Quaternion Dynamics, Part 3 – Pentuple Inversion

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Summary

This text continues the development of pentuples begun in Part – 2 of these works. Matrix formulations are presented that are easily inverted. The presentation of a pentuple is similar to the form of a quaternion. A functionality is presented in **Equation 4.2** that mimics wave-function collapse. Octonion multiplication is shown to be very similar irrespective of whether the complex i commutes normally or anti-commutes.

Discussion

Pentuples:

A quaternion **Q** is defined as follows:

$$\mathbf{Q} = q_0 + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$$

Let us now define a pentuple $\mathbf{P_Q}$ as follows:

Equation 1.0:

$$\mathbf{P_Q} = qi + \mathbf{Q} = q_0 + qi + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k} = Q_0\begin{bmatrix}1\\i\end{bmatrix} + q_i\mathbf{i} + q_j\mathbf{j} + q_k\mathbf{k}$$

Where

Equation 1.1:

$$Q_0 = \begin{bmatrix}q_0 & 0\\0 & q\end{bmatrix}$$

By expressing a pentuple as shown in the Right-Hand-Side of **Equation 1.0**, the complex plane is given the same mathematical status as the unit vectors. Herein, the complex i anti-commutes with the unit vectors.

This presentation allows a pentuple multiplication to be presented in matrix form as follows:

Equation 2.0:

$$\mathbf{P_A P_B} = \begin{bmatrix}+A_0 & -a_i & -a_j & -a_k\\+a_i & +A_0 & -a_k & +a_j\\+a_j & +a_k & +A_0 & -a_i\\+a_k & -a_j & +a_i & +A_0\end{bmatrix}\begin{bmatrix}B_0\\b_i\\b_j\\b_k\end{bmatrix}$$

This form is nearly identical to that of quaternion multiplication with the difference being that A_0 is substituted for a_0. All of the terms follow the format presented in **Equation 1.0** and **Equation 1.1**.

It must be remembered that the rows that result from the matrix multiplication are associated with the column vector [1, i], and the unit vectors **i**, **j**, and **k**, respectively. When performing this multiplication, the following identity is needed:

Equation 2.1:

$$A_0\begin{bmatrix}1\\i\end{bmatrix}B_0\begin{bmatrix}1\\i\end{bmatrix} = (a_0 + ai)(b_0 + bi) = \begin{bmatrix}(a_0b_0 - ab) & 0\\0 & (ab_0 + a_0b)\end{bmatrix}\begin{bmatrix}1\\i\end{bmatrix}$$

The cases wherein either the real term is zero or the complex term is zero are of special interest.

Equation 2.1.1:

$$a_0 b_0 - ab = 0\,; real\ term\ is\ zero\ therefore\ solution\ is\ complex$$

Solution 1:

$$a_0 = a \ and \ b_0 = b$$

Solution 2:

$$a_0 = -a \ and \ b_0 = -b$$

Solution 3:

$$a_0 = b \ and \ b_0 = a$$

Solution 4:

$$a_0 = -b \ and \ b_0 = -a$$

Equation 2.1.2:

$$ab_0 + \ a_0 b = 0\,; complex\ term\ is\ zero\ therefore\ solution\ is\ real$$

Solution 1:

$$a_0 = \ b_0 \ and \ a = \ -b$$

Solution 2:

$$a_0 = \ -b_0 \ and \ a = \ b$$

Solution 3:

$$a_0 = \ a \ and \ b_0 = \ -b$$

Solution 4:

$$a_0 = \ -a \ and \ b_0 = \ b$$

Next, let us define the complex conjugate of A_0 (**Equation 2.1.2**, Solution 1) as follows:

Equation 2.2:

$$A_0^* = \begin{bmatrix} a_0 & 0 \\ 0 & -a \end{bmatrix}$$

This makes it simple to produce an inverse for the coefficient matrix presented in **Equation 2.0**.

Equation 3.0:

$$\begin{bmatrix} +A_0 & -a_i & -a_j & -a_k \\ +a_i & +A_0 & -a_k & +a_j \\ +a_j & +a_k & +A_0 & -a_i \\ +a_k & -a_j & +a_i & +A_0 \end{bmatrix}^{-1} = \frac{1}{\|\mathbf{P_A}\|^2} \begin{bmatrix} +A_0^* & +a_i & +a_j & +a_k \\ -a_i & +A_0^* & +a_k & -a_j \\ -a_j & -a_k & +A_0^* & +a_i \\ -a_k & +a_j & -a_i & +A_0^* \end{bmatrix}$$

Where

Equation 3.1:

$$\|\mathbf{P_A}\|^2 = \ a_0^2 + \ a^2 + \ a_i^2 + \ a_j^2 + \ a_k^2$$

When the inverse matrix is multiplied by the coefficient matrix, several identities become apparent. These are as follows:

Equation 3.2:

$$-a_i A_0 + \ A_0^* a_i = 0$$

Equation 3.3:

$$-a_j A_0 + \ A_0^* a_j = 0$$

Equation 3.4:

$$-a_k A_0 + \ A_0^* a_k = 0$$

These identities are true because the complex i anti-commutes with the unit vectors but the scalar value commutes normally. These expressions appear in the non-diagonal terms of the matrix that results from multiplying the coefficient matrix of **Equation 2.0** with the inverse matrix of **Equation 3.0**.

Bi-Quaternion:

Now let us consider a bi-quaternion form of octonion $\mathbf{O_Q}$ as follows:

Equation 4.0:

$$\mathbf{O_Q} = \ e^{i\omega}\mathbf{Q} = \ [\cos(\omega) \ + \ i\sin(\omega)]\mathbf{Q} \ = \ \cos(\omega)\,\mathbf{Q} \ + \ i\sin(\omega)\,\mathbf{Q}$$

The conjugate of this expression is:

Equation 4.1:

$$\mathbf{O}_{\mathbf{Q}}^{*} = \cos(\omega)\,\mathbf{Q}^{*} - i\sin(\omega)\,\mathbf{Q}$$

Note that this uses both **Q** and **Q***. This is easily proven by multiplication of these two expressions.

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = [\cos(\omega)\,\mathbf{Q}^{*} - i\sin(\omega)\,\mathbf{Q}][\cos(\omega)\,\mathbf{Q} + i\sin(\omega)\,\mathbf{Q}]$$

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = \cos\omega\,\mathbf{Q}^{*}\cos\omega\,\mathbf{Q} - i\sin\omega\,\mathbf{Q}\cos\omega\,\mathbf{Q} + \cos\omega\,\mathbf{Q}^{*}i\sin\omega\,\mathbf{Q} - i\sin\omega\,\mathbf{Q}i\sin\omega\,\mathbf{Q}$$

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = cos^{2}\omega\,\mathbf{Q}^{*}\mathbf{Q} - i\sin\omega\cos\omega\,\mathbf{Q}\mathbf{Q} + i\cos\omega\sin\omega\,\mathbf{Q}\mathbf{Q} - i^{2}sin^{2}\omega\mathbf{Q}^{*}\mathbf{Q}$$

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = (cos^{2}\omega + sin^{2}\omega)\,\mathbf{Q}^{*}\mathbf{Q} + i(-\sin\omega\cos\omega + \cos\omega\sin\omega)\mathbf{Q}\mathbf{Q}$$

Equation 4.2:

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = \mathbf{Q}^{*}\mathbf{Q} = q_{0}^{2} + q_{i}^{2} + q_{j}^{2} + q_{k}^{2}$$

There is something that is very interesting about this expression. The phase angle ω has disappeared from the expression. Therefore, the result is the same irrespective of the phase angle ω that is used. In fact, the complex plane has disappeared from the expression. Essentially, the phase angle ω has become a hidden variable. In the author's opinion, this behavior mimics the behavior of wave-function collapse.

Now let us consider a more general bi-quaternion constructed of two quaternions as follows:

$$\mathbf{O} = \mathbf{A} + i\mathbf{B}$$

Based upon **Equation 4.1**. the conjugate ***should*** be:

$$\mathbf{O}^{*} = \mathbf{A}^{*} - i\mathbf{B}$$

However, multiplication of these two expressions does not produce a scalar value.

$$\mathbf{O}^{*}\mathbf{O} = (\mathbf{A}^{*} - i\mathbf{B})(\mathbf{A} + i\mathbf{B})$$

$$\mathbf{O}^{*}\mathbf{O} = \mathbf{A}^{*}\mathbf{A} - i\mathbf{B}\mathbf{A} + \mathbf{A}^{*}i\mathbf{B} - i\mathbf{B}i\mathbf{B}$$

$$\mathbf{O}^{*}\mathbf{O} = \mathbf{A}^{*}\mathbf{A} - i\mathbf{B}\mathbf{A} + i\mathbf{A}\mathbf{B} - i^{2}\mathbf{B}^{*}\mathbf{B}$$

$$\mathbf{O}^{*}\mathbf{O} = (\mathbf{A}^{*}\mathbf{A} + \mathbf{B}^{*}\mathbf{B}) + i(-\mathbf{B}\mathbf{A} + \mathbf{A}\mathbf{B})$$

$$\mathbf{O}^{*}\mathbf{O} = (\mathbf{A}^{*}\mathbf{A} + \mathbf{B}^{*}\mathbf{B}) + i(2\mathbf{ab})$$

The complex term is 2**ab**. Since this is generally not equal to zero, the proposed conjugate is incorrect. It appears that there is no simple conjugate for the generalized bi-quaternion form. This observation is consistent with what was observed in the section on Octonions in Part – 2 of this work.

Given that the generalized bi-quaternion does not appear to have a simple conjugate, the author will focus on the version presented in **Equation 4.0**.

$$\mathbf{O_Q} = e^{i\omega}\mathbf{Q} = [\cos(\omega) + i\sin(\omega)]\mathbf{Q} = \cos(\omega)\,\mathbf{Q} + i\sin(\omega)\,\mathbf{Q}$$

The author proposes to rewrite this expression as follows:

Equation 5.0:

$$\mathbf{O_Q} = Q_0\begin{bmatrix}1\\ i\end{bmatrix} + Q_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} + Q_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} + Q_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}$$

Where

Equation 5.1:

$$Q_0 = q_0E\,;\; Q_i = q_iE\,;\; Q_j = q_jE\,;\; Q_k = q_kE$$

And

Equation 5.2:

$$E = \begin{bmatrix}\cos\omega & 0\\ 0 & \sin\omega\end{bmatrix};\; E^* = \begin{bmatrix}\cos\omega & 0\\ 0 & -\sin\omega\end{bmatrix}$$

The author is tentatively referring to **Equation 5.2** as Euler's Matrix – hence the letter E. **Equation 5.0** is similar to the form presented by the author in the section Pentuples in Part – 2 of this work. However, here the author has not included the column matrix [1, i] as part of the coefficients Q_x. This resolves some difficulties associated with the previous form.

The next step is to multiply two octonions $\mathbf{O_A}$ and $\mathbf{O_C}$ that both satisfy **Equation 5.0** and to express it in matrix form. The octonions will have phase angles ω_A and ω_C respectively. The author will move all of the unit vectors to the right-side of each group of terms. This will allow **Equation 2.1** to be used if needed.

$$\mathbf{O_A O_C} = \left(A_0\begin{bmatrix}1\\ i\end{bmatrix} + A_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} + A_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} + A_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}\right)\left(C_0\begin{bmatrix}1\\ i\end{bmatrix} + C_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} + C_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} + C_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}\right)$$

$$\mathbf{O_A O_C} = A_0\begin{bmatrix}1\\ i\end{bmatrix}C_0\begin{bmatrix}1\\ i\end{bmatrix} + A_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i}C_0\begin{bmatrix}1\\ i\end{bmatrix} + A_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j}C_0\begin{bmatrix}1\\ i\end{bmatrix} + A_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}C_0\begin{bmatrix}1\\ i\end{bmatrix} +$$

$$A_0\begin{bmatrix}1\\ i\end{bmatrix}C_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} + A_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i}C_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} + A_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j}C_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} + A_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}C_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i} +$$

$$A_0\begin{bmatrix}1\\ i\end{bmatrix}C_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} + A_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i}C_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} + A_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j}C_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} + A_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}C_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j} +$$

$$A_0\begin{bmatrix}1\\ i\end{bmatrix}C_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k} + A_i\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{i}C_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k} + A_j\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{j}C_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k} + A_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}C_k\begin{bmatrix}1\\ i\end{bmatrix}\mathbf{k}$$

Let us first move all of the unit vectors to the right-side of each group of terms.

$$\mathbf{O_A O_C} = A_0 \begin{bmatrix}1\\i\end{bmatrix} C_0 \begin{bmatrix}1\\i\end{bmatrix} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} +$$

$$A_0 \begin{bmatrix}1\\i\end{bmatrix} C_i \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{ii} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{ji} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{ki} +$$

$$A_0 \begin{bmatrix}1\\i\end{bmatrix} C_j \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{ij} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{jj} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{kj} +$$

$$A_0 \begin{bmatrix}1\\i\end{bmatrix} C_k \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{ik} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{jk} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{kk}$$

Next, we will simplify the unit vector terms.

$$\mathbf{O_A O_C} = A_0 \begin{bmatrix}1\\i\end{bmatrix} C_0 \begin{bmatrix}1\\i\end{bmatrix} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} +$$

$$A_0 \begin{bmatrix}1\\i\end{bmatrix} C_i \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} - A_i \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix} - A_j \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} +$$

$$A_0 \begin{bmatrix}1\\i\end{bmatrix} C_j \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} - A_j \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix} - A_k \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} +$$

$$A_0 \begin{bmatrix}1\\i\end{bmatrix} C_k \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} - A_i \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} - A_k \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}$$

Lastly, we will rearrange the terms. The scalar terms will be grouped together and the respective vector terms will be grouped together.

$$\mathbf{O_A O_C} = A_0 \begin{bmatrix}1\\i\end{bmatrix} C_0 \begin{bmatrix}1\\i\end{bmatrix} - A_i \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix} - A_j \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix} - A_k \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix} +$$

$$A_i \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} + A_0 \begin{bmatrix}1\\i\end{bmatrix} C_i \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} - A_k \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} + A_j \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{i} +$$

$$A_j \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_k \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} + A_0 \begin{bmatrix}1\\i\end{bmatrix} C_j \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} - A_i \begin{bmatrix}1\\i\end{bmatrix} C_k^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{j} +$$

$$A_k \begin{bmatrix}1\\i\end{bmatrix} C_0^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} - A_j \begin{bmatrix}1\\i\end{bmatrix} C_i^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} + A_i \begin{bmatrix}1\\i\end{bmatrix} C_j^* \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k} + A_0 \begin{bmatrix}1\\i\end{bmatrix} C_k \begin{bmatrix}1\\i\end{bmatrix}\mathbf{k}$$

This can now be represented as:

Equation 5.3:

$$\mathbf{O_A O_C} = \begin{bmatrix} 0 & -A_i & -A_j & -A_k \\ +A_i & 0 & -A_k & +A_j \\ +A_j & +A_k & 0 & -A_i \\ +A_k & -A_j & +A_i & 0 \end{bmatrix} \begin{bmatrix} C_0^* \\ C_i^* \\ C_j^* \\ C_k^* \end{bmatrix} + \begin{bmatrix} +A_0 & 0 & 0 & 0 \\ 0 & +A_0 & 0 & 0 \\ 0 & 0 & +A_0 & 0 \\ 0 & 0 & 0 & +A_0 \end{bmatrix} \begin{bmatrix} C_0 \\ C_i \\ C_j \\ C_k \end{bmatrix}$$

Now let us test **Equation 5.3** by multiplying a pair of conjugates as presented by **Equation 4.0** and **Equation 4.1**.

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = \begin{bmatrix} 0 & +Q_i & +Q_j & +Q_k \\ -Q_i & 0 & +Q_k & -Q_j \\ -Q_j & -Q_k & 0 & +Q_i \\ -Q_k & +Q_j & -Q_i & 0 \end{bmatrix}\begin{bmatrix} Q_0^* \\ Q_i^* \\ Q_j^* \\ Q_k^* \end{bmatrix} + \begin{bmatrix} +Q_0^* & 0 & 0 & 0 \\ 0 & +Q_0^* & 0 & 0 \\ 0 & 0 & +Q_0^* & 0 \\ 0 & 0 & 0 & +Q_0^* \end{bmatrix}\begin{bmatrix} Q_0 \\ Q_i \\ Q_j \\ Q_k \end{bmatrix}$$

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = \begin{bmatrix} +Q_i\,Q_i^* + Q_j\,Q_j^* + Q_k\,Q_k^* \\ -Q_i\,Q_0^* + Q_k\,Q_j^* - Q_j\,Q_k^* \\ -Q_j\,Q_0^* - Q_k\,Q_i^* + Q_i\,Q_k^* \\ -Q_k\,Q_0^* + Q_j\,Q_i^* - Q_i\,Q_j^* \end{bmatrix} + \begin{bmatrix} +Q_0^*\,Q_0 \\ +Q_0^*\,Q_i \\ +Q_0^*\,Q_j \\ +Q_0^*\,Q_k \end{bmatrix}$$

$$\mathbf{O}_{\mathbf{Q}}^{*}\mathbf{O}_{\mathbf{Q}} = \begin{bmatrix} q_0^2 + q_i^2 + q_j^2 + q_k^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This result agrees with **Equation 4.2**. Several identities present themselves here. These are as follows:

Equation 5.3.1:

$$-Q_iQ_0^* + Q_0^*Q_i = 0\,;\; -Q_jQ_0^* + Q_0^*Q_j = 0\,;\; -Q_kQ_0^* + Q_0^*Q_k = 0$$

Equation 5.3.2:

$$+Q_kQ_j^* - Q_jQ_k^* = 0\,;\; -Q_kQ_i^* + Q_iQ_k^* = 0\,;\; +Q_jQ_i^* - Q_iQ_j^* = 0$$

Bi-Quaternion Matrix Inverse:

Given that multiplication of two bi-quaternions as described by **Equation 4.0** produces **Equation 5.3** which contains two matrix multiplications, how can there be a single matrix that represents the inverse matrix for this operation? Yet, there must be an inverse matrix because **Equation 4.1** is shown to be the conjugate of **Equation 4.0**. To resolve this, we must return to the discussion of Octonions in Part – 2 of this work. In that work, the author had not yet had the insight associated with **Equation 4.1** of this work. Therefore, it seems appropriate to revisit that previous effort in light of the new concept. The objective will be to restate **Equation 5.3** as an 8x8 matrix multiplication.

The form of the relationship just prior to **Equation 5.3** is:

$$\mathbf{O_A O_C} = A_0 \begin{bmatrix} 1 \\ i \end{bmatrix} C_0 \begin{bmatrix} 1 \\ i \end{bmatrix} - A_i \begin{bmatrix} 1 \\ i \end{bmatrix} C_i^* \begin{bmatrix} 1 \\ i \end{bmatrix} - A_j \begin{bmatrix} 1 \\ i \end{bmatrix} C_j^* \begin{bmatrix} 1 \\ i \end{bmatrix} - A_k \begin{bmatrix} 1 \\ i \end{bmatrix} C_k^* \begin{bmatrix} 1 \\ i \end{bmatrix} +$$

$$A_i \begin{bmatrix} 1 \\ i \end{bmatrix} C_0^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{i} + A_0 \begin{bmatrix} 1 \\ i \end{bmatrix} C_i \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{i} - A_k \begin{bmatrix} 1 \\ i \end{bmatrix} C_j^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{i} + A_j \begin{bmatrix} 1 \\ i \end{bmatrix} C_k^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{i} +$$

$$A_j \begin{bmatrix} 1 \\ i \end{bmatrix} C_0^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{j} + A_k \begin{bmatrix} 1 \\ i \end{bmatrix} C_i^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{j} + A_0 \begin{bmatrix} 1 \\ i \end{bmatrix} C_j \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{j} - A_i \begin{bmatrix} 1 \\ i \end{bmatrix} C_k^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{j} +$$

$$A_k \begin{bmatrix} 1 \\ i \end{bmatrix} C_0^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{k} - A_j \begin{bmatrix} 1 \\ i \end{bmatrix} C_i^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{k} + A_i \begin{bmatrix} 1 \\ i \end{bmatrix} C_j^* \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{k} + A_0 \begin{bmatrix} 1 \\ i \end{bmatrix} C_k \begin{bmatrix} 1 \\ i \end{bmatrix} \mathbf{k}$$

This must be equal to the following:

$$\mathbf{O_A O_C} = (\mathbf{A} + i\mathbf{B})(\mathbf{C} + i\mathbf{D})$$

Where the various terms of **A**, **B**, **C**, and **D** are specified to coincide with **Equation 4.0**. Expanding this gives he following:

$$\mathbf{O_A O_C} = \mathbf{AC} + i\mathbf{BC} + \mathbf{A}i\mathbf{D} + i\mathbf{B}i\mathbf{D}$$

$$\mathbf{O_A O_C} = \mathbf{AC} + \mathbf{B}^* i\mathbf{C} + \mathbf{A}i\mathbf{D} - \mathbf{B}^*\mathbf{D}$$

$$\mathbf{O_A O_C} = \mathbf{AC} - \mathbf{B}^*\mathbf{D} +$$

$$\mathbf{B}^* i\mathbf{C} + \mathbf{A}i\mathbf{D}$$

Equation 6.0:

$$\mathbf{O_A O_C} = \begin{bmatrix} +\mathbf{A} & -\mathbf{B}^* \\ +\mathbf{B}^* & +\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}$$

Look at how simple it is to express this by using Hamilton's quaternions! This can then be presented as an 8x8 matrix multiplication by substituting the submatrix that is associated with each quaternion.

In Part – 2 of this work, the author showed in **4.4.1** that an octonion multiplication wherein the complex i commutes normally could be represented as follows:

$$\mathbf{O_A O_C} = \begin{bmatrix} +\mathbf{A} & -\mathbf{B} \\ +\mathbf{B} & +\mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}$$

Therefore, it appears that complex i anti-commutation causes **B** to become **B*** in the matrix form.

Therefore, the 8x8 matrix representation for the matrix multiplication wherein the complex i anti-commutes is as follows:

Equation 6.1:

$$\mathbf{O_A O_C} = \begin{bmatrix} +a_0 & -a_i & -a_j & -a_k & -b_0 & -b_i & -b_j & -b_k \\ +a_i & +a_0 & -a_k & +a_j & +b_i & -b_0 & -b_k & +b_j \\ +a_j & +a_k & +a_0 & -a_i & +b_j & +b_k & -b_0 & -b_i \\ +a_k & -a_j & +a_i & +a_0 & +b_k & -b_j & +b_i & -b_0 \\ +b_0 & +b_i & +b_j & +b_k & +a_0 & -a_i & -a_j & -a_k \\ -b_i & +b_0 & +b_k & -b_j & +a_i & +a_0 & -a_k & +a_j \\ -b_j & -b_k & +b_0 & +b_i & +a_j & +a_k & +a_0 & -a_i \\ -b_k & +b_j & -b_i & +b_0 & +a_k & -a_j & +a_i & +a_0 \end{bmatrix} \begin{bmatrix} +c_0 \\ +c_i \\ +c_j \\ +c_k \\ +d_0 \\ +d_i \\ +d_j \\ +d_k \end{bmatrix}$$

And the inverse matrix is as follows:

Equation 6.2:

$$[m]^{-1} = \frac{1}{\|\mathbf{Q}\|^2} \begin{bmatrix} +a_0 & +a_i & +a_j & +a_k & +b_0 & -b_i & -b_j & -b_k \\ -a_i & +a_0 & +a_k & -a_j & +b_i & +b_0 & -b_k & +b_j \\ -a_j & -a_k & +a_0 & +a_i & +b_j & +b_k & +b_0 & -b_i \\ -a_k & +a_j & -a_i & +a_0 & +b_k & -b_j & +b_i & +b_0 \\ -b_0 & +b_i & +b_j & +b_k & +a_0 & +a_i & +a_j & +a_k \\ -b_i & -b_0 & +b_k & -b_j & -a_i & +a_0 & +a_k & -a_j \\ -b_j & -b_k & -b_0 & +b_i & -a_j & -a_k & +a_0 & +a_i \\ -b_k & +b_j & -b_i & -b_0 & -a_k & +a_j & -a_i & +a_0 \end{bmatrix}$$

Where:

Equation 6.2.1:

$$\|\mathbf{Q}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2$$

As a reminder, the coefficients in the above matrices must satisfy the following:

Equation 6.3:

$$a_0 = \cos(\omega)\, q_0;\; a_i = \cos(\omega)\, q_i;\; a_j = \cos(\omega)\, q_j;\; a_k = \cos(\omega)\, q_k$$

$$b_0 = \sin(\omega)\, q_0;\; b_i = \sin(\omega)\, q_i;\; b_j = \sin(\omega)\, q_j;\; b_k = \sin(\omega)\, q_k$$

This was shown in Part – 2 of this work to be necessary for the coefficient matrix to be invertible.

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