnitude but opposite in direction".

And from (4.24):

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n})=n_x\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x)+n_y\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y)+n_z\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_{\boldsymbol{z}}) \tag{4.28}$$

This means that if we have the traction vectors on the three orthogonal surfaces at a given point and time then we can get the traction vector on any surface that passes through that point at that time by having the unit normal vector of this surface and using this linear relation. So, we must define the traction vectors on the three orthogonal surfaces at any point and at any time. The traction vector on the surface with unit normal vector $\boldsymbol{e}_x$ by its components, defines as:

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x)=T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x+T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y+T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z \tag{4.29}$$

here $T_{xx}(\boldsymbol{r},t)$, $T_{xy}(\boldsymbol{r},t)$, and $T_{xz}(\boldsymbol{r},t)$ are scalars that depend only on $\boldsymbol{r}$ and $t$. In each case the first subscript indicates the direction of normal unit vector of the surface that this case acts on it, and the second subscript indicates the direction of this component of traction vector. Similarly, the traction vectors on the surfaces with unit normal vectors $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$, define as:

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y)=T_{yx}(\boldsymbol{r},t)\,\boldsymbol{e}_x+T_{yy}(\boldsymbol{r},t)\,\boldsymbol{e}_y+T_{yz}(\boldsymbol{r},t)\,\boldsymbol{e}_z \tag{4.30}$$

and

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z)=T_{zx}(\boldsymbol{r},t)\,\boldsymbol{e}_x+T_{zy}(\boldsymbol{r},t)\,\boldsymbol{e}_y+T_{zz}(\boldsymbol{r},t)\,\boldsymbol{e}_z \tag{4.31}$$

By substituting these equations in (4.28)

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n})=&\,n_x\big\{T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x+T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y+T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\big\}\\&+n_y\big\{T_{yx}(\boldsymbol{r},t)\,\boldsymbol{e}_x+T_{yy}(\boldsymbol{r},t)\,\boldsymbol{e}_y+T_{yz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\big\}\\&+n_z\big\{T_{zx}(\boldsymbol{r},t)\,\boldsymbol{e}_x+T_{zy}(\boldsymbol{r},t)\,\boldsymbol{e}_y+T_{zz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\big\}\end{aligned}$$

by rearranging the equation

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n}) &= \big\{n_xT_{xx}(\boldsymbol{r},t)+n_yT_{yx}(\boldsymbol{r},t)+n_zT_{zx}(\boldsymbol{r},t)\big\}\boldsymbol{e}_x\\ &+\big\{n_xT_{xy}(\boldsymbol{r},t)+n_yT_{yy}(\boldsymbol{r},t)+n_zT_{zy}(\boldsymbol{r},t)\big\}\boldsymbol{e}_y\\ &+\big\{n_xT_{xz}(\boldsymbol{r},t)+n_yT_{yz}(\boldsymbol{r},t)+n_zT_{zz}(\boldsymbol{r},t)\big\}\boldsymbol{e}_z\end{aligned}$$

this can be shown as

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n}) = \begin{bmatrix} t_x(\boldsymbol{r},t,\boldsymbol{n})\\ t_y(\boldsymbol{r},t,\boldsymbol{n})\\ t_z(\boldsymbol{r},t,\boldsymbol{n})\end{bmatrix} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz}\\ T_{yx} & T_{yy} & T_{yz}\\ T_{zx} & T_{zy} & T_{zz}\end{bmatrix}^T \begin{bmatrix} n_x\\ n_y\\ n_z\end{bmatrix} \tag{4.32}$$

using the vector relations, this becomes

$$\boldsymbol{t} = \boldsymbol{T}^T.\boldsymbol{n} \tag{4.33}$$

where $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r},t)$ is a second order tensor and is called stress tensor. This tensor depends only on the position vector and time. This relation means that "for describing the state of stress on any surface at a given point and time we need the 9 components of the stress tensor at that point and time". So, the second local relation (4.3) for linear momentum leads to the existence of stress tensor.

Let us apply the third local relation (4.4) for linear momentum, where $B = \partial(\rho\boldsymbol{v})/\partial t - \rho\boldsymbol{b}$ and $\phi = \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n}) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}$. Thus:

$$\begin{aligned}\frac{\partial(\rho\boldsymbol{v})}{\partial t} - \rho\boldsymbol{b} &= \frac{\partial}{\partial x}\big\{\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x\big\} + \frac{\partial}{\partial y}\big\{\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y\big\}\\ &+ \frac{\partial}{\partial z}\big\{\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z\big\}\end{aligned} \tag{4.34}$$

Using the relations (4.29), (4.30), and (4.31), we have:

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x &= (T_{xx}-\rho v_xv_x)\boldsymbol{e}_x + (T_{xy}-\rho v_yv_x)\boldsymbol{e}_y + (T_{xz}-\rho v_zv_x)\boldsymbol{e}_z\\ \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y &= (T_{yx}-\rho v_xv_y)\boldsymbol{e}_x + (T_{yy}-\rho v_yv_y)\boldsymbol{e}_y + (T_{yz}-\rho v_zv_y)\boldsymbol{e}_z\\ \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z &= (T_{zx}-\rho v_xv_z)\boldsymbol{e}_x + (T_{zy}-\rho v_yv_z)\boldsymbol{e}_y + (T_{zz}-\rho v_zv_z)\boldsymbol{e}_z\end{aligned}$$

Substituting these equations into the equation (4.34) and rearranging it, yields:

$$\begin{aligned}\frac{\partial(\rho\boldsymbol{v})}{\partial t} - \rho\boldsymbol{b} &= \frac{\partial}{\partial x}\big\{T_{xx}\boldsymbol{e}_x + T_{xy}\boldsymbol{e}_y + T_{xz}\boldsymbol{e}_z\big\} + \frac{\partial}{\partial y}\big\{T_{yx}\boldsymbol{e}_x + T_{yy}\boldsymbol{e}_y + T_{yz}\boldsymbol{e}_z\big\}\\ &+ \frac{\partial}{\partial z}\big\{T_{zx}\boldsymbol{e}_x + T_{zy}\boldsymbol{e}_y + T_{zz}\boldsymbol{e}_z\big\} - \frac{\partial}{\partial x}\big\{\rho v_xv_x\boldsymbol{e}_x + \rho v_yv_x\boldsymbol{e}_y + \rho v_zv_x\boldsymbol{e}_z\big\}\\ &- \frac{\partial}{\partial y}\big\{\rho v_xv_y\boldsymbol{e}_x + \rho v_yv_y\boldsymbol{e}_y + \rho v_zv_y\boldsymbol{e}_z\big\} - \frac{\partial}{\partial z}\big\{\rho v_xv_z\boldsymbol{e}_x + \rho v_yv_z\boldsymbol{e}_y + \rho v_zv_z\boldsymbol{e}_z\big\}\end{aligned}$$

therefore

$$\begin{aligned}\frac{\partial(\rho\boldsymbol{v})}{\partial t} - \rho\boldsymbol{b} &= \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix} T_{xx} & T_{xy} & T_{xz}\\ T_{yx} & T_{yy} & T_{yz}\\ T_{zx} & T_{zy} & T_{zz}\end{bmatrix} - \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\rho v_xv_x & \rho v_xv_y & \rho v_xv_z\\ \rho v_yv_x & \rho v_yv_y & \rho v_yv_z\\ \rho v_zv_x & \rho v_zv_y & \rho v_zv_z\end{bmatrix}\\ &= \nabla.\boldsymbol{T} - \nabla.(\rho\boldsymbol{v}\boldsymbol{v})\end{aligned}$$

where $\rho\boldsymbol{v}\boldsymbol{v} = \rho v_i v_j$ is the last second order tensor in the first line of the equation. By rearranging the equation:

$$\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \nabla.(\rho\boldsymbol{v}\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{4.35}$$

This is the differential equation of linear momentum conservation law in continuum mechanics and is called the general equation of motion or Cauchy equation of motion. Using the mass continuity equation (4.18), it becomes:

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}.\nabla)\boldsymbol{v} = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{4.36}$$

4.3. **Conservation of angular momentum.**

The basic law of conservation of angular momentum of a control volume $\mathcal{M}$ about point $\boldsymbol{r}_0$ says:

*The total angular momentum about point $\boldsymbol{r}_0$ over the control volume $\mathcal{M}$ at time $(t_0+\Delta t)$ equals the total angular momentum about point $\boldsymbol{r}_0$ over $\mathcal{M}$ at time $t_0$ plus the net of angular momentum flow about point $\boldsymbol{r}_0$ into $\mathcal{M}$ from $t_0$ to $(t_0+\Delta t)$ plus the total moment of surface and body forces about point $\boldsymbol{r}_0$ over $\mathcal{M}$ from $t_0$ to $(t_0+\Delta t)$.* So:

$$\begin{aligned}\left\{\int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{v})\,dV\right\}_{t_0+\Delta t} &= \left\{\int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{v})\,dV\right\}_{t_0} + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}}\boldsymbol{\phi}_{am}\,dS\right\}d\tau \\ &+ \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}}(\boldsymbol{r}'\times\boldsymbol{t})\,dS\right\}d\tau + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{b})\,dV\right\}d\tau\end{aligned} \tag{4.37}$$

by $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}_0$ then $\boldsymbol{r}'\times\rho\boldsymbol{v}$ is the angular momentum about $\boldsymbol{r}_0$ per unit volume, $\boldsymbol{\phi}_{am} = \boldsymbol{\phi}_{am}(\boldsymbol{r},t,\boldsymbol{n})$ is the angular momentum flow about $\boldsymbol{r}_0$ into the surface that it acts, $\boldsymbol{r}'\times\boldsymbol{t}$ is the moment of surface force about $\boldsymbol{r}_0$ per unit area, and $\boldsymbol{r}'\times\rho\boldsymbol{b}$ is the moment of body force about $\boldsymbol{r}_0$ per unit volume. By using the general equation (4.8) for the flow of angular momentum about $\boldsymbol{r}_0$, we have $\boldsymbol{\phi}_{am} = -(\boldsymbol{r}'\times\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n} = \boldsymbol{r}'\times\boldsymbol{\phi}_{lm}$, where $\boldsymbol{\phi}_{lm} = -(\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}$ is the linear momentum flow into $\mathcal{M}$. By rearranging the equation:

$$\begin{aligned}&\left\{\int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{v})\,dV\right\}_{t_0+\Delta t} - \left\{\int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{v})\,dV\right\}_{t_0} = \\ &\qquad\int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}}\left\{\boldsymbol{r}'\times\big(\boldsymbol{t}-(\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\big)\right\}dS + \int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{b})\,dV\right\}d\tau\end{aligned} \tag{4.38}$$

This is similar to the general equation (4.5). So, by using (4.6) it becomes:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{v})\,dV = \int_{\partial\mathcal{M}}\left\{\boldsymbol{r}'\times\big(\boldsymbol{t}-(\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\big)\right\}dS + \int_{\mathcal{M}}(\boldsymbol{r}'\times\rho\boldsymbol{b})\,dV \tag{4.39}$$

This is the integral equation of angular momentum conservation law in continuum mechanics. Using (4.7) and rearranging the equation:

$$\int_{\mathcal{M}}\left\{\frac{\partial}{\partial t}(\boldsymbol{r}'\times\rho\boldsymbol{v}) - (\boldsymbol{r}'\times\rho\boldsymbol{b})\right\}dV = \int_{\partial\mathcal{M}}\left\{\boldsymbol{r}'\times\big(\boldsymbol{t}-(\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\big)\right\}dS \tag{4.40}$$

This is similar to the integral equation (4.1) by the vector forms of $B$ and $\phi$, where $B = \partial(\boldsymbol{r}' \times \rho\boldsymbol{v})/\partial t - (\boldsymbol{r}' \times \rho\boldsymbol{b})$ and $\phi = \boldsymbol{r}' \times \big(\boldsymbol{t} - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\big) = \boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{n})\big)$. So, the three general local relations (4.2), (4.3), and (4.4) hold for it. The first and second local relations (4.2) and (4.3) lead to:

$$\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{n})\big) = -\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n}) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, -\boldsymbol{n})\big) \tag{4.41}$$

and

$$\begin{aligned}\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{n})\big) &= n_x\big\{\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{e}_x)\big)\big\} \\ &+ n_y\big\{\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{e}_y)\big)\big\} + n_z\big\{\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) + \boldsymbol{\phi}_{lm}(\boldsymbol{r}, t, \boldsymbol{e}_z)\big)\big\}\end{aligned} \tag{4.42}$$

But these equations are the cross product of $\boldsymbol{r}'$ and the equations (4.25) and (4.26), respectively, that already were obtained in the subsection of the linear momentum. So, these equations do not give us new results. The third local relation (4.4) for $B = \partial(\boldsymbol{r}' \times \rho\boldsymbol{v})/\partial t - (\boldsymbol{r}' \times \rho\boldsymbol{b})$ and $\phi = \boldsymbol{r}' \times \big(\boldsymbol{t} - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{n}\big)$, leads to:

$$\begin{aligned}\frac{\partial}{\partial t}(\boldsymbol{r}' \times \rho\boldsymbol{v}) - (\boldsymbol{r}' \times \rho\boldsymbol{b}) &= \frac{\partial}{\partial x}\big\{\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x\big)\big\} \\ &+ \frac{\partial}{\partial y}\big\{\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y\big)\big\} + \frac{\partial}{\partial z}\big\{\boldsymbol{r}' \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z\big)\big\}\end{aligned} \tag{4.43}$$

In the Eulerian approach for $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}_0$, we have:

$$\frac{\partial \boldsymbol{r}'}{\partial t} = \boldsymbol{0}, \qquad \frac{\partial \boldsymbol{r}'}{\partial x} = \boldsymbol{e}_x, \qquad \frac{\partial \boldsymbol{r}'}{\partial y} = \boldsymbol{e}_y, \qquad \frac{\partial \boldsymbol{r}'}{\partial z} = \boldsymbol{e}_z \tag{4.44}$$

By using these relations, the equation (4.43) becomes:

$$\begin{aligned}\boldsymbol{r}' \times \left\{\frac{\partial(\rho\boldsymbol{v})}{\partial t} - \rho\boldsymbol{b}\right\} &= \boldsymbol{r}' \times \left\{\frac{\partial}{\partial x}\big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x\big) + \frac{\partial}{\partial y}\big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y\big)\right. \\ &\left.+ \frac{\partial}{\partial z}\big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z\big)\right\} \\ &+ \Big\{\boldsymbol{e}_x \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x\big) + \boldsymbol{e}_y \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y\big) \\ &+ \boldsymbol{e}_z \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z\big)\Big\}\end{aligned} \tag{4.45}$$

But the first two lines of this equation is the cross product of $\boldsymbol{r}'$ and the equation (4.34) that already was obtained in the subsection of the linear momentum. So, these parts remove from the equation and we have:

$$\begin{aligned}\boldsymbol{e}_x \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x\big) + \boldsymbol{e}_y \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y\big) \\ + \boldsymbol{e}_z \times \big(\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) - (\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z\big) = \boldsymbol{0}\end{aligned} \tag{4.46}$$

thus

$$\begin{aligned}
&\boldsymbol{e}_x \times \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x) + \boldsymbol{e}_y \times \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y) + \boldsymbol{e}_z \times \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z) = \\
&\qquad \boldsymbol{e}_x \times \big((\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_x\big) + \boldsymbol{e}_y \times \big((\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_y\big) + \boldsymbol{e}_z \times \big((\rho\boldsymbol{v})\boldsymbol{v}.\boldsymbol{e}_z\big) = \\
&\qquad \rho v_x(\boldsymbol{e}_x \times \boldsymbol{v}) + \rho v_y(\boldsymbol{e}_y \times \boldsymbol{v}) + \rho v_z(\boldsymbol{e}_z \times \boldsymbol{v}) = \\
&\qquad \rho v_x(-v_z\boldsymbol{e}_y + v_y\boldsymbol{e}_z) + \rho v_y(v_z\boldsymbol{e}_x - v_x\boldsymbol{e}_z) + \rho v_z(-v_y\boldsymbol{e}_x + v_x\boldsymbol{e}_y) = \\
&\qquad (\rho v_y v_z - \rho v_y v_z)\boldsymbol{e}_x + (-\rho v_x v_z + \rho v_x v_z)\boldsymbol{e}_y + (\rho v_x v_y - \rho v_x v_y)\boldsymbol{e}_z = \boldsymbol{0}
\end{aligned}$$

so

$$\boldsymbol{e}_x \times \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x) + \boldsymbol{e}_y \times \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y) + \boldsymbol{e}_z \times \boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z) = \boldsymbol{0} \tag{4.47}$$

substituting the components of the traction vectors from (4.29), (4.30), and (4.31) into the equation, yields:

$$\begin{aligned}
\boldsymbol{e}_x \times (T_{xx}\,\boldsymbol{e}_x + T_{xy}\,\boldsymbol{e}_y + T_{xz}\,\boldsymbol{e}_z) + \boldsymbol{e}_y \times (T_{yx}\,\boldsymbol{e}_x + T_{yy}\,\boldsymbol{e}_y + T_{yz}\,\boldsymbol{e}_z)& \\
+ \boldsymbol{e}_z \times (T_{zx}\,\boldsymbol{e}_x + T_{zy}\,\boldsymbol{e}_y + T_{zz}\,\boldsymbol{e}_z) = \boldsymbol{0}&
\end{aligned}$$

this implies

$$\begin{aligned}
(-T_{xz}\,\boldsymbol{e}_y + T_{xy}\,\boldsymbol{e}_z) + (T_{yz}\,\boldsymbol{e}_x - T_{yx}\,\boldsymbol{e}_z) + (-T_{zy}\,\boldsymbol{e}_x + T_{zx}\,\boldsymbol{e}_y) =& \\
(T_{yz} - T_{zy})\,\boldsymbol{e}_x + (T_{zx} - T_{xz})\,\boldsymbol{e}_y + (T_{xy} - T_{yx})\,\boldsymbol{e}_z = \boldsymbol{0}&
\end{aligned}$$

So, we have

$$T_{xy} = T_{yx}, \qquad T_{xz} = T_{zx}, \qquad T_{yz} = T_{zy} \tag{4.48}$$

or

$$\boldsymbol{T} = \boldsymbol{T}^T \tag{4.49}$$

therefore, the third local relation (4.4) for conservation of angular momentum leads to the symmetry of stress tensor. By (4.49) we can tell "for describing the state of stress on any surface at a given point and time we need the 6 components of the symmetric stress tensor at that point and time".

4.4. **Conservation of energy.**
The basic law of conservation of energy of a control volume $\mathcal{M}$ says:
*The total energy over the control volume $\mathcal{M}$ at time $(t_0 + \Delta t)$ equals the total energy over $\mathcal{M}$ at time $t_0$ plus the net of energy flow into $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$ plus the total surface heat into $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$ plus the total heat generation over $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$ plus the total work done by surface and body forces over $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$.* So:

$$\begin{aligned}
\left\{\int_{\mathcal{M}} (\rho e + \frac{1}{2}\rho v^2)\,dV\right\}_{t_0+\Delta t} &= \left\{\int_{\mathcal{M}} (\rho e + \frac{1}{2}\rho v^2)\,dV\right\}_{t_0} + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \phi_{en}\,dS\right\}d\tau \\
&+ \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} q_s\,dS\right\}d\tau + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\mathcal{M}} \rho\dot{q}_g\,dV\right\}d\tau \\
&+ \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \boldsymbol{t}.\boldsymbol{v}\,dS\right\}d\tau + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\mathcal{M}} (\rho\boldsymbol{b}).\boldsymbol{v}\,dV\right\}d\tau
\end{aligned} \tag{4.50}$$

where $e = e(\boldsymbol{r}, t)$ is the internal energy per unit mass, and $\rho e + 1/2\rho v^2$ is the total energy (internal energy + kinetic energy) per unit volume. Here $v^2 = v_i v_i = v_x^2 + v_y^2 + v_z^2$. On the right hand side, $\phi_{en} = \phi_{en}(\boldsymbol{r}, t, \boldsymbol{n})$ is the energy flow into the surface that it acts,

$q_s = q_s(\boldsymbol{r}, t, \boldsymbol{n})$ is the rate of surface heat into $\mathcal{M}$ per unit area, $\dot{q}_g = \dot{q}_g(\boldsymbol{r}, t)$ is the rate of heat generation per unit mass, $\boldsymbol{t}.\boldsymbol{v}$ and $(\rho\boldsymbol{b}).\boldsymbol{v}$ are the rates of work done by the surface force per unit area and body force per unit volume, respectively. By using the general equation (4.8) for the flow of energy we have $\phi_{en} = -(\rho e + 1/2\rho v^2)\boldsymbol{v}.\boldsymbol{n}$, and by rearranging the equation:

$$\left\{ \int_{\mathcal{M}} (\rho e + \frac{1}{2}\rho v^2)\, dV \right\}_{t_0+\Delta t} - \left\{ \int_{\mathcal{M}} (\rho e + \frac{1}{2}\rho v^2)\, dV \right\}_{t_0} = \int_{t_0}^{t_0+\Delta t} \left\{ \int_{\partial\mathcal{M}} \{\boldsymbol{t}.\boldsymbol{v} + q_s - (\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}.\boldsymbol{n}\}\, dS + \int_{\mathcal{M}} \{(\rho\boldsymbol{b}).\boldsymbol{v} + \rho\dot{q}_g\}\, dV \right\} d\tau \tag{4.51}$$

This is similar to the general equation (4.5). So, by using (4.6) it becomes:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}} (\rho e + \frac{1}{2}\rho v^2)\, dV = \int_{\partial\mathcal{M}} \{\boldsymbol{t}.\boldsymbol{v} + q_s - (\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}.\boldsymbol{n}\}\, dS + \int_{\mathcal{M}} \{(\rho\boldsymbol{b}).\boldsymbol{v} + \rho\dot{q}_g\}\, dV \tag{4.52}$$

This is the integral equation of energy conservation law in continuum mechanics. Using (4.7) and rearranging the equation:

$$\int_{\mathcal{M}} \left\{ \frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho v^2) - (\rho\boldsymbol{b}).\boldsymbol{v} - \rho\dot{q}_g \right\} dV = \int_{\partial\mathcal{M}} \{\boldsymbol{t}.\boldsymbol{v} + q_s - (\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}.\boldsymbol{n}\}\, dS \tag{4.53}$$

This is similar to the integral equation (4.1), where $B = \partial(\rho e + 1/2\rho v^2)/\partial t - (\rho\boldsymbol{b}).\boldsymbol{v} - \rho\dot{q}_g$ and $\phi = \boldsymbol{t}.\boldsymbol{v} + q_s - (\rho e + 1/2\rho v^2)\boldsymbol{v}.\boldsymbol{n} = \boldsymbol{t}.\boldsymbol{v} + q_s + \phi_{en}$. So, the three general local relations (4.2), (4.3), and (4.4) hold for it. The first and second local relations (4.2) and (4.3) lead to:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}).\boldsymbol{v} + q_s(\boldsymbol{r}, t, \boldsymbol{n}) + \phi_{en}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n}).\boldsymbol{v} - q_s(\boldsymbol{r}, t, -\boldsymbol{n}) - \phi_{en}(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{4.54}$$

and

$$\begin{aligned} \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}).\boldsymbol{v} + q_s(\boldsymbol{r}, t, \boldsymbol{n}) + \phi_{en}(\boldsymbol{r}, t, \boldsymbol{n}) &= n_x\{\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x).\boldsymbol{v} + q_s(\boldsymbol{r}, t, \boldsymbol{e}_x) + \phi_{en}(\boldsymbol{r}, t, \boldsymbol{e}_x)\} \\ &+ n_y\{\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y).\boldsymbol{v} + q_s(\boldsymbol{r}, t, \boldsymbol{e}_y) + \phi_{en}(\boldsymbol{r}, t, \boldsymbol{e}_y)\} \\ &+ n_z\{\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z).\boldsymbol{v} + q_s(\boldsymbol{r}, t, \boldsymbol{e}_z) + \phi_{en}(\boldsymbol{r}, t, \boldsymbol{e}_z)\} \end{aligned} \tag{4.55}$$

But as we showed in (4.9) and (4.10), the energy flow $\phi_{en} = -(\rho e + 1/2\rho v^2)\boldsymbol{v}.\boldsymbol{n}$ satisfies the two local relations (4.2) and (4.3), and their meanings. Therefore, the energy flow terms remove from the two above equations. Also, in (4.27) and (4.28), we saw that the traction vector $\boldsymbol{t}$ satisfies the two local relations (4.2) and (4.3). As a result, $\boldsymbol{t}.\boldsymbol{v}$ satisfies that equations, as well. So, these terms remove from the two above equations and we have from (4.54):

$$q_s(\boldsymbol{r}, t, \boldsymbol{n}) = -q_s(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{4.56}$$

This is the general Cauchy lemma for surface heat and states that "the surface heats acting on opposite sides of the same surface at a given point and time are equal in magnitude but opposite in sign". From (4.55), we have:

$$q_s(\boldsymbol{r}, t, \boldsymbol{n}) = n_x q_s(\boldsymbol{r}, t, \boldsymbol{e}_x) + n_y q_s(\boldsymbol{r}, t, \boldsymbol{e}_y) + n_z q_s(\boldsymbol{r}, t, \boldsymbol{e}_z) \tag{4.57}$$

This means that if we have the surface heats on the three orthogonal surfaces at a given point and time then we can get the surface heat on any surface that passes through that point at that time by having the unit normal vector of this surface and using this linear relation. So, we must define the scalar surface heats into the three orthogonal surfaces with unit normal vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$, respectively, as:

$$q_s(\boldsymbol{r},t,\boldsymbol{e}_x) = -q_x(\boldsymbol{r},t), \qquad q_s(\boldsymbol{r},t,\boldsymbol{e}_y) = -q_y(\boldsymbol{r},t), \qquad q_s(\boldsymbol{r},t,\boldsymbol{e}_z) = -q_z(\boldsymbol{r},t) \quad (4.58)$$

the negative sign is due to the fact that we suppose for example $q_x(\boldsymbol{r},t)$ is the exit heat from the surface with unit normal vector $\boldsymbol{e}_x$ but $q_s(\boldsymbol{r},t,\boldsymbol{e}_x)$ is the surface heat into that surface. Here the subscripts in $q_x$, $q_y$, and $q_z$ indicate the direction of unit normal vector of the surfaces that they act on them. So, we have from (4.57):

$$q_s(\boldsymbol{r},t,\boldsymbol{n}) = -n_x q_x(\boldsymbol{r},t) - n_y q_y(\boldsymbol{r},t) - n_z q_z(\boldsymbol{r},t) = -\begin{bmatrix} q_x & q_y & q_z \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \quad (4.59)$$

thus

$$q_s(\boldsymbol{r},t,\boldsymbol{n}) = -\boldsymbol{q}(\boldsymbol{r},t).\boldsymbol{n} \quad (4.60)$$

where $\boldsymbol{q}(\boldsymbol{r},t)$ is a vector that depends only on the position vector and time and is called heat flux vector. So, the first and second local relations (4.2) and (4.3) for the conservation of energy lead to the existence of heat flux vector $\boldsymbol{q}(\boldsymbol{r},t)$. This means that for describing the surface heat on any surface at a given point and time we need the 3 components of $\boldsymbol{q}(\boldsymbol{r},t)$ at that point and time. The third local relation (4.4) for energy conservation is:

$$\begin{aligned}\frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho v^2) - (\rho\boldsymbol{b}).\boldsymbol{v} - \rho\dot{q}_g = \frac{\partial}{\partial x}\big\{&\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x).\boldsymbol{v} - \boldsymbol{q}.\boldsymbol{e}_x - (\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}.\boldsymbol{e}_x\big\} \\ + \frac{\partial}{\partial y}\big\{&\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y).\boldsymbol{v} - \boldsymbol{q}.\boldsymbol{e}_y - (\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}.\boldsymbol{e}_y\big\} \quad (4.61) \\ + \frac{\partial}{\partial z}\big\{&\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z).\boldsymbol{v} - \boldsymbol{q}.\boldsymbol{e}_z - (\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}.\boldsymbol{e}_z\big\}\end{aligned}$$

By using the equation (4.33), i.e., $\boldsymbol{t} = \boldsymbol{T}^T.\boldsymbol{n}$, the above equation can be shown as:

$$\frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho v^2) - (\rho\boldsymbol{b}).\boldsymbol{v} - \rho\dot{q}_g = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} T_{xx}v_x + T_{xy}v_y + T_{xz}v_z \\ T_{yx}v_x + T_{yy}v_y + T_{yz}v_z \\ T_{zx}v_x + T_{zy}v_y + T_{zz}v_z \end{bmatrix}$$
$$- \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix} - \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} (\rho e + 1/2\rho v^2)v_x \\ (\rho e + 1/2\rho v^2)v_y \\ (\rho e + 1/2\rho v^2)v_z \end{bmatrix}$$

by vector relations, this becomes

$$\frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho v^2) - (\rho\boldsymbol{b}).\boldsymbol{v} - \rho\dot{q}_g = \nabla.(\boldsymbol{T}.\boldsymbol{v}) - \nabla.\boldsymbol{q} - \nabla.\big((\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}\big)$$

by rearranging the equation, we have

$$\frac{\partial}{\partial t}(\rho e + \frac{1}{2}\rho v^2) + \nabla.\big((\rho e + \frac{1}{2}\rho v^2)\boldsymbol{v}\big) = \nabla.(\boldsymbol{T}.\boldsymbol{v}) - \nabla.\boldsymbol{q} + (\rho\boldsymbol{b}).\boldsymbol{v} + \rho\dot{q}_g \quad (4.62)$$

This is the differential equation of energy conservation law in continuum mechanics. Also, there are some other forms of energy equation that are obtained from the above equation. We have:

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\frac{1}{2}\rho v^2) = \frac{1}{2}\frac{\partial}{\partial t}(\rho \boldsymbol{v}.\boldsymbol{v}) = \boldsymbol{v}.\Big\{\frac{\partial}{\partial t}(\rho \boldsymbol{v})\Big\} \\
&\nabla.\big((\frac{1}{2}\rho v^2)\boldsymbol{v}\big) = \frac{1}{2}\nabla.\big((\rho \boldsymbol{v}.\boldsymbol{v})\boldsymbol{v}\big) = \boldsymbol{v}.\Big\{\nabla.(\rho \boldsymbol{v}\boldsymbol{v})\Big\} \\
&\nabla.(\boldsymbol{T}.\boldsymbol{v}) = \boldsymbol{v}.\Big\{\nabla.\boldsymbol{T}\Big\} + \boldsymbol{T} : \nabla \boldsymbol{v}
\end{aligned}
\tag{4.63}
$$

where $\boldsymbol{T} : \nabla \boldsymbol{v}$ is the following scalar

$$
\begin{aligned}
\boldsymbol{T} : \nabla \boldsymbol{v} = T_{ij}\frac{\partial v_j}{\partial x_i} &= T_{xx}\frac{\partial v_x}{\partial x} + T_{xy}\frac{\partial v_y}{\partial x} + T_{xz}\frac{\partial v_z}{\partial x} + T_{yx}\frac{\partial v_x}{\partial y} + T_{yy}\frac{\partial v_y}{\partial y} + T_{yz}\frac{\partial v_z}{\partial y} \\
&+ T_{zx}\frac{\partial v_x}{\partial z} + T_{zy}\frac{\partial v_y}{\partial z} + T_{zz}\frac{\partial v_z}{\partial z}
\end{aligned}
$$

By using the relations (4.63), the equation (4.62) becomes:

$$
\frac{\partial(\rho e)}{\partial t} + \boldsymbol{v}.\Big\{\frac{\partial}{\partial t}(\rho \boldsymbol{v})\Big\} + \nabla.(\rho e \boldsymbol{v}) + \boldsymbol{v}.\Big\{\nabla.(\rho \boldsymbol{v}\boldsymbol{v})\Big\} = \boldsymbol{v}.\Big\{\nabla.\boldsymbol{T}\Big\} + \boldsymbol{T} : \nabla \boldsymbol{v} - \nabla.\boldsymbol{q} + (\rho \boldsymbol{b}).\boldsymbol{v} + \rho \dot{q}_g
$$

by rearranging this equation

$$
\begin{aligned}
\frac{\partial(\rho e)}{\partial t} + \nabla.(\rho e \boldsymbol{v}) &= \boldsymbol{T} : \nabla \boldsymbol{v} - \nabla.\boldsymbol{q} + \rho \dot{q}_g \\
&\quad - \boldsymbol{v}.\Big\{\frac{\partial(\rho \boldsymbol{v})}{\partial t} + \nabla.(\rho \boldsymbol{v}\boldsymbol{v}) - \nabla.\boldsymbol{T} - \rho \boldsymbol{b}\Big\}
\end{aligned}
$$

but due to the differential equation of linear momentum conservation law (4.35), the expression in the braces in the second line of the above equation is equal to zero, therefore this line removes from the equation and we have:

$$
\frac{\partial(\rho e)}{\partial t} + \nabla.(\rho e \boldsymbol{v}) = \boldsymbol{T} : \nabla \boldsymbol{v} - \nabla.\boldsymbol{q} + \rho \dot{q}_g \tag{4.64}
$$

this is the differential equation of internal energy balance. Using the mass continuity equation (4.18) it becomes:

$$
\rho\frac{\partial e}{\partial t} + \rho \boldsymbol{v}.\nabla e = \boldsymbol{T} : \nabla \boldsymbol{v} - \nabla.\boldsymbol{q} + \rho \dot{q}_g \tag{4.65}
$$

4.5. **Entropy law.**

The basic law of entropy of a control volume $\mathcal{M}$ says:

*The total entropy over the control volume $\mathcal{M}$ at time $(t_0 + \Delta t)$ is greater than or equal to the total entropy over $\mathcal{M}$ at time $t_0$ plus the net of entropy flow into $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$ plus the total surface heat per temperature into $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$ plus the total heat generation per temperature over $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$.* So:

$$
\begin{aligned}
\Big\{\int_{\mathcal{M}} \rho s \, dV\Big\}_{t_0+\Delta t} &\geqslant \Big\{\int_{\mathcal{M}} \rho s \, dV\Big\}_{t_0} + \int_{t_0}^{t_0+\Delta t}\Big\{\int_{\partial\mathcal{M}} \phi_{ent} \, dS\Big\} d\tau \\
&+ \int_{t_0}^{t_0+\Delta t}\Big\{\int_{\partial\mathcal{M}} \frac{q_s}{T} \, dS\Big\} d\tau + \int_{t_0}^{t_0+\Delta t}\Big\{\int_{\mathcal{M}} \frac{\rho \dot{q}_g}{T} \, dV\Big\} d\tau
\end{aligned}
\tag{4.66}
$$

where $s = s(\boldsymbol{r}, t)$ is the entropy per unit mass and $\rho s$ is the entropy per unit volume. On the right hand side, $\phi_{ent} = \phi_{ent}(\boldsymbol{r}, t, \boldsymbol{n})$ is the entropy flow into the surface that it acts, $q_s = q_s(\boldsymbol{r}, t, \boldsymbol{n})$ is the rate of surface heat into $\mathcal{M}$ per unit area, $\dot{q}_g = \dot{q}_g(\boldsymbol{r}, t)$ is the rate of heat generation per unit mass, and $T = T(\boldsymbol{r}, t)$ is the absolute temperature. In order to convert this inequality to an equation, we may define the rate of entropy generation per unit mass as $\dot{s}_g = \dot{s}_g(\boldsymbol{r}, t)$, where $\dot{s}_g \geqslant 0$, and add the total entropy generation over $\mathcal{M}$ from $t_0$ to $(t_0 + \Delta t)$ to the right hand side of the above inequality. So, we have the following equation:

$$\begin{aligned}\left\{\int_{\mathcal{M}} \rho s\, dV\right\}_{t_0+\Delta t} &= \left\{\int_{\mathcal{M}} \rho s\, dV\right\}_{t_0} + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \phi_{ent}\, dS\right\}d\tau \\ &+ \int_{t_0}^{t_0+\Delta t}\left\{\int_{\partial\mathcal{M}} \frac{q_s}{T}\, dS\right\}d\tau + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\mathcal{M}} \frac{\rho\dot{q}_g}{T}\, dV\right\}d\tau + \int_{t_0}^{t_0+\Delta t}\left\{\int_{\mathcal{M}} \rho\dot{s}_g\, dV\right\}d\tau\end{aligned} \tag{4.67}$$

By using the general equation (4.8) for the flow of energy we have $\phi_{ent} = -(\rho s)\boldsymbol{v}.\boldsymbol{n}$, and by rearranging the equation:

$$\begin{aligned}\left\{\int_{\mathcal{M}} \rho s\, dV\right\}_{t_0+\Delta t} - \left\{\int_{\mathcal{M}} \rho s\, dV\right\}_{t_0} = \int_{t_0}^{t_0+\Delta t}&\left\{\int_{\partial\mathcal{M}} \{\frac{q_s}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{n}\}\, dS\right. \\ &\left. + \int_{\mathcal{M}} \{\frac{\rho\dot{q}_g}{T} + \rho\dot{s}_g\}\, dV\right\}d\tau\end{aligned} \tag{4.68}$$

this is similar to the general equation (4.5). So, by using (4.6) it becomes:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}} \rho s\, dV = \int_{\partial\mathcal{M}} \{\frac{q_s}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{n}\}\, dS + \int_{\mathcal{M}} \{\frac{\rho\dot{q}_g}{T} + \rho\dot{s}_g\}\, dV \tag{4.69}$$

This is the integral equation of entropy law in continuum mechanics. Since $\dot{s}_g \geqslant 0$, by removing the integral of $\rho\dot{s}_g$ from the equation, we have:

$$\frac{\partial}{\partial t}\int_{\mathcal{M}} \rho s\, dV \geqslant \int_{\partial\mathcal{M}} \{\frac{q_s}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{n}\}\, dS + \int_{\mathcal{M}} \frac{\rho\dot{q}_g}{T}\, dV \tag{4.70}$$

This inequality is called the Clausius-Duhem inequality. Using (4.7) and rearranging the equation (4.69), we have:

$$\int_{\mathcal{M}} \left\{\frac{\partial(\rho s)}{\partial t} - \frac{\rho\dot{q}_g}{T} - \rho\dot{s}_g\right\} dV = \int_{\partial\mathcal{M}} \{\frac{q_s}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{n}\}\, dS \tag{4.71}$$

This is similar to the integral equation (4.1), where $B = \partial(\rho s)/\partial t - \rho\dot{q}_g/T - \rho\dot{s}_g$ and $\phi = q_s/T - (\rho s)\boldsymbol{v}.\boldsymbol{n} = q_s/T + \phi_{ent}$. So, the three general local relations (4.2), (4.3), and (4.4) hold for it. The first and second local relations (4.2) and (4.3) lead to:

$$q_s(\boldsymbol{r}, t, \boldsymbol{n})/T + \phi_{ent}(\boldsymbol{r}, t, \boldsymbol{n}) = -q_s(\boldsymbol{r}, t, -\boldsymbol{n})/T - \phi_{ent}(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{4.72}$$

and

$$\begin{aligned}q_s(\boldsymbol{r}, t, \boldsymbol{n})/T + \phi_{ent}(\boldsymbol{r}, t, \boldsymbol{n}) &= n_x\{q_s(\boldsymbol{r}, t, \boldsymbol{e}_x)/T + \phi_{ent}(\boldsymbol{r}, t, \boldsymbol{e}_x)\} \\ &+ n_y\{q_s(\boldsymbol{r}, t, \boldsymbol{e}_y)/T + \phi_{ent}(\boldsymbol{r}, t, \boldsymbol{e}_y)\} + n_z\{q_s(\boldsymbol{r}, t, \boldsymbol{e}_z) + \phi_{ent}(\boldsymbol{r}, t, \boldsymbol{e}_z)\}\end{aligned} \tag{4.73}$$

But we have from (4.60) that $q_s(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{q}(\boldsymbol{r}, t).\boldsymbol{n}$, therefore $q_s/T$ satisfies the two local relations (4.2) and (4.3). Also, as we showed in (4.9) and (4.10), entropy flow $\phi_{ent} = -(\rho s)\boldsymbol{v}.\boldsymbol{n}$ satisfies the two local relations (4.2) and (4.3) and their meanings.

Thus, the two above equations do not give us new results. Applying the third local relation (4.4) to the entropy integral equation (4.71), where $B = \partial(\rho s)/\partial t - \rho\dot{q}_g/T - \rho\dot{s}_g$ and $\phi = q_s/T + \phi_{ent} = -\boldsymbol{q}(\boldsymbol{r}, t).\boldsymbol{n}/T - (\rho s)\boldsymbol{v}.\boldsymbol{n}$, leads to:

$$\begin{aligned}\frac{\partial(\rho s)}{\partial t} - \frac{\rho\dot{q}_g}{T} - \rho\dot{s}_g = \frac{\partial}{\partial x}\Big\{-\frac{\boldsymbol{q}.\boldsymbol{e}_x}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{e}_x\Big\} + \frac{\partial}{\partial y}\Big\{-\frac{\boldsymbol{q}.\boldsymbol{e}_y}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{e}_y\Big\} \\ + \frac{\partial}{\partial z}\Big\{-\frac{\boldsymbol{q}.\boldsymbol{e}_z}{T} - (\rho s)\boldsymbol{v}.\boldsymbol{e}_z\Big\}\end{aligned} \tag{4.74}$$

so, we have

$$\begin{aligned}\frac{\partial(\rho s)}{\partial t} - \frac{\rho\dot{q}_g}{T} - \rho\dot{s}_g &= -\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}q_x/T \\ q_y/T \\ q_z/T\end{bmatrix} - \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}\rho s v_x \\ \rho s v_y \\ \rho s v_z\end{bmatrix} \\ &= -\nabla.(\frac{\boldsymbol{q}}{T}) - \nabla.(\rho s\boldsymbol{v})\end{aligned}$$

by rearranging this equation

$$\frac{\partial(\rho s)}{\partial t} + \nabla.(\rho s\boldsymbol{v}) = -\nabla.(\frac{\boldsymbol{q}}{T}) + \frac{\rho\dot{q}_g}{T} + \rho\dot{s}_g \tag{4.75}$$

This is the differential equation of entropy law in continuum mechanics. Using the mass continuity equation (4.18) it becomes:

$$\rho\frac{\partial s}{\partial t} + \rho\boldsymbol{v}.\nabla s = -\nabla.(\frac{\boldsymbol{q}}{T}) + \frac{\rho\dot{q}_g}{T} + \rho\dot{s}_g \tag{4.76}$$

Since $\dot{s}_g \geqslant 0$, by removing $\rho\dot{s}_g$ from the equation (4.75) we have:

$$\frac{\partial(\rho s)}{\partial t} + \nabla.(\rho s\boldsymbol{v}) \geqslant -\nabla.(\frac{\boldsymbol{q}}{T}) + \frac{\rho\dot{q}_g}{T} \tag{4.77}$$

This is the differential form of the Clausius-Duhem inequality.

## 5. Conclusion

We considered the general integral equation on the control volume $\mathcal{M}$, as the form:

$$\int_{\mathcal{M}} B\, dV = \int_{\partial\mathcal{M}} \phi\, dS$$

where $B = B(\boldsymbol{r}, t)$ is called body term and $\phi = \phi(\boldsymbol{r}, t, \boldsymbol{n})$ is called surface term. These functions are continuous over the volume and the surface of $\mathcal{M}$, respectively. Here if $B$ is scalar then $\phi$ must be scalar, and if $B$ is vector then $\phi$ must be vector. We wanted to determine how many local relations can be derived from this general integral equation and what they are.

We first derived the general Cauchy lemma for surface term from the above integral equation as the first local relation. So:

The first local relation:

$$\phi(\boldsymbol{r}, t, \boldsymbol{n}) = -\phi(\boldsymbol{r}, t, -\boldsymbol{n})$$

Then by a new general exact tetrahedron argument we showed that applying the general integral equation to a tetrahedron control volume leads to the following fundamental equation:

$$E_0 + E_1\frac{1}{3}h + E_2\frac{1}{12}h^2 + E_3\frac{1}{60}h^3 + \ldots + E_m\frac{2}{(m+2)!}h^m + \ldots = 0$$

where $h$ is the altitude of the tetrahedron. $E_m$'s are expressions that contain the surface term, body term, their derivatives, and the powers of the components of unit normal vector of the base face of tetrahedron. Then we showed that the only solution of this equation is:

$$E_m = 0, \qquad m = 0, 1, 2, \ldots, \infty$$

i.e., all of the $E_m$'s must be equal to zero. By these, we proved that $E_0 = 0$ leads to the second local relation that obtains from the general integral equation as:

The second local relation:

$$\phi(\boldsymbol{r}, t, \boldsymbol{n}) = n_x\phi(\boldsymbol{r}, t, \boldsymbol{e}_x) + n_y\phi(\boldsymbol{r}, t, \boldsymbol{e}_y) + n_z\phi(\boldsymbol{r}, t, \boldsymbol{e}_z)$$

and $E_1 = 0$ leads to the third local relation that is a partial differential equation as:

The Third local relation:

$$B(\boldsymbol{r}, t) = \frac{\partial\phi(\boldsymbol{r}, t, \boldsymbol{e}_x)}{\partial x} + \frac{\partial\phi(\boldsymbol{r}, t, \boldsymbol{e}_y)}{\partial y} + \frac{\partial\phi(\boldsymbol{r}, t, \boldsymbol{e}_z)}{\partial z}$$

In other equations $E_m = 0$, for $m = 2, 3, \ldots, \infty$, the results of $E_0 = 0$ and $E_1 = 0$ are repeated. Then we showed that all the fundamental laws of continuum mechanics can be shown in the form of the general integral equation that we considered it. So, the three general local relations hold for the integral forms of the fundamental laws.

These three local relations for the conservation of mass lead to the properties of mass flow and derivation of the mass continuity equation. For the conservation of linear momentum, the first local relation leads to the Cauchy lemma for traction vectors, the second local relation leads to the existence of stress tensor, and the third local relation leads to the general equation of motion. For the conservation of angular momentum, the first and second local relations repeat the results of these two local relations in the conservation of linear momentum but the third local relation leads to the symmetry of stress tensor. For the conservation of energy, the first local relation leads to the Cauchy lemma for surface heat, the second local relation leads to the existence of heat flux vector, and the third local relation leads to the differential conservation equation of total energy. For the entropy law the first and second local relations repeat the results of these two local relations in the conservation of energy and the third local relation leads to the differential form of entropy law and the Clausius-Duhem inequality.

***Dedication:*** *This article is dedicated to my mother B. Hussaini, my father M. Azadi, and my sisters and brothers.*

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