Effects of Coordinate Curvature on Integration

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Abstract

In this paper, the integration of a function over a curved manifold is examined in the case where the curvature of the manifold results in a varying density of coordinates over which the function is being integrated where the upper bound of the of integration is infinity. It is shown that when the coordinate density varies in such a case, the true area under the curve is not correctly calculated by traditional techniques of integration. This situation is then applied to the Schwarzschild metric and geodesic equation of General Relativity to examine the proper time taken for a freefalling observer to reach the event horizon of a black hole.

Integration and Coordinate Density

Consider a velocity $\frac{dx}{dt}$ defined by some function that is parameterized by a variable *r* where *r* increases (or decreases) as *t* increases such that $\frac{dx}{dt} = f(r)$. We will begin by integrating this function between t_0 and some finite *t*. In Figure 1, we see this function plotted in two different cases: on the left, we have a flat manifold where the time coordinate density is constant along the length of the function, and on the right we have a curved manifold which causes the density of the time coordinates to increase as *r* increases.

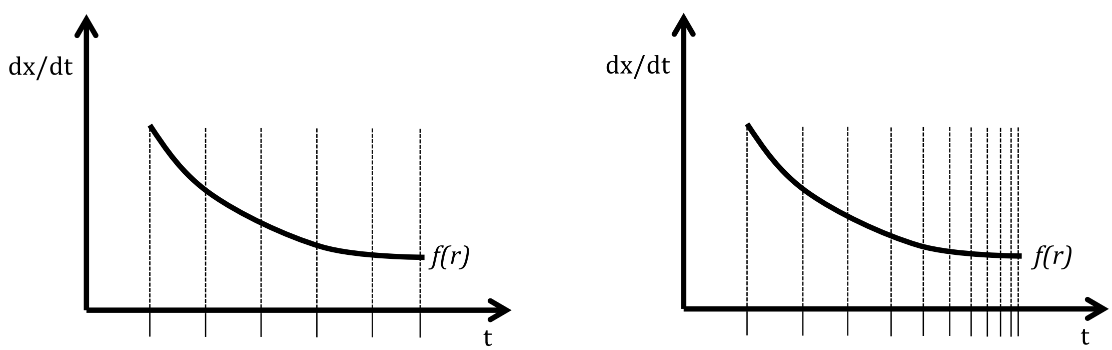

Figure 1 – Velocity vs. Time on Flat (left) and Curved (right) Manifolds (Finite Upper Bound)

On both the left and right sides of Figure 1, the numerical Δt between tick marks on the time axis is the same between any two adjacent tick marks. Now let us examine what happens when we approximate the integrals by summing the areas of the rectangles (where each rectangle has area $A = \left(\frac{dx}{dt}(t)\right)(\Delta t)$). When comparing the left and right images in Figure 1, we see that since the Δt between tick marks in both images is the same, the approximate integral on the right side will give a larger value than the integral on the left as a result of the increasing density of the time coordinate as r increases (we have more rectangles of equal Δt on the right side). Now, it is not entirely surprising that the integral on the right would be larger than that on the left since we are essentially integrating the function over a larger time interval on the right side (we could just stretch the coordinates on the right side such that they have equal density and then integrate normally over a larger time interval). The purpose of Figure 1 is to introduce the effect of increasing coordinate density on an integral to emphasize that when we approximate the integral by summing rectangles when the density increases, we get a larger value for the integral because we are summing more rectangles of equal Δt.

Now consider the same type of velocity function integrated from some finite time to *infinite* time in flat and curved spacetime. Figure 2 shows both of these cases for a velocity function that decreases to zero as t goes to infinity.

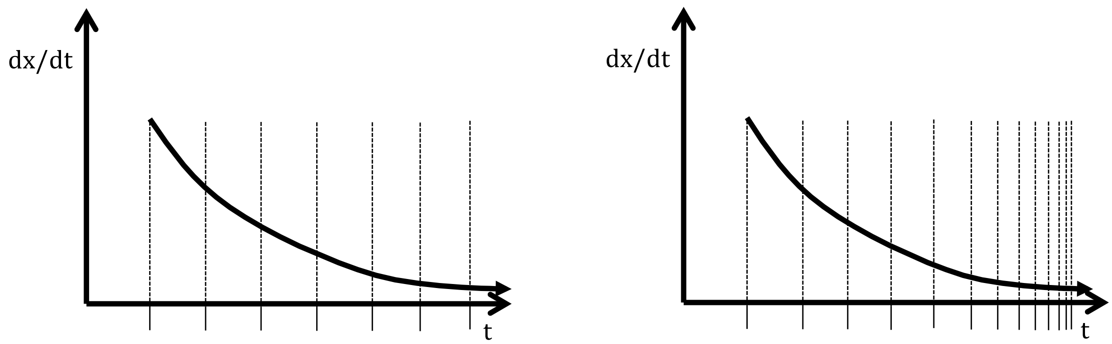

Figure 2 – Velocity vs. Time on Flat (left) and Curved (right) Manifolds (Infinite Upper Bound)

Just as was the case in Figure 1, we can see that when we approximate the integrals in Figure 2 by summing the areas of rectangles, the integral on the right will give a larger than the integral on the left. However, in this case, since the upper bound of t is infinite in both cases, we can't attribute the increase in area to an increase in the time interval. If we suppose that $f(r)$ decreases in such a way that the integral from t_0 to ∞ on the left side (flat manifold) gives a finite value, we can see that the integral on the right side will give a value greater than that and if the coordinate density goes to infinity as $\frac{dx}{dt}$ goes to zero, the integral can even be infinite. This idea of coordinate density can be thought of as being analogous to a dynamic unit change. For instance, in the flat manifold case, suppose x and t were measured in the same units and we multiply the integral by a constant. That would essentially be a change of units (minutes to seconds or mm to meters). But a change of units is really just a rescaling of the axes. So since the coordinate density is describing how the coordinates are scaled over the manifold, it is as if the coordinate units are being changed as you move along the manifold. This is essentially what length contraction and time dilation in General Relativity is, a relative stretching or squeezing of the coordinate axes.

Figure 3 illustrates this concept by examining an observer moving over both flat and curved coordinates:

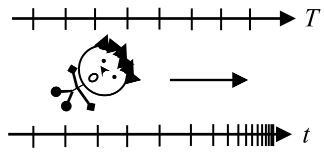

Figure 3 – Motion Over Flat and Curved Coordinates

In Figure 3 our observer, Scout, is moving inertially over both flat coordinates (T) and curved coordinates (t). Since Scout is inertial and the T coordinates are flat, this means that constant intervals of T correspond to constant intervals of time measured by the clock she is carrying with her $\left(\frac{\Delta T}{\Delta \tau} = const\right)$. But we can see by inspection that as she moves, the amount of time ticked by her clock relative to the number of t ticks she passes decreases over time $\left(\frac{\Delta \tau}{\Delta t} = f(t)\right)$. In fact, we could construct the t coordinate axis such that it extends infinitely off to the right while each successive t tick gets closer, making $f(t) \to 0$ as Scout moves to infinite τ. We can then imagine shrinking the intervals of T and t to an infinitesimal size such that the coordinates are continuous, and we would find that it must be that the sizes of the infinitesimals dT and dt are different, namely that dt becomes increasingly smaller than dT as Scout moves.

We will examine observers at rest and in freefall in a gravitational field to assess this situation in more detail.

Radial Motion in the Schwarzschild Field

The well-known Schwarzschild metric is given in (1) below (note we will be using units where the speed of light is 1 and we will drop the angular term of the metric since we will only be examining radial motion):

$$d\tau^2 = \left[1 - \frac{2GM}{r}\right] dt^2 - \left[1 - \frac{2GM}{r}\right]^{-1} dr^2 \tag{1}$$

The r coordinate represents some notion of distance from the center of the gravitational source. Thus, this radial coordinate gives circles around the source where, in a top-down view of the source, the circle radii increase linearly as one moves away from the center. Let's now consider the coordinate speed of a freefalling observer (who starts to fall from rest at infinity) [1]:

$$\frac{dr}{dt} = -\sqrt{\frac{2GM}{r}}\left[1 - \frac{2GM}{r}\right] \tag{2}$$

Let us now substitute (2) into (1) to examine the proper time of the freefalling observer:

$$\frac{d\tau}{dt} = \left[1 - \frac{2GM}{r}\right] \tag{3}$$

Now, the freefalling observer is inertial and is therefore similar to Scout from Figure 3. It was shown in Figure 3 that Scout's acceleration through the *t* dimension was only due to the curvature of the *t* coordinates relative to the flat coordinates. Let us consider the coordinate acceleration of particles in the Schwarzschild field [2]:

$$\frac{d^2x^\lambda}{d\tau^2} = A^\lambda - \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{4}$$

Equation 4 is a re-arranged equation for proper acceleration. What equation 4 tells us is that the coordinate acceleration of a particle is the real acceleration of the particle minus the effects of coordinate curvature. The *A* term can be thought of as the flat space acceleration, resulting in a force that an observer objectively feels as she moves. The second term is the part of the coordinate acceleration caused by the coordinate curvature. This is what is responsible for Scout's acceleration. For the freefalling observer (*A = 0*), we get the following geodesic equations [2]:

$$\frac{d^2t}{d\tau^2} = -\frac{2GM}{r\sqrt{r-2GM}}\frac{dr}{d\tau}\frac{dt}{d\tau} \tag{5}$$

$$\frac{d^2r}{d\tau^2} = \frac{GM}{r^2}\left[\frac{2r}{\sqrt{r-2GM}}\left(\frac{dr}{d\tau}\right)^2 - \frac{(r-2GM)}{r}\left(\frac{dt}{d\tau}\right)^2\right] \tag{6}$$

We notice from Equation 6 that an inertial observer starting from any finite $r > 2GM$ with $\frac{dr}{d\tau} = 0$ must begin accelerating through *r* due to the second term in Equation 6. This means that $\frac{dr}{d\tau}$ will become non-zero and therefore $\frac{d^2t}{d\tau^2}$ will become non-zero. Thus, the inertial observer must accelerate through the time dimension, but from Equation 4, we know that this acceleration is purely a coordinate artifact as depicted in Figure 3 ($A^t = 0$ for the inertial observer). Thus the time coordinate in the Schwarzschild metric must be compressed relative to the flat time coordinate akin to the *t* coordinate in Figure 3.

Next, let's consider an observer at rest at some *r* in the gravitational field. In this case $\frac{d^2r}{d\tau^2} = \frac{dr}{d\tau} = 0$ for all time and therefore $\frac{d^2t}{d\tau^2} = 0$ for all time (we can say that $A^t = 0$ because a non-zero A^t would correspond to some change in rest energy which we do not have in this case). But $A^r = -\frac{GM}{r^2}$ (we know from Equation 1 that an observer at rest has $\frac{dt}{d\tau} = \sqrt{\frac{r}{r-2GM}}$) and therefore the rest observer does not follow a geodesic, she objectively feels a force as if she were accelerating in flat space. We can see why this would be the case again from Figure 3. If Scout were the rest observer, she would not accelerate relative to the *t* coordinate (her $\frac{\Delta\tau}{\Delta t}$ would be constant). This means that she would have to accelerate relative to the flat *T* coordinate, and therefore she feels a force due to her acceleration (from our perspective, she would slow down as she crossed the screen).

The event horizon at $r = 2GM$ lies at $t = \infty$. The argument being made here is that integrating Equation 3 in the usual way is incorrect because it assumes a constant magnitude *dt*. But from Figures 1 to 3, it has been suggested that *dt* must be treated as a function of *t*. Given that the coordinate acceleration for the freefalling observer comes entirely from the manifold curvature and not from any real flat-space acceleration, we might re-express Equation 4 as:

$$\frac{d^2x^\lambda}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dx^\lambda}{d\tau}\right) - \frac{d}{d\tau}\left(\frac{dx^\lambda}{dT}\right) \tag{7}$$

Equation 7 states that the coordinate acceleration is equal to the actual rate of change of the velocity (as would be felt in flat space) minus the rate of change of the coordinate curvature relative to flat space.

Figure 4 shows worldlines of rest observers and the inertial observer plotted against *t*, *T*, and τ showing how the coordinate curvature curves the worldlines of rest observers relative to flat space (rest observers are the solid curved lines of constant $\frac{dt}{d\tau}$, the inertial observer is the straight dashed line):

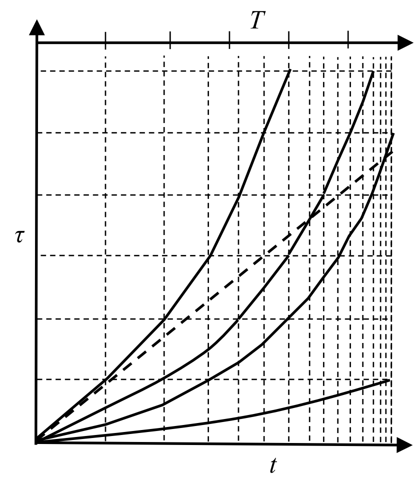

Figure 4 – τ vs. *t* for Observers at Rest in a Gravitational Field

If we consider the rest observers from the perspective of an inertial observer freefalling from infinity, we can view it as follows. First, the rest observers will appear to be moving *away* from the inertial observer with a velocity proportional to their distance from the center of gravity (their $\frac{d\tau}{dt} < 1$ results in a gravitational redshift of the light they emit observed by the observer at infinity). As time passes for the freefalling observer, the rest observers will accelerate toward her with a constant acceleration proportional to their distance from the CG. This acceleration will initially appear as a reduction in redshift from their initial velocity (they still appear to be moving away, but are slowing down). At each time in the freefaller's frame, there will be one observer whose signals will no longer be redshifted, and will therefore seem momentarily at rest relative to her (on Figure 4, this would be when a particular rest curve has a 45-degree slope. Specifically, this will happen when $1 - \frac{2GM}{r_{freefall}} = \sqrt{1 - \frac{2GM}{r_{rest}}}$). When a rest observer has passed that point, they will appear to be moving toward the freefaller with increasing speed and their signals will be blueshifted in her frame. The rest observer at r will appear to pass the freefalling observer with speed $\frac{dr}{d\tau} = \sqrt{\frac{2GM}{r}}$ (such that $\frac{dr}{d\tau} = 1$ corresponds to the speed of light). The rest observers will pass the freefaller and their signals will become increasingly redshifted over time in her frame as they move away.

Given this light signal analysis and Figure 4, we can construct a τ vs. r diagram for the freefalling observer:

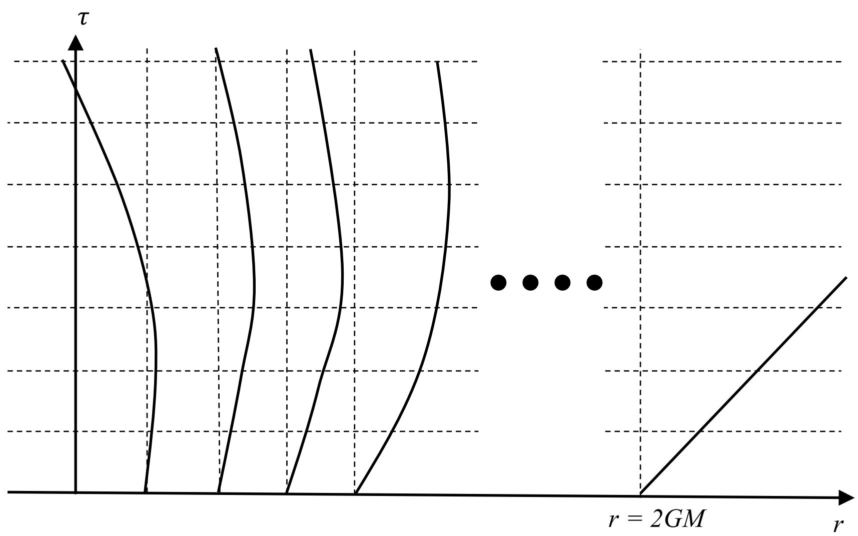

Figure 5 - τ vs. r for the Inertial Observer

In Figure 5, the inertial observer moves along the τ-axis. The solid curved lines represent curves of constant *r* in the inertial frame. Figures 4 and 5 emphasize the fact that it is the rest observers that are accelerating relative to the inertial observer and not the other way around. In Figure 5, the inertial observer reaches a particular rest observer when the rest observer curve intersects the τ-axis. What we see here is that the $r = 2GM$ curve is a 45-degree line (light-speed motion away from the inertial observer – infinite redshift) that never turns to intersect the τ-axis. Similarly, in Figure 4, the $r = 2GM$ curve is a flat horizontal line that never curves up.

These Figures suggest that the inertial observer will never reach the event horizon in finite time according to her clock. If we integrate Equation 3 to get the total proper time to the horizon, we get a finite number. This is because although we are integrating over an infinite time, the increase in proper time per unit time decreases in such a way that the integral is asymptotic. But given that the decreasing derivative comes entirely from the coordinate curvature, we can see from Figures 3, 4, and 5 that we could get an infinite proper time in spite of the derivative. This is most clear from Figure 3. We can see from Figure 3 that it is the rest observers that will approach some kind of finite condition. In this case, a given rest observer (not accelerating relative to the *t* coordinate) will asymptotically approach some finite *T* as their clock goes to infinity. This illustrates the problem with the traditional integral in curved space. Because in Figure 3, if we talk about moving to the right in units of *t*, we will inevitably asymptote since the distance between intervals of t decreases (relative to intervals of t further to the left). But Scout can just move constantly relative to the T coordinate which has a constant spacing (and therefore accelerate relative to the t coordinate) and will therefore move off infinitely to the right, regardless of how compressed the t coordinate gets. This idea is shown more explicitly in Figures 4 and 5, where the horizon worldline never curves into the path of the inertial worldline. Given that the T and t axes are continuous, we can always take a small region of T at different locations and find that the average density of t in that region is arbitrarily large (no matter how dense the axis is in one region, it can always be denser in some other region). Thus, in Figure 3, rest observers will all appear to slow down asymptotically as they move (constant $\frac{d\tau}{dt}$) while the freefaller moves with constant speed and *all* observers will experience infinite proper time because the t-axis extends inifinitely. That the size of the infinitessimals are variable over the manifold suggests that this issue may involve countable vs. uncountable infinites, but that will not be analyzed here.

<u>Radial Coordinate Transformation</u>

It is desirable at this point to make a coordinate change for the radial coordinate such that it is better able to capture the curvature near the horizon in the same way the time coordinate does. We will choose coordinate R such that $\frac{dR}{dr} = \frac{r}{r-2GM}$. This coordinate varies identically to the r coordinate for large r (this is good because r is a good physical coordinate at large r) and then diverges from it at the horizon. Note that $R \to \infty$ as $r \to \infty$ and $R \to -\infty$ as $r \to 2GM$. Making this coordinate substitution in (2) gives:

$$\frac{dR}{dt} = -\sqrt{\frac{2GM}{r}} \tag{9}$$

This coordinate choice is also useful because the speed of light in these coordinates is 1 independent of R and t. The Schwarzschild metric with the new coordinate becomes:

$$d\tau^2 = \frac{r-2GM}{r}[dt^2 - dR^2] \tag{10}$$

In terms of the R coordinate, the τ-R grid will look very much like Figure 4 because the t and R coordinates are curved by the same factor. However, we from Equation 9 that $\frac{d\tau}{dR} = -\frac{d\tau}{dt}\sqrt{\frac{r}{2GM}}$. Therefore, rather than getting a straight line for the freefalling observer as was the case in Figure 4, the $\sqrt{\frac{r}{2GM}}$ factor gives us the worldline shown in Figure 6 (recall that $r \to 2GM$ as $R \to -\infty$):

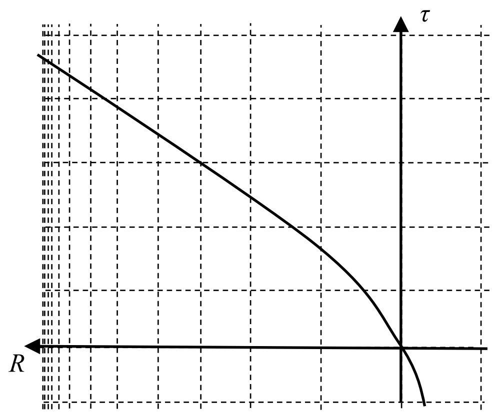

Figure 6 – τ vs. R for the Freefalling Observer

Combining Equations 1 and 2, we see that $\frac{d\tau}{dr} \to 1$ as $r \to 2GM$. This is because as r goes to $2GM$, both $\frac{dr}{dt}$ and $\frac{d\tau}{dt}$ for the freefalling observer go to zero, where $\frac{d\tau}{dt}$ goes to zero for the reasons discussed above, and $\frac{dr}{dt}$ goes to zero because of the extreme curvature of space near $r = 2GM$ (the factor $\left[1 - \frac{2GM}{r}\right]$).

Observations from the Central Observer

Further evidence for an infinite proper time to the horizon will be given by considering an observer at rest at the center of a collapsing spherically symmetric shell. According to Birkhoff's theorem, the space inside the shell, where the central observer is, will be flat. Therefore, according to the clock of an observer at infinity, light within the shell will travel just like it does at infinity. Therefore, as the collapsing shell approaches its Schwarzschild radius (say 1 light-second), the observer at infinity will find that according to her clock, it will take just over 1 second for a signal to travel from the central observer to an observer on the shell. But the clocks of both the central observer and shell observer will slow to a near stop relative to the observer at infinity. Thus in the frames of the central and shell observers, signals exchanged between them will be received almost instantly as the shell approaches its Schwarzschild radius. Thus, in their frame, it will appear as though the space between $r = 0$ and $r = 2GM$ contracts to zero proper distance as the shell reaches its Schwarzschild radius. In other words, in the collapsing frame, $r = 2GM$ will correspond to the center of gravity (there will be nowhere else to fall after that in the freefall frame). It is also notable that the clock of the central observer ticks at the same rate as an observer at rest at the location of the shell. Therefore, if the shell were actually able to reach the horizon, the central observer's clock would stop ticking and signals from it would be infinitely blueshifted when received by the collapsing shell (it is easily shown that the relative velocity between the central and freefalling observer is $V = \frac{dr}{d\tau} = \sqrt{\frac{2GM}{r}}$ and the time dilation between the freefalling and central clocks is governed by the $\sqrt{1 - V^2}$ factor which goes to zero at $r = 2GM$). This is yet another example as to why it is nonsensical for the shell to be able to reach $r = 2GM$ in a finite time.

Conclusion

It has been shown that when accounting for curved spacetime while integrating the freefall geodesic, the freefaller experiences an infinite amount of proper time before reaching the horizon. We also know that the freefalling worldline approaches a null geodesic asymptotically, as can be deduced from Figure 6. This means that there will be a final light signal receivable by the freefaller from rest observers. Therefore, we must conclude that in the frame of the freefalling observer near the horizon, when she looks out to signals coming from the rest observers, those observers will appear to her to be slowing down since she experiences infinite proper time in her frame while receiving a finite number of light signals from the rest observers. What we find is that the rest observers will see the freefalling observer slow *exponentially* as their times go to infinity, while the freefaller will see the rest observers slow *asymptotically* as her time goes to infinity. This means that in the rest observer frame, the freefaller will have an open future, unfolding at an exponentially slower rate over time, while in the freefalling frame the rest observers will have a closed future, where the rest observers will appear to evolve toward a finite future time at an asymptotically slower rate. These features are shown in Figure 7 below:

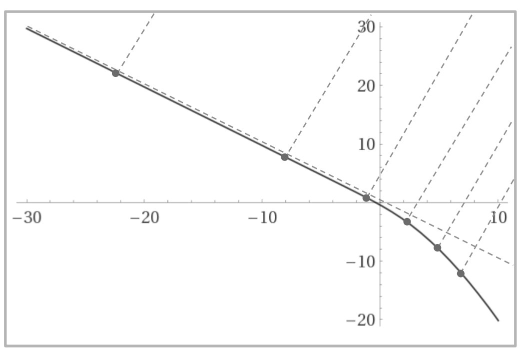

Figure 7 – Light Signals on *t*-*R* Chart

Figure 7 is a *t*-*R* chart that shows a single infalling signal representing the signal to which the freefall worldline is asymptotic. The freefalling observer will receive this signal after an infinite proper time and will receive no signals lying above that one on the chart. If at any time the freefaller accelerates in a direction away from the black hole, he will receive more future signals from the rest observers beyond this asymptotic signal since his worldline will curve upwards on Figure 7 as a result of his acceleration. Then if he stops accelerating and begins freefall again, there will be a new light signal to which his worldline will be asymptotic. The dots in Figure 7 represent intervals of equal proper time along the worldline and we can see that since the worldline is infinite (with tangents always below the speed of light) on this chart, there will be an infinite number of dots on the line spaced increasingly far apart and rest observers will receive an infinite number of signals from the freefalling observer at longer and longer intervals.

References

[1] Raine, D., Thomas, E.: Black Holes: A Student Text. Imperial College Press, (2015).

[2] “Lecture Notes on General Relativity”: https://arxiv.org/pdf/gr-qc/9712019.pdf. Cited July 17, 2017.