EXACT TETRAHEDRON ARGUMENT FOR THE EXISTENCE OF STRESS TENSOR AND GENERAL EQUATION OF MOTION

EHSAN AZADI[1]

Abstract. The birth of modern continuum mechanics is the Cauchy's idea for traction vectors and his achievements of the existence of stress tensor and derivation of the general equation of motion. He gave a proof of the existence of stress tensor that is called Cauchy tetrahedron argument. But there are some challenges on the different versions of tetrahedron argument and the proofs of the existence of stress tensor. We give a new proof of the existence of stress tensor and derivation of the general equation of motion. The exact tetrahedron argument gives us, for the first time, a clear and deep insight into the origins and the nature of these fundamental concepts and equations of continuum mechanics. This new approach leads to the exact definition and derivation of these fundamental parameters and relations of continuum mechanics. By the exact tetrahedron argument we derived the relation for the existence of stress tensor and the general equation of motion, simultaneously. In this new proof, there is no limited, average, or approximate process and all of the effective parameters are exact values. Also in this proof, we show that all the challenges on the previous tetrahedron arguments and the proofs of the existence of stress tensor are removed.

1. Introduction

The existence of stress tensor and the general equation of motion form the main part of the foundation of continuum mechanics. In 1822 to 1828, Cauchy introduced the basic idea of *traction vector* and presented a proof of the existence of *stress tensor* that is called *Cauchy tetrahedron argument* and by using another process he obtained the general equation of motion that is called *Cauchy equation of motion*. He also derived some important properties of the state of stress, e.g. the *symmetry of stress tensor* [3,4, 7,8]. The basic idea of Cauchy was that the internal forces on the surface in continuum media in addition to the normal component can have the tangential components. From Truesdell in (1968, [8]), on pages 336 and 338:

> *Thus it might seem that CAUCHY's achievement in formulating and developing the general theory of stress was an easy one. It was not. CAUCHY's concept has the simplicity of genius. Its deep and thorough originality is fully outlined only against the background of the century of achievement by the brilliant geometers who preceded, treating special*

[1]Department of Mechanical Engineering, Sharif University of Technology, Iran.
E-mail address: eazadi65@gmail.com, Submitted to vixra.org 4 July 2017.

> *kinds and cases of deformable bodies by complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became and has remained the foundation of the mechanics of gross bodies.*

We already provided a comprehensive review of the different versions of tetrahedron argument and the proofs of the existence of stress tensor presented in the published books during about two centuries from the birth of the Cauchy's idea (1822) to this time and we considered the important challenges and the improvements of each one (2017, [1]).

In this article, for the first time, we give the exact tetrahedron argument that removes all the stated challenges and opens a new and deep insight into the foundation of continuum mechanics and the nature of the traction vector, the stress tensor, and the general equation of motion.

In order to present the exact tetrahedron argument, first we give the general forms of the conservation of linear momentum for a mass element and prove the important relation that is called *Cauchy lemma* for the traction vectors that act on the opposite sides of the same surface. Then, the exact tetrahedron argument will be presented. We also study some aspects of this new proof and discuss the challenges that hold for the previous tetrahedron arguments and the proofs of the existence of stress tensor, on this new proof.

The integral equation of conservation of linear momentum on a mass element in continuum media is:

$$\frac{d}{dt}\int_{\mathcal{M}} \rho\boldsymbol{v}\,dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\,dS + \int_{\mathcal{M}} \rho\boldsymbol{b}\,dV \tag{1.1}$$

where $\rho = \rho(\boldsymbol{r}, t)$ is the density, $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, t)$ is the velocity vector, and $\rho\boldsymbol{v}$ is the linear momentum per unit volume of the mass element $\mathcal{M}$. On the right hand side, $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ is the surface force per unit area that is called traction vector and acts on the surface of the mass element, i.e., $\partial\mathcal{M}$, and $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{r}, t)$ is the body force per unit mass. Here $\boldsymbol{r}$ is the position vector, t is time, and $\boldsymbol{n}$ is the outward unit normal vector on the surface of mass element. By using the transport theorem and the conservation of mass [5, 9], the left hand side of the equation becomes:

$$\frac{d}{dt}\int_{\mathcal{M}} \rho\boldsymbol{v}\,dV = \int_{\mathcal{M}} \left\{\frac{\partial(\rho\boldsymbol{v})}{\partial t} + \nabla.(\rho\boldsymbol{v})\right\} dV = \int_{\mathcal{M}} \left\{\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}.\nabla)\boldsymbol{v}\right\} dV = \int_{\mathcal{M}} \rho\boldsymbol{a}\,dV \tag{1.2}$$

where $\boldsymbol{a} = \partial\boldsymbol{v}/\partial t + (\boldsymbol{v}.\nabla)\boldsymbol{v}$ is the acceleration vector. By rearranging the equation (1.1):

$$\int_{\mathcal{M}} (\rho\boldsymbol{a} - \rho\boldsymbol{b})\,dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\,dS \tag{1.3}$$

for simplicity, we use $\boldsymbol{B} = (\rho\boldsymbol{a} - \rho\boldsymbol{b})$ that is called *body term* within the proof. So, the equation (1.3) rewrites as:

$$\int_{\mathcal{M}} \boldsymbol{B}\,dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\,dS \tag{1.4}$$

In general, $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r}, t)$ and $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ are continuous functions in their scope in continuum media.

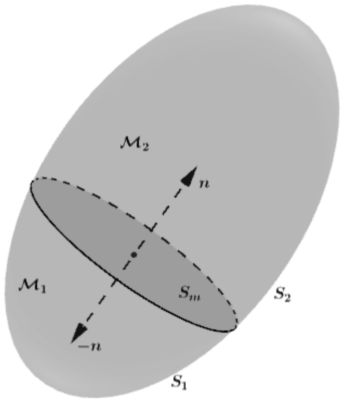

FIGURE 1. The mass elements $\mathcal{M}_1$ and $\mathcal{M}_2$, where $\partial\mathcal{M}_1 = S_1 \cup S_m$ and $\partial\mathcal{M}_2 = S_2 \cup S_m$, and the mass element $\mathcal{M}$ such that $V_{\mathcal{M}} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$ and $\partial\mathcal{M} = S_1 \cup S_2$.

2. Cauchy Lemma

Cauchy lemma deals with the traction vectors that act on the opposite sides of the same surface at a given point and time. There are some approaches to prove this lemma in the literature. Here we present a proof of the Cauchy lemma that is nearly similar to the proofs in [2, 6]. Suppose the mass element $\mathcal{M}$ splits into $\mathcal{M}_1$ and $\mathcal{M}_2$ by the surface S_m in the way that $V_{\mathcal{M}} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$, $\partial\mathcal{M}_1 = S_1 \cup S_m$, $\partial\mathcal{M}_2 = S_2 \cup S_m$, and $\partial\mathcal{M} = S_1 \cup S_2$, see Figure 1. If the equation (1.4) applies to $\mathcal{M}_1$ and $\mathcal{M}_2$, then the sum of these equations is:

$$\int_{\mathcal{M}_1} \boldsymbol{B}_1\, dV + \int_{\mathcal{M}_2} \boldsymbol{B}_2\, dV = \int_{\partial\mathcal{M}_1} \boldsymbol{t}_1\, dS + \int_{\partial\mathcal{M}_2} \boldsymbol{t}_2\, dS$$

By $V_{\mathcal{M}} = V_{\mathcal{M}_1} \cup V_{\mathcal{M}_2}$, the sum of the body term integrals is equal to the integral of the body term on $\mathcal{M}$. In addition, by $\partial\mathcal{M}_1 = S_1 \cup S_m$ and $\partial\mathcal{M}_2 = S_2 \cup S_m$, the surface integrals split as:

$$\int_{\mathcal{M}} \boldsymbol{B}\, dV = \int_{S_1} \boldsymbol{t}_1\, dS + \int_{S_m} \boldsymbol{t}_1\, dS + \int_{S_2} \boldsymbol{t}_2\, dS + \int_{S_m} \boldsymbol{t}_2\, dS$$

By $\partial\mathcal{M} = S_1 \cup S_2$, the sum of the surface integrals on S_1 and S_2 is equal to the surface integral of $\boldsymbol{t}$ on $\partial\mathcal{M}$, so:

$$\int_{\mathcal{M}} \boldsymbol{B}\, dV = \int_{\partial\mathcal{M}} \boldsymbol{t}\, dS + \int_{S_m} \boldsymbol{t}_1\, dS + \int_{S_m} \boldsymbol{t}_2\, dS$$

Comparing this integral equation with the integral equation (1.4), implies that:

$$\int_{S_m} \boldsymbol{t}_1\, dS + \int_{S_m} \boldsymbol{t}_2\, dS = \boldsymbol{0}$$

But $\boldsymbol{t}_1$ on S_m is $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$, and $\boldsymbol{t}_2$ on S_m is $\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$, so:

$$\int_{S_m} \big\{\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) + \boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})\big\}\, dS = \boldsymbol{0}$$

therefore, we have

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n}) \tag{2.1}$$

This is the Cauchy lemma that is derived by using the integral equation of conservation of linear momentum (1.4). It states *"the traction vectors acting on opposite sides of the same surface at a given point and time are equal in magnitude but opposite in direction"*.

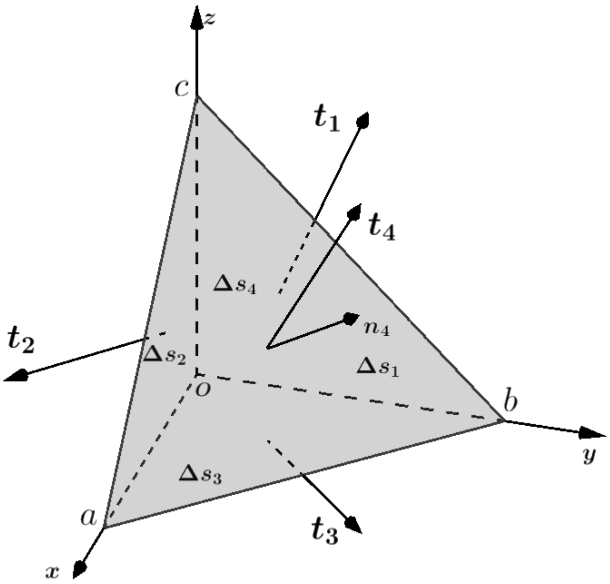

FIGURE 2. The geometry of tetrahedron element and the exact traction vectors on the faces.

3. Exact Tetrahedron Argument

Today there is a belief that the foundation of mechanics is a dead subject, but this is not correct. Here, for the first time, we present and prove the exact tetrahedron argument.

Consider a tetrahedron element in continuum media that its vortex is at the point $\boldsymbol{o}$ and its three orthogonal faces are parallel to the three orthogonal planes of the Cartesian coordinate system. The fourth surface of the tetrahedron, i.e., its base, has the outward unit normal vector $\boldsymbol{n}_4$. For simplicity, the vortex point is at the origin of the coordinate system. The geometrical parameters are shown in Figure 2. The vector $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ is the position vector from the origin of the coordinate system. Now the integral equation of conservation of linear momentum (1.4) applies to this tetrahedron mass element:

$$\int_{\Delta s_4} \boldsymbol{t}_4 \, dS + \int_{\Delta s_1} \boldsymbol{t}_1 \, dS + \int_{\Delta s_2} \boldsymbol{t}_2 \, dS + \int_{\Delta s_3} \boldsymbol{t}_3 \, dS = \int_{\mathcal{M}} \boldsymbol{B} \, dV \tag{3.1}$$

The key idea of this proof is to write the variables of this equation in terms of the exact Taylor series about a point in the domain. Here, we derive these series about the vortex point of tetrahedron (point $\boldsymbol{o}$), where the three orthogonal faces pass through it. Note that time (t) is the same in the all terms, so it does not exist in the Taylor series. For $\boldsymbol{B}(\boldsymbol{r}, t)$ at any point in the domain of the mass element, we have:

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_o + \frac{\partial \boldsymbol{B}_o}{\partial x} x + \frac{\partial \boldsymbol{B}_o}{\partial y} y + \frac{\partial \boldsymbol{B}_o}{\partial z} z \\ &+ \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{B}_o}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{B}_o}{\partial y^2} y^2 + \frac{\partial^2 \boldsymbol{B}_o}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{B}_o}{\partial x \partial y} xy + 2\frac{\partial^2 \boldsymbol{B}_o}{\partial x \partial z} xz + 2\frac{\partial^2 \boldsymbol{B}_o}{\partial y \partial z} yz \Big) \\ &+ \ldots = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!n!k!} \frac{\partial^{(m+n+k)} \boldsymbol{B}}{\partial x^m \partial y^n \partial z^k}\Big|_o x^m y^n z^k \end{aligned} \tag{3.2}$$

Here $\boldsymbol{B}_o$ and $\partial \boldsymbol{B}_o / \partial x$ are the exact values of $\boldsymbol{B}$ and $\partial \boldsymbol{B} / \partial x$ at the point $\boldsymbol{o}$, respectively. Similarly, the other derivatives are the exact values of the corresponding derivatives of

$\boldsymbol{B}$ at the point $\boldsymbol{o}$. On the surface Δs_1, $x=0$ and $\boldsymbol{n}_1$ does not change, so:

$$\begin{aligned}\boldsymbol{t}_1 &= \boldsymbol{t}_{1_o} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial y} y + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z} z + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2} y^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z} yz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)} \boldsymbol{t}_1}{\partial y^m \partial z^k}\Big|_o y^m z^k\end{aligned} \tag{3.3}$$

where $\boldsymbol{t}_{1_o}$ is the exact value of the traction vector $\boldsymbol{t}_1$ on Δs_1 at the point $\boldsymbol{o}$. On the surface Δs_2, $y=0$ and $\boldsymbol{n}_2$ does not change, and on the surface Δs_3, $z=0$ and $\boldsymbol{n}_3$ does not change, so:

$$\begin{aligned}\boldsymbol{t}_2 &= \boldsymbol{t}_{2_o} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z} z + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z} xz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)} \boldsymbol{t}_2}{\partial x^m \partial z^k}\Big|_o x^m z^k\end{aligned} \tag{3.4}$$

$$\begin{aligned}\boldsymbol{t}_3 &= \boldsymbol{t}_{3_o} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y} y + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2} y^2 + 2\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y} xy\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{m!k!} \frac{\partial^{(m+k)} \boldsymbol{t}_3}{\partial x^m \partial y^k}\Big|_o x^m y^k\end{aligned} \tag{3.5}$$

Similarly, $\boldsymbol{t}_{2_o}$ and $\boldsymbol{t}_{3_o}$ are the exact values of $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ at the point $\boldsymbol{o}$ on Δs_2 and Δs_3, respectively.

For the traction vector on the surface Δs_4 a more explanation is needed. The traction vector on Δs_4 expands based on the traction vector on the inclined surface that is parallel to Δs_4 and passes through the vortex point of tetrahedron (point $\boldsymbol{o}$). Because the unit normal vectors of these two surfaces are the same, see Figure 3. Therefore:

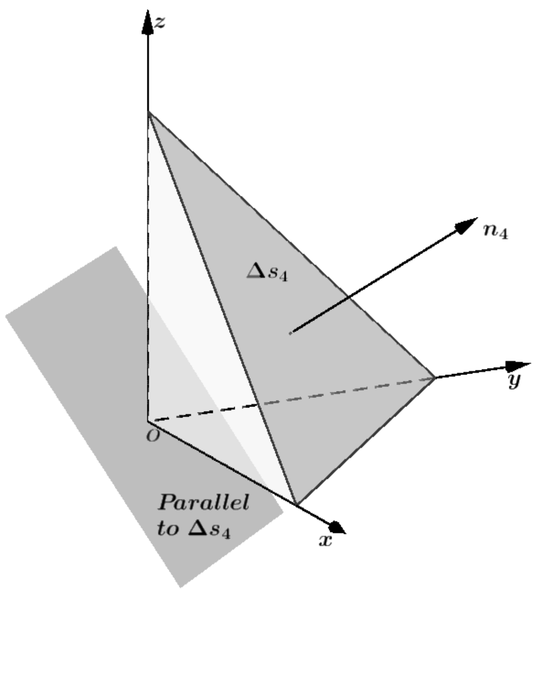

FIGURE 3. Inclined surface that is parallel to Δs_4 and passes through point $\boldsymbol{o}$.

$$\begin{aligned}\boldsymbol{t}_4 &= \boldsymbol{t}_{4_o} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial x}x + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}y + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}z \\ &+ \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}x^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}y^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}z^2 + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}xy + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}xz + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}yz\Big) \\ &+ \ldots = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m!n!k!}\frac{\partial^{(m+n+k)}\boldsymbol{t}_4}{\partial x^m \partial y^n \partial z^k}\Big|_o x^m y^n z^k\end{aligned} \tag{3.6}$$

where $\boldsymbol{t}_{4_o}$ is the exact traction vector at the point $\boldsymbol{o}$ on the inclined surface with unit normal vector $\boldsymbol{n}_4$, that this surface passes exactly through point $\boldsymbol{o}$, the vertex point of tetrahedron element. Here x, y, and z are the components of the position vector $\boldsymbol{r}$ on the surface Δs_4.

Note that $\boldsymbol{t}_{1_o}$, $\boldsymbol{t}_{2_o}$, $\boldsymbol{t}_{3_o}$, and $\boldsymbol{t}_{4_o}$ are the exact traction vectors at the point $\boldsymbol{o}$ but on the different surfaces with unit normal vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$, and $\boldsymbol{n}_4$, respectively. The body term $\boldsymbol{B}_o$ is exactly defined at the point $\boldsymbol{o}$. So, all the traction vectors and the body term vector with subscript o and all their derivatives, such as $\partial^2\boldsymbol{t}_{4_o}/\partial x\partial y$, are exactly defined at the point $\boldsymbol{o}$ and are bounded. As a result, for the convergence of the above Taylor series it is enough that we have $|\boldsymbol{r}| \leqslant 1$ in the domain of the mass element $\mathcal{M}$. But the scale of the coordinate system is arbitrary and we can define this scale such that the greatest distance in the domain of the mass element from the origin, is equal to one, i.e., $|\boldsymbol{r}|_{max} = 1$. By this scale, in the entire of the tetrahedron mass element we have $|\boldsymbol{r}| \leqslant 1$, that leads to the convergence condition for the above Taylor series.

Now all of the variables are prepared for integration in the integral equation (3.1). The integration of $\boldsymbol{B}$ on the volume of $\mathcal{M}$:

$$\begin{aligned}\int_{\mathcal{M}} \boldsymbol{B}\, dV &= \int_0^c \int_0^{b(1-\frac{z}{c})} \int_0^{a(1-\frac{y}{b}-\frac{z}{c})} \Big\{\boldsymbol{B}_o + \frac{\partial \boldsymbol{B}_o}{\partial x}x + \frac{\partial \boldsymbol{B}_o}{\partial y}y + \frac{\partial \boldsymbol{B}_o}{\partial z}z + \ldots\Big\}\, dx\, dy\, dz \\ &= \frac{1}{6}abc\Big\{\boldsymbol{B}_o + \frac{1}{4}\Big(\frac{\partial \boldsymbol{B}_o}{\partial x}a + \frac{\partial \boldsymbol{B}_o}{\partial y}b + \frac{\partial \boldsymbol{B}_o}{\partial z}c\Big) + \ldots\Big\}\end{aligned} \tag{3.7}$$

The integration of $\boldsymbol{t}_4$ on Δs_4:

$$\begin{aligned}\int_{\Delta s_4} \boldsymbol{t}_4\, dS &= \int_0^b \int_0^{a(1-\frac{y}{b})} \Big\{\sqrt{\big(-\frac{c}{a}\big)^2 + \big(-\frac{c}{b}\big)^2 + 1}\Big(\boldsymbol{t}_{4_o} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial x}x + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}y \\ &+ \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\big(c(1-\frac{x}{a}-\frac{y}{b})\big) + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}x^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}y^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\big(c(1-\frac{x}{a}-\frac{y}{b})\big)^2 \\ &+ 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}xy + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}x\big(c(1-\frac{x}{a}-\frac{y}{b})\big) + 2\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}y\big(c(1-\frac{x}{a}-\frac{y}{b})\big)\Big) \\ &+ \ldots\Big)\Big\}\, dx\, dy \\ &= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}\Big\{\boldsymbol{t}_{4_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}a + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}b + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}c\Big) \\ &+ \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}a^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}b^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}c^2 + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}ab + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}ac + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}bc\Big) + \ldots\Big\}\end{aligned} \tag{3.8}$$

The integration of $\boldsymbol{t}_1$ on Δs_1:

$$\begin{aligned}\int_{\Delta s_1} \boldsymbol{t}_1\, dS &= \int_0^c \int_0^{b(1-\frac{z}{c})} \Big\{ \boldsymbol{t}_{1_o} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial y} y + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z} z \\ &\quad + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2} y^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z} yz \Big) + \dots \Big\}\, dy\, dz \\ &= \frac{1}{2} bc \Big\{ \boldsymbol{t}_{1_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y} b + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z} c \Big) + \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2} b^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2} c^2 + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z} bc \Big) + \dots \Big\}\end{aligned} \tag{3.9}$$

The integration of $\boldsymbol{t}_2$ on Δs_2:

$$\begin{aligned}\int_{\Delta s_2} \boldsymbol{t}_2\, dS &= \int_0^c \int_0^{a(1-\frac{z}{c})} \Big\{ \boldsymbol{t}_{2_o} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z} z \\ &\quad + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2} z^2 + 2\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z} xz \Big) + \dots \Big\}\, dx\, dz \\ &= \frac{1}{2} ac \Big\{ \boldsymbol{t}_{2_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x} a + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z} c \Big) + \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2} a^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2} c^2 + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z} ac \Big) + \dots \Big\}\end{aligned} \tag{3.10}$$

The integration of $\boldsymbol{t}_3$ on Δs_3:

$$\begin{aligned}\int_{\Delta s_3} \boldsymbol{t}_3\, dS &= \int_0^b \int_0^{a(1-\frac{y}{b})} \Big\{ \boldsymbol{t}_{3_o} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial x} x + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y} y \\ &\quad + \frac{1}{2!}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2} x^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2} y^2 + 2\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y} xy \Big) + \dots \Big\}\, dx\, dy \\ &= \frac{1}{2} ab \Big\{ \boldsymbol{t}_{3_o} + \frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x} a + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y} b \Big) + \frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2} a^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2} b^2 + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y} ab \Big) + \dots \Big\}\end{aligned} \tag{3.11}$$

The geometrical relations for the area of faces and the volume of the tetrahedron are:

$$\begin{aligned}&\Delta s_1 = \frac{1}{2} bc, \qquad \Delta s_2 = \frac{1}{2} ac, \qquad \Delta s_3 = \frac{1}{2} ab \\ &\Delta s_4 = \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}, \qquad \Delta V = \frac{1}{6} abc\end{aligned} \tag{3.12}$$

By substituting the obtained equations for the integrals of the traction vectors and the body term into the equation (3.1) and using the above geometrical relations, we have:

$$\begin{aligned}
&\Delta s_4\Big\{\boldsymbol{t}_{4_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}a+\frac{\partial \boldsymbol{t}_{4_o}}{\partial y}b+\frac{\partial \boldsymbol{t}_{4_o}}{\partial z}c\Big)\\
&\quad+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}a^2+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}b^2+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}c^2+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}ab+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}ac+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}bc\Big)+\dots\Big\}\\
&\quad+\Delta s_1\Big\{\boldsymbol{t}_{1_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}b+\frac{\partial \boldsymbol{t}_{1_o}}{\partial z}c\Big)+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}b^2+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}c^2+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}bc\Big)+\dots\Big\}\\
&\quad+\Delta s_2\Big\{\boldsymbol{t}_{2_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}a+\frac{\partial \boldsymbol{t}_{2_o}}{\partial z}c\Big)+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}a^2+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}c^2+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}ac\Big)+\dots\Big\}\\
&\quad+\Delta s_3\Big\{\boldsymbol{t}_{3_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}a+\frac{\partial \boldsymbol{t}_{3_o}}{\partial y}b\Big)+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}a^2+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}b^2+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}ab\Big)+\dots\Big\}\\
&\quad-\Delta V\Big\{\boldsymbol{B}_o+\frac{1}{4}\Big(\frac{\partial \boldsymbol{B}_o}{\partial x}a+\frac{\partial \boldsymbol{B}_o}{\partial y}b+\frac{\partial \boldsymbol{B}_o}{\partial z}c\Big)+\dots\Big\}=\boldsymbol{0}
\end{aligned} \tag{3.13}$$

In the geometry of tetrahedron, h is the height of the vertex $\boldsymbol{o}$ from the base face, i.e., Δs_4. So, we have the following geometrical relations for a tetrahedron with $\boldsymbol{n}_4 = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$, where a, b, and c are greater than zero, see Figure 2.

$$\begin{aligned}
&h=n_x a, \qquad h=n_y b, \qquad h=n_z c\\
&\frac{1}{h^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}, \qquad \Delta s_4=\frac{abc}{2h}\\
&\Delta s_1=n_x\Delta s_4, \qquad \Delta s_2=n_y\Delta s_4, \qquad \Delta s_3=n_z\Delta s_4\\
&\Delta V=\frac{1}{6}abc=\frac{1}{3}h\Delta s_4
\end{aligned} \tag{3.14}$$

If we divide the equation (3.13) by Δs_4 and use the relations (3.14) for the faces and volume of the tetrahedron, then substitute the relations $a = h/n_x$, $b = h/n_y$, and $c = h/n_z$ into the equation, it becomes:

$$\begin{aligned}
&\Big\{\boldsymbol{t}_{4_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y}+\frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big)h\\
&+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}\frac{1}{n_xn_y}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}\frac{1}{n_xn_z}+\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}\frac{1}{n_yn_z}\Big)h^2+\dots\Big\}\\
&+n_x\Big\{\boldsymbol{t}_{1_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y}+\frac{\partial \boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big)h+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}\frac{1}{n_yn_z}\Big)h^2+\dots\Big\}\\
&+n_y\Big\{\boldsymbol{t}_{2_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big)h+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}\frac{1}{n_xn_z}\Big)h^2+\dots\Big\}\\
&+n_z\Big\{\boldsymbol{t}_{3_o}+\frac{1}{3}\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big)h+\frac{1}{12}\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}\frac{1}{n_xn_y}\Big)h^2+\dots\Big\}\\
&-\frac{1}{3}h\Big\{\boldsymbol{B}_o+\frac{1}{4}\Big(\frac{\partial \boldsymbol{B}_o}{\partial x}\frac{1}{n_x}+\frac{\partial \boldsymbol{B}_o}{\partial y}\frac{1}{n_y}+\frac{\partial \boldsymbol{B}_o}{\partial z}\frac{1}{n_z}\Big)h+\dots\Big\}=\boldsymbol{0}
\end{aligned} \tag{3.15}$$

Now by rearranging the equation based on the powers of h, we have:

$$\begin{aligned}
&\Big\{\boldsymbol{t}_{4_o}+n_x\boldsymbol{t}_{1_o}+n_y\boldsymbol{t}_{2_o}+n_z\boldsymbol{t}_{3_o}\Big\}\\
&\quad+\Bigg\{\Big(\frac{\partial\boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x}+\frac{\partial\boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y}+\frac{\partial\boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big)+n_x\Big(\frac{\partial\boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y}+\frac{\partial\boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big)\\
&\quad+n_y\Big(\frac{\partial\boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x}+\frac{\partial\boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big)+n_z\Big(\frac{\partial\boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x}+\frac{\partial\boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big)-\boldsymbol{B}_o\Bigg\}\frac{1}{3}h\\
&\quad+\Bigg\{\Big(\frac{\partial^2\boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2\boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2\boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2\boldsymbol{t}_{4_o}}{\partial x\partial y}\frac{1}{n_xn_y}+\frac{\partial^2\boldsymbol{t}_{4_o}}{\partial x\partial z}\frac{1}{n_xn_z}+\frac{\partial^2\boldsymbol{t}_{4_o}}{\partial y\partial z}\frac{1}{n_yn_z}\Big)\\
&\quad+n_x\Big(\frac{\partial^2\boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2\boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2\boldsymbol{t}_{1_o}}{\partial y\partial z}\frac{1}{n_yn_z}\Big)+n_y\Big(\frac{\partial^2\boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2\boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2}+\frac{\partial^2\boldsymbol{t}_{2_o}}{\partial x\partial z}\frac{1}{n_xn_z}\Big)\\
&\quad+n_z\Big(\frac{\partial^2\boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2}+\frac{\partial^2\boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2}+\frac{\partial^2\boldsymbol{t}_{3_o}}{\partial x\partial y}\frac{1}{n_xn_y}\Big)-\Big(\frac{\partial\boldsymbol{B}_o}{\partial x}\frac{1}{n_x}+\frac{\partial\boldsymbol{B}_o}{\partial y}\frac{1}{n_y}+\frac{\partial\boldsymbol{B}_o}{\partial z}\frac{1}{n_z}\Big)\Bigg\}\frac{1}{12}h^2\\
&\quad+\ldots=\mathbf{0}
\end{aligned}\tag{3.16}$$

Note that by the coordinate system here and by $\Delta V\neq 0$, no one of n_x, n_y, and n_z is exactly zero. So, all of the expressions in the braces {} of the equation (3.16) exist. We can rename the expressions in the braces and rewrite the equation as:

$$\boldsymbol{E}_0+\boldsymbol{E}_1\frac{1}{3}h+\boldsymbol{E}_2\frac{1}{12}h^2+\ldots=\mathbf{0}\tag{3.17}$$

If we continue to integrate the higher order derivatives of all terms based on their Taylor series that is a long time and complicated process that it does not present here, we have:

$$\boldsymbol{E}_0+\boldsymbol{E}_1\frac{1}{3}h+\boldsymbol{E}_2\frac{1}{12}h^2+\boldsymbol{E}_3\frac{1}{60}h^3+\ldots+\boldsymbol{E}_m\frac{2}{(m+2)!}h^m+\ldots=\mathbf{0}\tag{3.18}$$

or

$$\sum_{m=0}^{\infty}\boldsymbol{E}_m\frac{2}{(m+2)!}h^m=\mathbf{0}\tag{3.19}$$

This is a great equation in the foundation of continuum mechanics that is derived for the first time. $\boldsymbol{E}_0$, $\boldsymbol{E}_1$, and $\boldsymbol{E}_2$ are shown in the braces of the equation (3.16) and $\boldsymbol{E}_3$ and other $\boldsymbol{E}_m$'s will be presented. We now discuss some aspects of the equation (3.18):

- $\boldsymbol{E}_m$'s are formed by the expressions of traction vectors, body term and their derivatives, and the components of unit normal vector of the inclined surface.
- Each of the $\boldsymbol{E}_m$'s exists, because the surface terms, body term, and their derivatives are defined as continuous functions in continuum media and by the coordinate system here and by $\Delta V\neq 0$, no one of n_x, n_y, and n_z is exactly zero.
- Each of the $\boldsymbol{E}_m$'s depends on the variables at the point $\boldsymbol{o}$ and the components of unit normal vector of the inclined surface that is parallel to Δs_4 and passes through point $\boldsymbol{o}$. Because the surface terms, body term, and their derivatives are defined at the point $\boldsymbol{o}$.

- $\boldsymbol{E}_m$'s do not depend on the volume of tetrahedron.
- h is a geometrical variable and by the scale of the coordinate system on the tetrahedron mass element such that $|\boldsymbol{r}|_{max} \leqslant 1$, the altitude of the tetrahedron (h) is not greater than one.
- Note that $h = 0$ is not valid, because the integral equation of conservation of linear momentum (1.4) is defined for the mass elements with nonzero volume.

By these properties, we return to the equation (3.18).

$$\boldsymbol{E}_0 + \boldsymbol{E}_1 \frac{1}{3} h + \boldsymbol{E}_2 \frac{1}{12} h^2 + \boldsymbol{E}_3 \frac{1}{60} h^3 + \ldots + \boldsymbol{E}_m \frac{2}{(m+2)!} h^m + \ldots = \mathbf{0}$$

We must find $\boldsymbol{E}_m$'s. Since $\boldsymbol{E}_m$'s are independent of h, the series on the left hand side is a power series. A power series is identically equal to zero if and only if all of its coefficients are equal to zero. Therefore:

$$\boldsymbol{E}_m = \mathbf{0}, \qquad m = 0, 1, 2, \ldots, \infty \tag{3.20}$$

Note that these results are valid not only for $h \to 0$ but also for all values of h in the domain. In other words, the results (3.20) are valid not only for an infinitesimal tetrahedron but also for any tetrahedron in the scaled coordinate system in continuum media. In addition, we have not done any approximate process during derivation of the equations (3.18) and (3.20). So, the results (3.20) hold exactly, not approximately.

Furthermore, the subscript o in the expressions of $\boldsymbol{E}_m$'s in the equation (3.16) indicates the vortex point of the tetrahedron. But any point in the domain in continuum media can be regarded as the vertex point of a tetrahedron and we could consider that tetrahedron. So, the point $\boldsymbol{o}$ can be any point in the continuum domain. We conclude that $\boldsymbol{E}_m$'s are equal to zero at any point in continuum media. This implies that all their derivatives are equal to zero, as well. For example, we have for $\boldsymbol{E}_0$:

$$\frac{\partial \boldsymbol{E}_0}{\partial x} = \frac{\partial \boldsymbol{E}_0}{\partial y} = \frac{\partial \boldsymbol{E}_0}{\partial z} = \mathbf{0} \tag{3.21}$$

and the other higher derivatives of $\boldsymbol{E}_0$ are equal to zero. This trend holds for other $\boldsymbol{E}_m$'s.

But what are $\boldsymbol{E}_m$'s? In the following we will consider them and see that they lead to the important results.

For $\boldsymbol{E}_0 = \mathbf{0}$, from the equation (3.16):

$$\boldsymbol{E}_0 = \boldsymbol{t}_{4_o} + n_x \boldsymbol{t}_{1_o} + n_y \boldsymbol{t}_{2_o} + n_z \boldsymbol{t}_{3_o} = \mathbf{0} \tag{3.22}$$

This relation is similar to the relation of Cauchy tetrahedron argument. The Cauchy's relation was [1]:

$$\boldsymbol{t}_4 + n_x \boldsymbol{t}_1 + n_y \boldsymbol{t}_2 + n_z \boldsymbol{t}_3 = \mathbf{0} \tag{3.23}$$

But there are some important conceptual differences between them:

- In the Cauchy's relation (3.23), the traction vectors are not exactly defined at the point $\boldsymbol{o}$. They are the sequence of the limit $h \to 0$ on the tetrahedron volume. But here in (3.22), the traction vectors are exactly defined at the point $\boldsymbol{o}$.

- In the Cauchy's relation (3.23), the traction vectors are average values on the tetrahedron faces. But here in (3.22), the traction vectors are defined at the point $\boldsymbol{o}$ on the surfaces that pass exactly through point $\boldsymbol{o}$.

- In the Cauchy's relation (3.23), the traction vector $\boldsymbol{t}_4$ is defined on the surface Δs_4 of the tetrahedron. This surface does not pass through point $\boldsymbol{o}$ even in the limit $h \to 0$ for an infinitesimal tetrahedron. But here in (3.22), $\boldsymbol{t}_{4_o}$ is defined on the surface that passes through point $\boldsymbol{o}$ and is parallel to Δs_4, see Figure 3.

These differences are very important, because they imply that the relation (3.22) is exactly point-based but the relation (3.23) is average-based.

Let us return to the relation (3.22) for $\boldsymbol{E}_0 = \boldsymbol{0}$, we have:

$$\boldsymbol{t}_{4_o} + n_x \boldsymbol{t}_{1_o} + n_y \boldsymbol{t}_{2_o} + n_z \boldsymbol{t}_{3_o} = \boldsymbol{0}$$

The traction vector $\boldsymbol{t}_{1_o}$ is defined on the negative side of the coordinate plane yz, i.e., $\boldsymbol{n}_1 = -1\boldsymbol{e}_x$, at the point $\boldsymbol{o}$. If $\boldsymbol{t}_{x_o}$ is the traction vector on the positive side of the coordinate plane yz at the point $\boldsymbol{o}$, then by the equation (2.1), i.e., $\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$, we have:

$$\boldsymbol{t}_{1_o} = -\boldsymbol{t}_{x_o} \tag{3.24}$$

Similarly, for $\boldsymbol{t}_{2_o}$ and $\boldsymbol{t}_{3_o}$:

$$\boldsymbol{t}_{2_o} = -\boldsymbol{t}_{y_o}, \qquad \boldsymbol{t}_{3_o} = -\boldsymbol{t}_{z_o} \tag{3.25}$$

By substituting these relations into (3.22)

$$\boldsymbol{t}_{4_o} + n_x(-\boldsymbol{t}_{x_o}) + n_y(-\boldsymbol{t}_{y_o}) + n_z(-\boldsymbol{t}_{z_o}) = \boldsymbol{0}$$

therefore

$$\boldsymbol{t}_{4_o} = n_{x4} \boldsymbol{t}_{x_o} + n_{y4} \boldsymbol{t}_{y_o} + n_{z4} \boldsymbol{t}_{z_o} \tag{3.26}$$

where $n_{x4} = n_x$, $n_{y4} = n_y$, and $n_{z4} = n_z$. So, the traction vector $\boldsymbol{t}_{4_o}$ can be obtained by a linear relation between the traction vectors on the three orthogonal planes and the components of its unit normal vector. But can we use the equation (3.26) for any unit normal vector rather than $\boldsymbol{n}_{4_o}$?

By considering the equations (3.13) and (3.16), we find that the equation (3.26) is really the following equation:

$$\boldsymbol{t}_{4_o} = \frac{\Delta s_1}{\Delta s_4} \boldsymbol{t}_{x_o} + \frac{\Delta s_2}{\Delta s_4} \boldsymbol{t}_{y_o} + \frac{\Delta s_3}{\Delta s_4} \boldsymbol{t}_{z_o} \tag{3.27}$$

and this equation is

$$\boldsymbol{t}_{4_o} = |n_{x4}| \boldsymbol{t}_{x_o} + |n_{y4}| \boldsymbol{t}_{y_o} + |n_{z4}| \boldsymbol{t}_{z_o} \tag{3.28}$$

In Figure 2, by $a > 0$, $b > 0$, and $c > 0$, the components of unit normal vector on the inclined surface are greater than zero. So, the equation (3.26) is valid for these cases.

For the surfaces that their unit normal vector components are negative and are not zero, consider a tetrahedron mass element by the unit normal vector of its inclined surface (base face), $\boldsymbol{n}_{-4}$, that all of its components are negative. Therefore, we have $\boldsymbol{n}_{-4_o} = n_{x-4}\boldsymbol{e}_x + n_{y-4}\boldsymbol{e}_y + n_{z-4}\boldsymbol{e}_z = -n_x\boldsymbol{e}_x - n_y\boldsymbol{e}_y - n_z\boldsymbol{e}_z$, where $\boldsymbol{n}_{-4_o}$ is the outward unit normal vector of the surface that is parallel to the inclined surface and passes through the vortex point of this tetrahedron (point $\boldsymbol{o}$), and n_x, n_y, and n_z are positive values. Applying the process of exact tetrahedron argument to this new tetrahedron, leads to the following equation similar to the equation (3.22):

$$\boldsymbol{E}_0 = \boldsymbol{t}_{-4_o} + |n_{x-4}|\boldsymbol{t}_{x_o} + |n_{y-4}|\boldsymbol{t}_{y_o} + |n_{z-4}|\boldsymbol{t}_{z_o} = \boldsymbol{0} \tag{3.29}$$

As compared with the equation (3.22), in this equation we have $\boldsymbol{t}_{x_o}$, $\boldsymbol{t}_{y_o}$, and $\boldsymbol{t}_{z_o}$ rather than $\boldsymbol{t}_{1_o}$, $\boldsymbol{t}_{2_o}$, and $\boldsymbol{t}_{3_o}$, respectively. Because the outward sides of orthogonal faces of this new tetrahedron are in the positive directions of the coordinate system. By the equation (3.29) and the components of $\boldsymbol{n}_{-4_o}$, we have:

$$\begin{aligned}
\boldsymbol{t}_{-4_o} &= -|n_{x-4}|\boldsymbol{t}_{x_o} - |n_{y-4}|\boldsymbol{t}_{y_o} - |n_{z-4}|\boldsymbol{t}_{z_o} \\
&= -|-n_x|\boldsymbol{t}_{x_o} - |-n_y|\boldsymbol{t}_{y_o} - |-n_z|\boldsymbol{t}_{z_o} \\
&= -n_x\boldsymbol{t}_{x_o} - n_y\boldsymbol{t}_{y_o} - n_z\boldsymbol{t}_{z_o} \\
&= n_{x-4}\boldsymbol{t}_{x_o} + n_{y-4}\boldsymbol{t}_{y_o} + n_{z-4}\boldsymbol{t}_{z_o}
\end{aligned} \tag{3.30}$$

So, the traction vector $\boldsymbol{t}_{-4_o}$ can be obtained from a linear relation between the traction vectors on the three orthogonal planes and the components of its unit normal vector. For the surfaces that one or two components of their unit normal vectors are negative but the other ones are not zero, the same process can be done.

For the other surfaces that one or two components of their unit normal vectors are equal to zero, the tetrahedron does not form, but due to the continuous property of the traction vectors on $\boldsymbol{n}$ and the arbitrary choosing for any orthogonal basis for the coordinate system, the traction vectors on these surfaces can be described by the equation (3.26), as well. So, in general, the normal unit vector $\boldsymbol{n}_4$ can be related to any surface that passes through point $\boldsymbol{o}$ in three-dimensional continuum media. Thus, the subscript 4 removes from the equation (3.26) and we have for every $\boldsymbol{n} = n_x\boldsymbol{e}_x + n_y\boldsymbol{e}_y + n_z\boldsymbol{e}_z$:

$$\boldsymbol{t}_o = n_x\boldsymbol{t}_{x_o} + n_y\boldsymbol{t}_{y_o} + n_z\boldsymbol{t}_{z_o} \tag{3.31}$$

The subscript o in this equation indicates the vortex point of the tetrahedron. But any point in the domain in continuum media can be the vertex point of a tetrahedron and we could consider this tetrahedron. So, the point $\boldsymbol{o}$ can be any point in continuum media and the subscript o removes from the equation:

$$\boldsymbol{t} = n_x\boldsymbol{t}_x + n_y\boldsymbol{t}_y + n_z\boldsymbol{t}_z \tag{3.32}$$

or

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = n_x\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x) + n_y\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y) + n_z\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z) \tag{3.33}$$

This means that if we have the traction vectors on the three orthogonal surfaces at a given point and time, then we can get the traction vector on any surface that passes through that point at that time by using the unit normal vector of the surface and the linear relation (3.33).

So, we must define the traction vectors on the three orthogonal surfaces at any point and at any time. The traction vector on the surface with unit normal vector $\boldsymbol{e}_x$ by its components is:

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x) = T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z \tag{3.34}$$

here $T_{xx}(\boldsymbol{r},t)$, $T_{xy}(\boldsymbol{r},t)$, and $T_{xz}(\boldsymbol{r},t)$ are scalars that depend only on $\boldsymbol{r}$ and t. In each one, the first subscript indicates the direction of normal unit vector of the surface that this component acts on it and the second subscript indicates the direction of this component of traction vector. Similarly, we define the traction vectors on the surfaces with unit normal vectors $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$, respectively, as:

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y) = T_{yx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{yy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{yz}(\boldsymbol{r},t)\,\boldsymbol{e}_z \tag{3.35}$$

and

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z) = T_{zx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{zy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{zz}(\boldsymbol{r},t)\,\boldsymbol{e}_z \tag{3.36}$$

By substituting these equations in (3.33)

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n}) = n_x&\big\{T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\big\}\\ + n_y&\big\{T_{yx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{yy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{yz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\big\}\\ + n_z&\big\{T_{zx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{zy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{zz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\big\}\end{aligned}$$

by rearranging the equation

$$\begin{aligned}\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n}) =& \big\{n_xT_{xx}(\boldsymbol{r},t) + n_yT_{yx}(\boldsymbol{r},t) + n_zT_{zx}(\boldsymbol{r},t)\big\}\boldsymbol{e}_x\\ &+ \big\{n_xT_{xy}(\boldsymbol{r},t) + n_yT_{yy}(\boldsymbol{r},t) + n_zT_{zy}(\boldsymbol{r},t)\big\}\boldsymbol{e}_y\\ &+ \big\{n_xT_{xz}(\boldsymbol{r},t) + n_yT_{yz}(\boldsymbol{r},t) + n_zT_{zz}(\boldsymbol{r},t)\big\}\boldsymbol{e}_z\end{aligned}$$

this can be shown as

$$\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{n}) = \begin{bmatrix} t_x(\boldsymbol{r},t,\boldsymbol{n})\\ t_y(\boldsymbol{r},t,\boldsymbol{n})\\ t_z(\boldsymbol{r},t,\boldsymbol{n})\end{bmatrix} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz}\\ T_{yx} & T_{yy} & T_{yz}\\ T_{zx} & T_{zy} & T_{zz}\end{bmatrix}^T \begin{bmatrix} n_x\\ n_y\\ n_z\end{bmatrix} \tag{3.37}$$

using the vector relations, we have

$$\boldsymbol{t} = \boldsymbol{T}^T.\boldsymbol{n} \tag{3.38}$$

where $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{r},t)$ is a second order tensor that is called stress tensor. This tensor depends only on the position vector and time. This relation means that in order to describe the state of stress on any surface at a given point and time we need the 9 components of the stress tensor at that point and time. So, $\boldsymbol{E}_0 = \boldsymbol{0}$ leads to the existence of stress tensor.

Note that here the stress tensor $\boldsymbol{T}$ is exactly defined as point-based but in the previous tetrahedron arguments it was average-based. Because they used the average values of traction vectors on the surfaces that did not pass through the same point and by an approximate process the stress tensor was derived.

Let us see what $\boldsymbol{E}_1 = \mathbf{0}$ tells.
From the equation (3.16):

$$\begin{aligned}\boldsymbol{E}_1 = &\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big) + n_x\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big)\\ &+ n_y\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big) + n_z\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big) - \boldsymbol{B}_o\end{aligned} \tag{3.39}$$

As previously stated, for the tetrahedron element with $\Delta V \neq 0$, no one of n_x, n_y, and n_z is exactly zero. Therefore, $\boldsymbol{E}_1$ exists. Furthermore, the unit normal vector $\boldsymbol{n}_4$ does not change on Δs_4, so:

$$\frac{\partial \boldsymbol{n}_4}{\partial x} = \frac{\partial \boldsymbol{n}_4}{\partial y} = \frac{\partial \boldsymbol{n}_4}{\partial z} = \mathbf{0} \tag{3.40}$$

Using the relations (3.40) and the equation (3.22), i.e., $\boldsymbol{t}_{4_o} = \boldsymbol{E}_0 - n_x\boldsymbol{t}_{1_o} - n_y\boldsymbol{t}_{2_o} - n_z\boldsymbol{t}_{3_o}$, we have for (3.39):

$$\boldsymbol{E}_1 = \frac{1}{n_x}\frac{\partial \boldsymbol{E}_0}{\partial x} + \frac{1}{n_y}\frac{\partial \boldsymbol{E}_0}{\partial y} + \frac{1}{n_z}\frac{\partial \boldsymbol{E}_0}{\partial z} - \frac{\partial \boldsymbol{t}_{1_o}}{\partial x} - \frac{\partial \boldsymbol{t}_{2_o}}{\partial y} - \frac{\partial \boldsymbol{t}_{3_o}}{\partial z} - \boldsymbol{B}_o$$

If we define $\boldsymbol{E}$ as:

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{t}_{1_o}}{\partial x} - \frac{\partial \boldsymbol{t}_{2_o}}{\partial y} - \frac{\partial \boldsymbol{t}_{3_o}}{\partial z} - \boldsymbol{B}_o \tag{3.41}$$

therefore, we have

$$\boldsymbol{E}_1 = \frac{1}{n_x}\frac{\partial \boldsymbol{E}_0}{\partial x} + \frac{1}{n_y}\frac{\partial \boldsymbol{E}_0}{\partial y} + \frac{1}{n_z}\frac{\partial \boldsymbol{E}_0}{\partial z} + \boldsymbol{E} \tag{3.42}$$

But we saw in (3.21) that the derivatives of $\boldsymbol{E}_0$ were equal to zero. So, from (3.42) and $\boldsymbol{E}_1 = \mathbf{0}$, we have:

$$\boldsymbol{E}_1 = \boldsymbol{E} = \mathbf{0} \tag{3.43}$$

By (3.41), $\boldsymbol{E}$ is defined at the vertex point of tetrahedron. But as previously stated, the vertex point of the tetrahedron can be at any point in continuum media. Therefore, by (3.43), $\boldsymbol{E} = \mathbf{0}$ at any point in continuum media. This implies that all derivatives of $\boldsymbol{E}$ are equal to zero at any point in continuum media. So:

$$\frac{\partial \boldsymbol{E}}{\partial x} = \frac{\partial \boldsymbol{E}}{\partial y} = \frac{\partial \boldsymbol{E}}{\partial z} = \mathbf{0} \tag{3.44}$$

By using the relations (3.24) and (3.25), i.e., $\boldsymbol{t}_{1_o} = -\boldsymbol{t}_{x_o}$, $\boldsymbol{t}_{2_o} = -\boldsymbol{t}_{y_o}$, and $\boldsymbol{t}_{3_o} = -\boldsymbol{t}_{z_o}$, the equation (3.41) becomes:

$$\boldsymbol{E} = \frac{\partial \boldsymbol{t}_{x_o}}{\partial x} + \frac{\partial \boldsymbol{t}_{y_o}}{\partial y} + \frac{\partial \boldsymbol{t}_{z_o}}{\partial z} - \boldsymbol{B}_o \tag{3.45}$$

but $\boldsymbol{E} = \mathbf{0}$, so

$$\boldsymbol{B}_o = \frac{\partial \boldsymbol{t}_{x_o}}{\partial x} + \frac{\partial \boldsymbol{t}_{y_o}}{\partial y} + \frac{\partial \boldsymbol{t}_{z_o}}{\partial z} \tag{3.46}$$

As explained earlier, we can remove the subscript o from the equation and tell that this equation is valid at any point and at any time in the continuum domain. Therefore:

$$\boldsymbol{B} = \frac{\partial \boldsymbol{t}_x}{\partial x} + \frac{\partial \boldsymbol{t}_y}{\partial y} + \frac{\partial \boldsymbol{t}_z}{\partial z} \tag{3.47}$$

or

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\partial \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_x)}{\partial x} + \frac{\partial \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_y)}{\partial y} + \frac{\partial \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{e}_z)}{\partial z} \tag{3.48}$$

This partial differential equation means that if we have the first derivatives of the traction vectors on the three orthogonal surfaces at a given point and time, then we can get the body term at that point and time by using the equation (3.48). By substituting the definitions of $\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_x)$, $\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_y)$, and $\boldsymbol{t}(\boldsymbol{r},t,\boldsymbol{e}_z)$ from the equations (3.34), (3.35), and (3.36) into the equation (3.48), it becomes:

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r},t) &= \frac{\partial}{\partial x}\{T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\} \\ &+ \frac{\partial}{\partial y}\{T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\} \\ &+ \frac{\partial}{\partial z}\{T_{xx}(\boldsymbol{r},t)\,\boldsymbol{e}_x + T_{xy}(\boldsymbol{r},t)\,\boldsymbol{e}_y + T_{xz}(\boldsymbol{r},t)\,\boldsymbol{e}_z\}\end{aligned} \tag{3.49}$$

by rearranging the equation and using $\boldsymbol{B} = \rho\boldsymbol{a} - \rho\boldsymbol{b}$ from the equation (1.3), we have at any $\boldsymbol{r}$ and t:

$$\begin{aligned}\rho\boldsymbol{a} - \rho\boldsymbol{b} = &\left\{\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}\right\}\boldsymbol{e}_x + \left\{\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z}\right\}\boldsymbol{e}_y \\ &+ \left\{\frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z}\right\}\boldsymbol{e}_z\end{aligned} \tag{3.50}$$

this can be shown as

$$\rho\boldsymbol{a} - \rho\boldsymbol{b} = \begin{bmatrix} \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \\ \\ \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{zy}}{\partial z} \\ \\ \frac{\partial T_{xz}}{\partial x} + \frac{\partial T_{yz}}{\partial y} + \frac{\partial T_{zz}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix} = \nabla.\boldsymbol{T}$$

so, we have

$$\rho\boldsymbol{a} = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{3.51}$$

or

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho\boldsymbol{b} \tag{3.52}$$

So, $\boldsymbol{E}_1 = \boldsymbol{0}$ leads to the general equation of motion that is called Cauchy equation of motion. Cauchy obtained this important equation by applying the conservation of linear momentum to a "cubic element" and he did not obtain it from the tetrahedron argument. The tetrahedron arguments that are represented by most of the scientists and authors in continuum mechanics lead only to the equation (3.23), i.e., $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$, for the existence of stress tensor. But here in addition to the exact derivation of the stress tensor, the other fundamental equation of continuum mechanics, i.e., the Cauchy equation of motion, is exactly derived from this tetrahedron argument, simultaneously.

Let us see what $\boldsymbol{E}_2 = \boldsymbol{0}$ tells.
From the equation (3.16):

$$\begin{aligned}\boldsymbol{E}_2 =& \Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial y}\frac{1}{n_x n_y} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x \partial z}\frac{1}{n_x n_z} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y \partial z}\frac{1}{n_y n_z}\Big)\\ &+ n_x\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y \partial z}\frac{1}{n_y n_z}\Big) + n_y\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x \partial z}\frac{1}{n_x n_z}\Big)\\ &+ n_z\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x \partial y}\frac{1}{n_x n_y}\Big) - \Big(\frac{\partial \boldsymbol{B}_o}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{B}_o}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{B}_o}{\partial z}\frac{1}{n_z}\Big)\end{aligned} \tag{3.53}$$

For $\boldsymbol{E}_2$, similar to the process for $\boldsymbol{E}_1 = \boldsymbol{0}$, we have:

$$\begin{aligned}\boldsymbol{E}_2 =& \frac{1}{n_x^2}\frac{\partial^2 \boldsymbol{E}_0}{\partial x^2} + \frac{1}{n_y^2}\frac{\partial^2 \boldsymbol{E}_0}{\partial y^2} + \frac{1}{n_z^2}\frac{\partial^2 \boldsymbol{E}_0}{\partial z^2} + \frac{1}{n_x n_y}\frac{\partial^2 \boldsymbol{E}_0}{\partial x \partial y} + \frac{1}{n_x n_z}\frac{\partial^2 \boldsymbol{E}_0}{\partial x \partial z} + \frac{1}{n_y n_z}\frac{\partial^2 \boldsymbol{E}_0}{\partial y \partial z}\\ &+ \frac{1}{n_x}\frac{\partial \boldsymbol{E}}{\partial x} + \frac{1}{n_y}\frac{\partial \boldsymbol{E}}{\partial y} + \frac{1}{n_z}\frac{\partial \boldsymbol{E}}{\partial z}\end{aligned} \tag{3.54}$$

By the previous explanations, all derivatives of $\boldsymbol{E}_0$ and $\boldsymbol{E}$ were equal to zero. Therefore, the equation (3.54) is a correct result of $\boldsymbol{E}_2 = \boldsymbol{0}$.

For $\boldsymbol{E}_3 = \boldsymbol{0}$ we have:

$$\begin{aligned}\boldsymbol{E}_3 =& \Big(\frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x^3}\frac{1}{n_x^3} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial y^3}\frac{1}{n_y^3} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial z^3}\frac{1}{n_z^3} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x^2 \partial y}\frac{1}{n_x^2 n_y} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x^2 \partial z}\frac{1}{n_x^2 n_z} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial y^2 \partial z}\frac{1}{n_y^2 n_z}\\ &+ \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x \partial y^2}\frac{1}{n_x n_y^2} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x \partial z^2}\frac{1}{n_x n_z^2} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial y \partial z^2}\frac{1}{n_y n_z^2} + \frac{\partial^3 \boldsymbol{t}_{4_o}}{\partial x \partial y \partial z}\frac{1}{n_x n_y n_z}\Big)\\ &+ n_x\Big(\frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial y^3}\frac{1}{n_y^3} + \frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial z^3}\frac{1}{n_z^3} + \frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial y^2 \partial z}\frac{1}{n_y^2 n_z} + \frac{\partial^3 \boldsymbol{t}_{1_o}}{\partial y \partial z^2}\frac{1}{n_y n_z^2}\Big)\\ &+ n_y\Big(\frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial x^3}\frac{1}{n_x^3} + \frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial z^3}\frac{1}{n_z^3} + \frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial x^2 \partial z}\frac{1}{n_x^2 n_z} + \frac{\partial^3 \boldsymbol{t}_{2_o}}{\partial x \partial z^2}\frac{1}{n_x n_z^2}\Big)\\ &+ n_z\Big(\frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial x^3}\frac{1}{n_x^3} + \frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial y^3}\frac{1}{n_y^3} + \frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial x^2 \partial y}\frac{1}{n_x^2 n_y} + \frac{\partial^3 \boldsymbol{t}_{3_o}}{\partial x \partial y^2}\frac{1}{n_x n_y^2}\Big)\\ &- \Big(\frac{\partial^2 \boldsymbol{B}_o}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{B}_o}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{B}_o}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{B}_o}{\partial x \partial y}\frac{1}{n_x n_y} + \frac{\partial^2 \boldsymbol{B}_o}{\partial x \partial z}\frac{1}{n_x n_z} + \frac{\partial^2 \boldsymbol{B}_o}{\partial y \partial z}\frac{1}{n_y n_z}\Big)\end{aligned} \tag{3.55}$$

Similar to the previous processes for $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$, we have for $\boldsymbol{E}_3$:

$$\begin{aligned}\boldsymbol{E}_3 =& \frac{1}{n_x^3}\frac{\partial^3 \boldsymbol{E}_0}{\partial x^3} + \frac{1}{n_y^3}\frac{\partial^3 \boldsymbol{E}_0}{\partial y^3} + \frac{1}{n_z^3}\frac{\partial^3 \boldsymbol{E}_0}{\partial z^3} + \frac{1}{n_x^2 n_y}\frac{\partial^3 \boldsymbol{E}_0}{\partial x^2 \partial y} + \frac{1}{n_x^2 n_z}\frac{\partial^3 \boldsymbol{E}_0}{\partial x^2 \partial z} + \frac{1}{n_y^2 n_z}\frac{\partial^3 \boldsymbol{E}_0}{\partial y^2 \partial z}\\ &+ \frac{1}{n_x n_y^2}\frac{\partial^3 \boldsymbol{E}_0}{\partial x \partial y^2} + \frac{1}{n_x n_z^2}\frac{\partial^3 \boldsymbol{E}_0}{\partial x \partial z^2} + \frac{1}{n_y n_z^2}\frac{\partial^3 \boldsymbol{E}_0}{\partial y \partial z^2} + \frac{1}{n_x n_y n_z}\frac{\partial^3 \boldsymbol{E}_0}{\partial x \partial y \partial z}\\ &+ \frac{1}{n_x^2}\frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{1}{n_y^2}\frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{1}{n_z^2}\frac{\partial^2 \boldsymbol{E}}{\partial z^2} + \frac{1}{n_x n_y}\frac{\partial^2 \boldsymbol{E}}{\partial x \partial y} + \frac{1}{n_x n_z}\frac{\partial^2 \boldsymbol{E}}{\partial x \partial z} + \frac{1}{n_y n_z}\frac{\partial^2 \boldsymbol{E}}{\partial y \partial z}\end{aligned} \tag{3.56}$$

We saw that all derivatives of $\boldsymbol{E}_0$ and $\boldsymbol{E}$ were equal to zero. So, the equation (3.56) is a correct result of $\boldsymbol{E}_3 = \boldsymbol{0}$. This process for other $\boldsymbol{E}_m$'s leads to the expressions that contain the higher derivatives of $\boldsymbol{E}_0$ and $\boldsymbol{E}$ and the higher powers of the components of the unit normal vector and the results are equal to zero.

4. Discussion

In this section, we discuss some aspects of this new proof and compare it with the previous proofs of the existence of stress tensor and derivation of the Cauchy equation of motion. We gave a comprehensive review of the Cauchy tetrahedron argument and the proofs of the existence of stress tensor (2017, [1]). In that article [1], we stated some important and fundamental challenges on the previous proofs. In order to consider the stated challenges on this new proof, we start with the first challenge in [1].

The challenge 1 told us that applying the conservation of linear momentum to any mass element with any volume and shape must lead to the equation of motion. But in the previous proofs, this process on an infinitesimal tetrahedron mass element led to the equation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$ that differs from the equation of motion. In the previous proofs, the equation of motion is obtained by using the stress tensor relation and applying the conservation of linear momentum to a cubic element or by using the divergence theorem in the integral equation of conservation of linear momentum. But in the present proof, both the relation for the existence of stress tensor and the equation of motion are obtained, simultaneously. So, the challenge 1 is removed in this new proof.

The challenge 2 told us that the previous proofs of the existence of stress tensor were based on infinitesimal volumes by the expressions like "$\Delta V \to 0$", "$h \to 0$", *"when the tetrahedron shrinks to a point"*, or *"when the tetrahedron shrinks to zero volume"*, while it must be proved that the existence of stress tensor at a point does not depend on the size of the mass element. In other words, the stress tensor exists for any size of mass element in continuum media, where the volume of mass element increases, decreases or does not change. Therefore, in the previous proofs the result is only valid for the infinitesimal volumes and they do not show that the result can be applied to the mass elements with any volume in continuum media. But here we proved that the existence of stress tensor is independent of the volume of mass element and we did not use an infinitesimal volume or a limit to zero volume in the present proof. So, this challenge is removed in this new proof.

The challenge 3 is related to the average values of the traction vectors, body forces, and inertia terms on the surfaces and the volume of the mass element in the previous proofs. The average values lead to the approximate process even for the infinitesimal mass element. But in the present proof, the exact values are used and the results are exactly held, therefore the challenge 3 is removed.

The challenge 4 is related to the order of magnitude of the surface forces in the limit $\Delta V \to 0$ or $h \to 0$. In the previous proofs, it was told that in this limit the order of magnitude of the surface forces is h^2 and the order of magnitude of the body forces and inertia is h^3. They told that in the limit $h \to 0$ the body forces and inertia go to zero faster than the surface forces, therefore only the surface forces remain in the equation. Here based on the present proof, we clearly show that this is not correct, because the order of magnitude of the surface forces is h^3 similar to the order of magnitude of the body forces and inertia. In order to prove this, we use the equation (3.16) and extract the integral of the surface force over the control volume $\mathcal{M}$ from this equation as:

$$\begin{aligned}\int_{\partial\mathcal{M}} \boldsymbol{t}\, dS = &\Big\{\boldsymbol{t}_{4_o} + n_x\boldsymbol{t}_{1_o} + n_y\boldsymbol{t}_{2_o} + n_z\boldsymbol{t}_{3_o}\Big\}\Delta s_4 \\ &+ \Big\{\Big(\frac{\partial \boldsymbol{t}_{4_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{4_o}}{\partial z}\frac{1}{n_z}\Big) + n_x\Big(\frac{\partial \boldsymbol{t}_{1_o}}{\partial y}\frac{1}{n_y} + \frac{\partial \boldsymbol{t}_{1_o}}{\partial z}\frac{1}{n_z}\Big) \\ &+ n_y\Big(\frac{\partial \boldsymbol{t}_{2_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{2_o}}{\partial z}\frac{1}{n_z}\Big) + n_z\Big(\frac{\partial \boldsymbol{t}_{3_o}}{\partial x}\frac{1}{n_x} + \frac{\partial \boldsymbol{t}_{3_o}}{\partial y}\frac{1}{n_y}\Big)\Big\}\frac{1}{3}h\Delta s_4 \\ &+ \Big\{\Big(\frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial y}\frac{1}{n_x n_y} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial x\partial z}\frac{1}{n_x n_z} + \frac{\partial^2 \boldsymbol{t}_{4_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) \\ &+ n_x\Big(\frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{1_o}}{\partial y\partial z}\frac{1}{n_y n_z}\Big) + n_y\Big(\frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial z^2}\frac{1}{n_z^2} + \frac{\partial^2 \boldsymbol{t}_{2_o}}{\partial x\partial z}\frac{1}{n_x n_z}\Big) \\ &+ n_z\Big(\frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x^2}\frac{1}{n_x^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial y^2}\frac{1}{n_y^2} + \frac{\partial^2 \boldsymbol{t}_{3_o}}{\partial x\partial y}\frac{1}{n_x n_y}\Big)\Big\}\frac{1}{12}h^2\Delta s_4 + \dots\end{aligned} \tag{4.1}$$

but as we showed in the equation (3.22), the expression in the braces of the first line of the above equation is exactly zero, i.e., $\boldsymbol{t}_{4_o} + n_x\boldsymbol{t}_{1_o} + n_y\boldsymbol{t}_{2_o} + n_z\boldsymbol{t}_{3_o} = \boldsymbol{0}$. So, this expression removes from the equation. When the volume of the tetrahedron goes to the infinitesimal volume ($\Delta V \to 0$ or $h \to 0$), the order of magnitude of the remaining expressions on the right hand side of the above equation is:

$$O\Big(\int_{\partial\mathcal{M}} \boldsymbol{t}\, dS\Big) = O\Big(\frac{1}{3}h\Delta s_4\Big) = h^3 \tag{4.2}$$

therefore, the order of magnitude of the surface forces is h^3, not h^2.

In the challenges 5 and 6, it was told that to prove the existence of stress tensor as a point-based function from the equation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$, the four surfaces that the traction vectors are defined on them must pass through the same point. But according to the previous proofs, in this equation $\boldsymbol{t}_4$ is defined on Δs_4 and this surface, even for infinitesimal tetrahedron, does not pass through the vertex point of the tetrahedron where the other three faces pass through it. But in the present proof, in the equation $\boldsymbol{t}_{4_o} + n_x\boldsymbol{t}_{1_o} + n_y\boldsymbol{t}_{2_o} + n_z\boldsymbol{t}_{3_o} = \boldsymbol{0}$, we defined all the traction vectors at the same point $\boldsymbol{o}$, where the four surfaces pass exactly through it. So, the stress tensor is exactly obtained as a point-based function and these challenges are removed in this new proof.

The challenges 7 and 8 are related to the equation $\boldsymbol{t}_4 + n_x\boldsymbol{t}_1 + n_y\boldsymbol{t}_2 + n_z\boldsymbol{t}_3 = \boldsymbol{0}$, where the traction vectors are the average values on the surfaces of an infinitesimal tetrahedron. It was told that by multiplying this equation by Δs_4, we have $\boldsymbol{t}_4\Delta s_4 + \boldsymbol{t}_1\Delta s_1 + \boldsymbol{t}_2\Delta s_2 + \boldsymbol{t}_3\Delta s_3 = \boldsymbol{0}$, this means that the sum of the surface forces on the infinitesimal tetrahedron is zero. This is not correct, because from the conservation of linear momentum (1.4), the surface forces on any mass element are equal to the body terms on that element. But in the present proof, we used the exact traction vectors that this led to the equation $\boldsymbol{t}_{4_o} + n_x \boldsymbol{t}_{1_o} + n_y \boldsymbol{t}_{2_o} + n_z \boldsymbol{t}_{3_o} = \mathbf{0}$. In this equation, since all the traction vectors are defined at the point $\boldsymbol{o}$, the equation $\boldsymbol{t}_{4_o}\Delta s_4 + \boldsymbol{t}_{1_o}\Delta s_1 + \boldsymbol{t}_{2_o}\Delta s_2 + \boldsymbol{t}_{3_o}\Delta s_3 = \mathbf{0}$ does not mean the sum of the traction vectors on the surface of the mass element is equal to zero.

5. Conclusion

We considered the general integral equation of conservation of linear momentum as:

$$\int_{\mathcal{M}} \rho \boldsymbol{a}\, dV = \int_{\partial \mathcal{M}} \boldsymbol{t}\, dS + \int_{\mathcal{M}} \rho \boldsymbol{b}\, dV$$

where $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n})$ is the traction vector (surface force per unit area). From the above integral equation, first, we derived the Cauchy lemma for traction vectors:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{r}, t, -\boldsymbol{n})$$

Then by a new exact tetrahedron argument, we showed that applying the general integral equation of conservation of linear momentum to the tetrahedron mass element leads to the following fundamental equation:

$$\boldsymbol{E}_0 + \boldsymbol{E}_1 \frac{1}{3} h + \boldsymbol{E}_2 \frac{1}{12} h^2 + \boldsymbol{E}_3 \frac{1}{60} h^3 + \ldots + \boldsymbol{E}_m \frac{2}{(m+2)!} h^m + \ldots = \mathbf{0}$$

where h is the altitude of the tetrahedron. $\boldsymbol{E}_m$'s are expressions that contain the traction vectors, inertia, body force, the derivatives of these terms, and the powers of the components of unit normal vector of the tetrahedron's base face. We showed that the only solution of this equation is:

$$\boldsymbol{E}_m = \mathbf{0}, \qquad m = 0, 1, 2, \ldots, \infty$$

i.e., $\boldsymbol{E}_m$'s must be equal to zero. Then, we proved that $\boldsymbol{E}_0 = \mathbf{0}$ leads to the existence of stress tensor:

$$\boldsymbol{t}(\boldsymbol{r}, t, \boldsymbol{n}) = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}^T \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \boldsymbol{T}^T . \boldsymbol{n}$$

and $\boldsymbol{E}_1 = \mathbf{0}$ leads to the derivation of the general equation of motion:

$$\rho(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\nabla)\boldsymbol{v}) = \nabla.\boldsymbol{T} + \rho \boldsymbol{b}$$

In other equations $\boldsymbol{E}_m = \mathbf{0}$, for $m = 2, 3, \ldots, \infty$, the results of $\boldsymbol{E}_0 = \mathbf{0}$ and $\boldsymbol{E}_1 = \mathbf{0}$ are repeated. In this new proof, there is no limited, average, or approximate process and all of the parameters are exact point-based functions. This proof is not limited to $h \to 0$ for an infinitesimal tetrahedron mass element. Also in this proof, we showed that all of the challenges on the previous tetrahedron arguments and the proofs of existence of stress tensor are removed.

Historical note: *The manuscript of the exact tetrahedron argument was prepared before writing the review article [1].*

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