The Sequence : $z_{n+1}=\left(2+\sqrt{3}+i\right)e^{-z_n}$

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abstract

In this note we briefly explore the sequence:

$$z_{n+1}=\left(2+\sqrt{3}+i\right)e^{-z_n}\ ,z_1=0,i=\sqrt{-1},n\in\mathbb{N}$$

Introduction

The sequence $z_n\in\mathbb{C},n\in\mathbb{N}$:

$$z_{n+1}=\left(2+\sqrt{3}+i\right)e^{-z_n}\quad ,z_1=0,i=\sqrt{-1},n\in\mathbb{N} \tag{1}$$

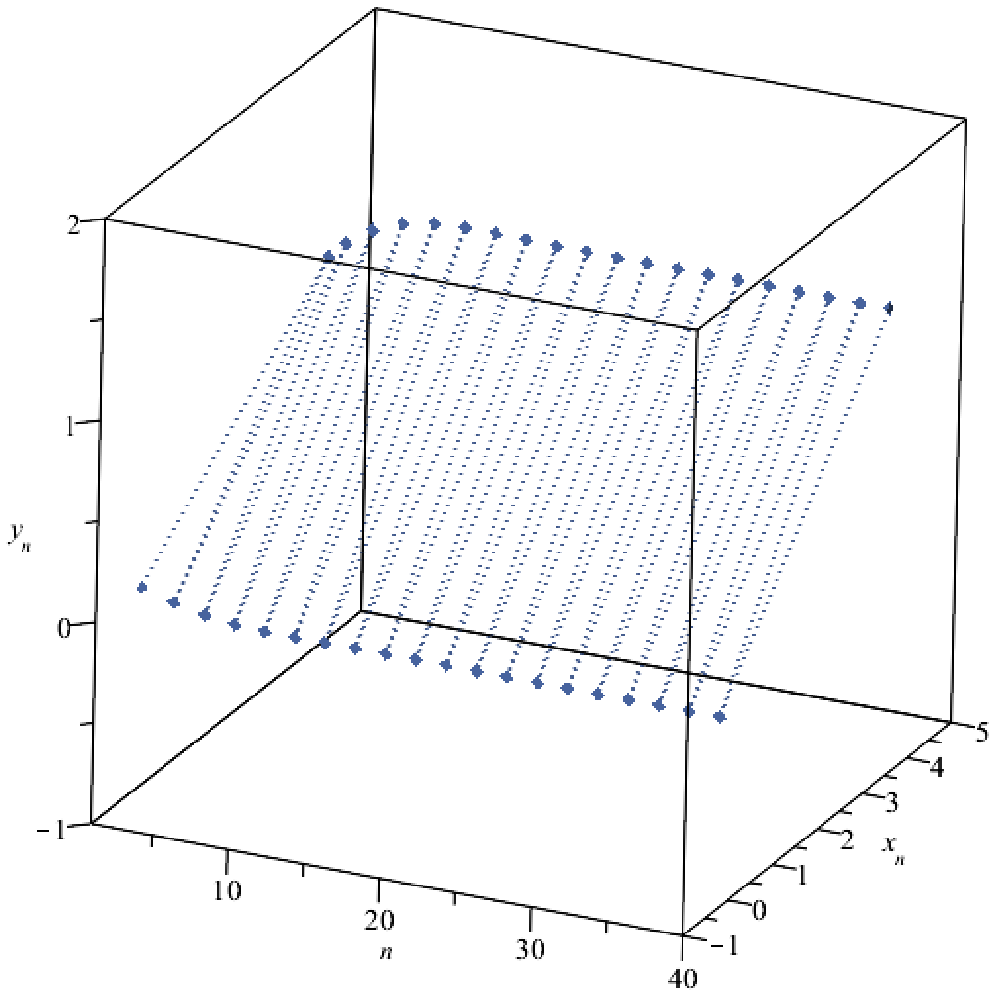
If $n\in\mathbb{N},z_n=x_n+i\,y_n\in\mathbb{C},x_n,y_n\in\mathbb{R}$, then:

$$\begin{cases} x_{n+1}=e^{-x_n}\left(\left(2+\sqrt{3}\right)\cos y_n+\sin y_n\right) \\ y_{n+1}=e^{-x_n}\left(\cos y_n-\left(2+\sqrt{3}\right)\sin y_n\right) \end{cases} , x_1=y_1=0 \tag{2}$$

Some Values

n	$z_n=x_n+y_n i$	n	$z_n=x_n+y_n i$
1	0	10	$3.35091+1.34521i$
2	$3.73205+i$	15	$0.06184-0.11966i$
3	$0.06842-0.06225i$	20	$3.37314+1.34165i$
4	$3.42035+1.14888i$	25	$0.06246-0.11688i$
5	$0.07980-0.09794i$	30	$3.37264+1.34195i$
6	$3.33896+1.25583i$	35	$0.06244-0.11688i$
7	$0.07474-0.11488i$	40	$3.37265+1.34193i$

Graphics 3D:



$P_n = (n, x_n, y_n)$; $z_n = x_n + i\,y_n$, $n \in \mathbb{N}$

Figure 1.

If $n \in \mathbb{N}, z_n = r_n e^{i\theta_n}$, then:

$$\begin{cases} r_{n+1} = 2\sqrt{2+\sqrt{3}}\; e^{-r_n \cos\theta_n} \\ \theta_{n+1} = \dfrac{\pi}{12} - r_n \sin\theta_n \end{cases} , r_1 = \theta_1 = 0 \tag{3}$$

Graphics 2D:

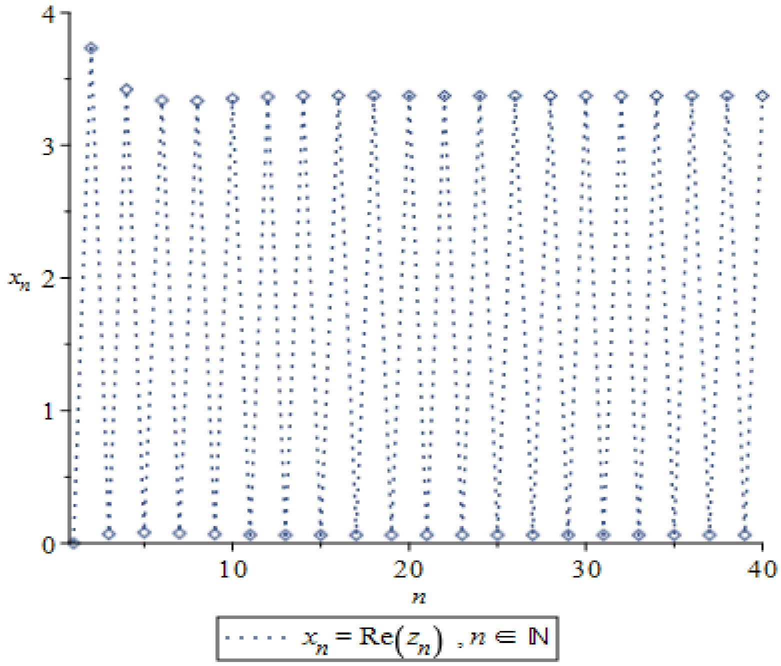


Figure 2.

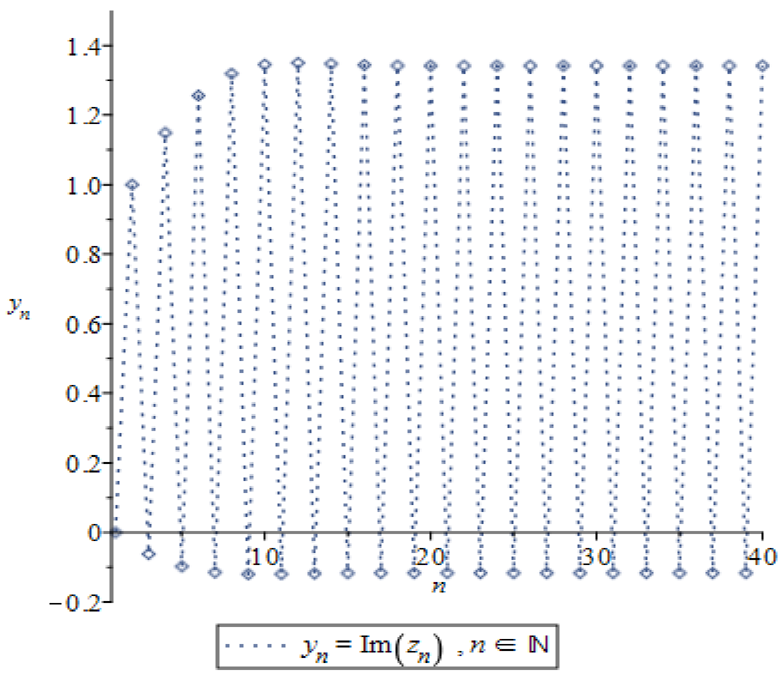


Figure 3.

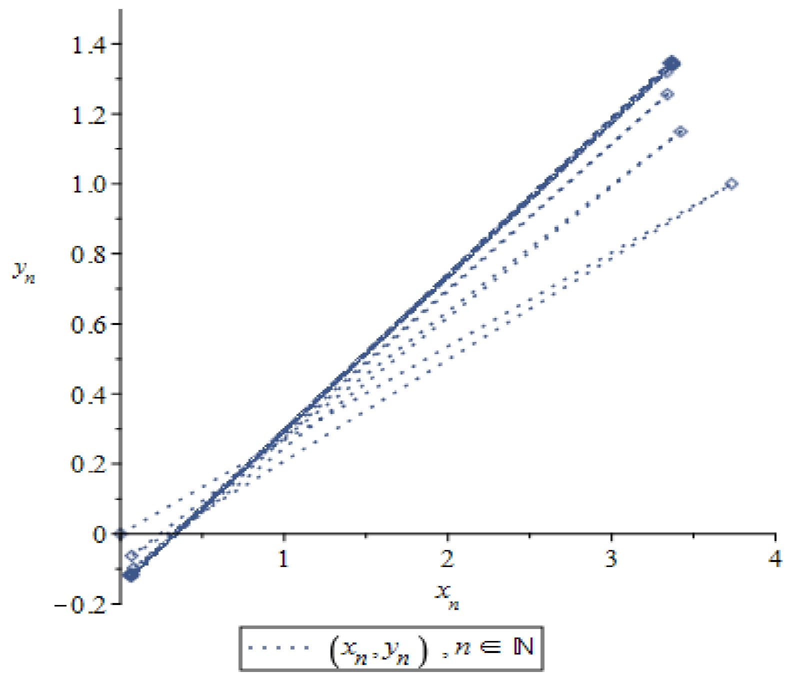


Figure 4.

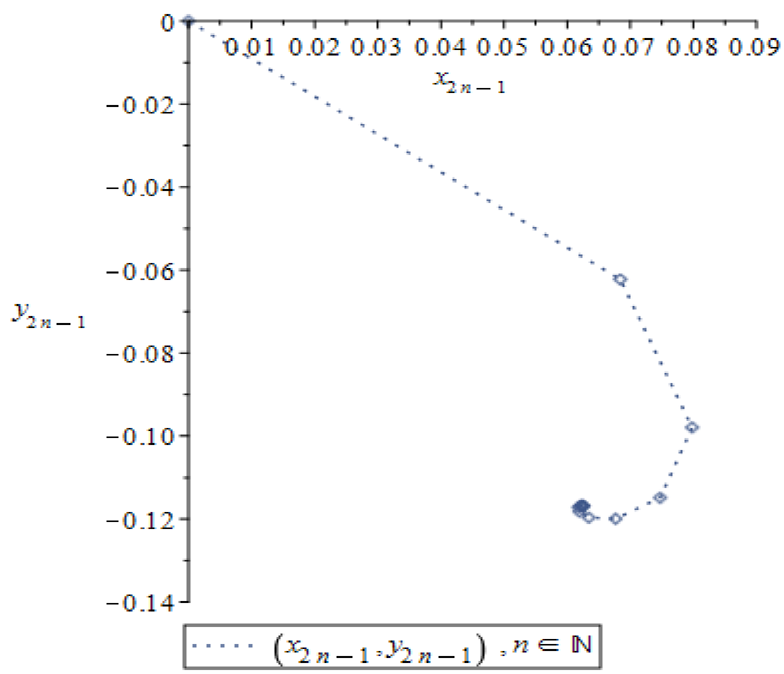


Figure 5.

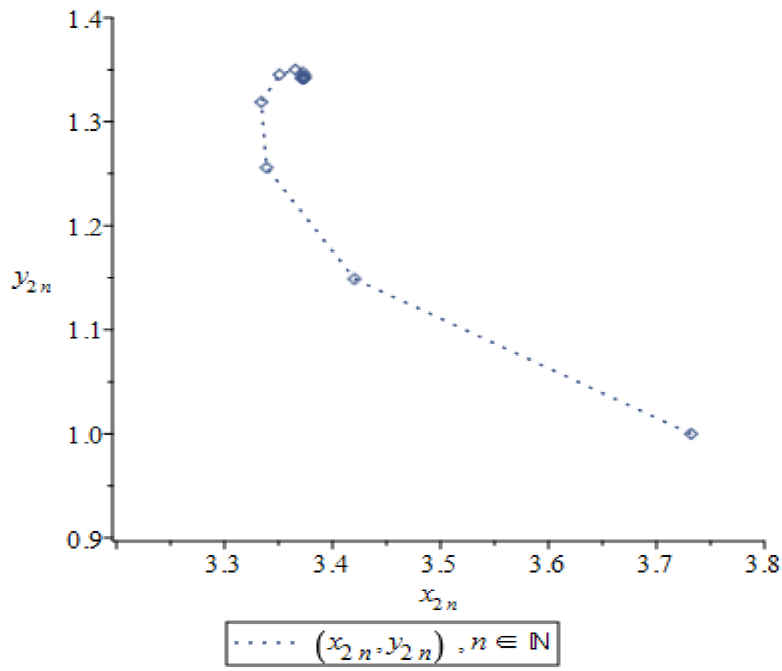


Figure 6.

Relations

$$z_n = \{z_1, z_2, z_3, z_4, \ldots, z_{2k-1}, z_{2k}, \ldots\} \quad (4)$$

$$z_n \to Divergent, Oscillating, Bounded \quad (5)$$

$$z_n \Rightarrow \begin{cases} z_{2n-1} = \{z_1, z_3, z_5, z_7, \ldots\}\ ,\ z_{2n-1} \text{ convergent sequence} \\ z_{2n} = \{z_2, z_4, z_6, z_8, \ldots\}\ ,\ z_{2n} \text{ convergent sequence} \end{cases} \quad (6)$$

$$z_{2n-1} \to w_1 = a + bi = 0.06244396\ldots - i\,0.11688549\ldots \quad (7)$$

$$z_{2n} \to w_2 = c + di = 3.37265018\ldots + i\,1.34193907\ldots \quad (8)$$

$$\pi = 12b + 12\tan^{-1}\left(\frac{d}{c}\right) \quad (9)$$

$$\pi = 12d + 12\tan^{-1}\left(\frac{b}{a}\right) = \frac{12}{7}d + \frac{12}{7}\tan^{-1}\left(\frac{a}{-b}\right) \quad (10)$$

Related Function

$$f(z) = \left(2+\sqrt{3}+i\right)e^{-\left(2+\sqrt{3}+i\right)e^{-z}} \quad , z \in \mathbb{C} \tag{11}$$

$$z_{n+1} = f(z_n) \quad , z_1 = 0 \Rightarrow z_n \to w_1 \tag{12}$$

$$z_{n+1} = f(z_n) \quad , z_1 = 3 \Rightarrow z_n \to w_2 \tag{13}$$

$$w_1 - f(w_1) = 0 \tag{14}$$

$$w_2 - f(w_2) = 0 \tag{15}$$

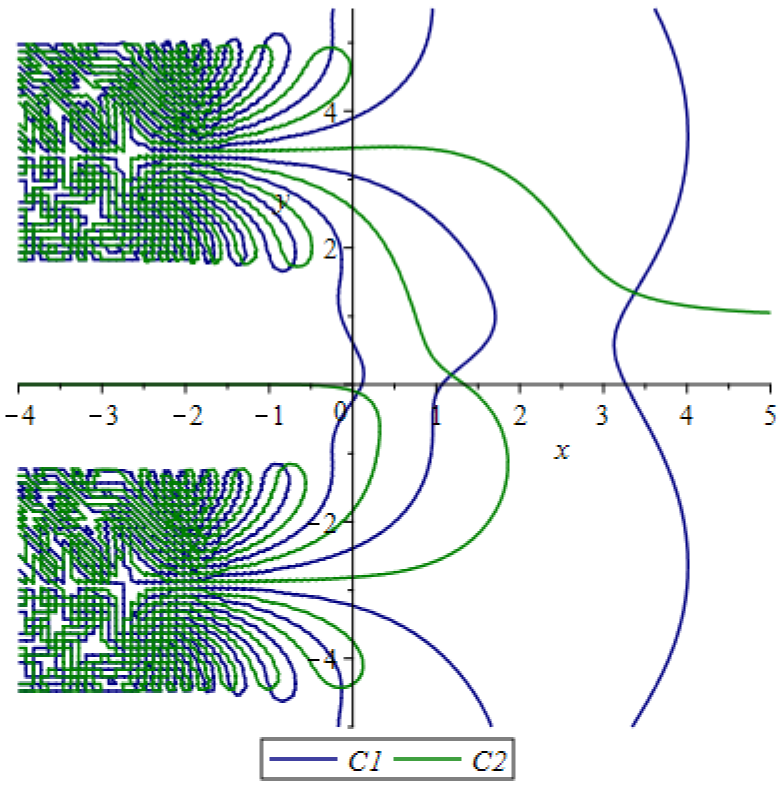


Figure 7.

$$C1: \mathrm{Re}\left(x+iy - f(x+iy)\right) = 0 \tag{16}$$

$$C2: \mathrm{Im}\left(x+iy - f(x+iy)\right) = 0 \tag{17}$$

Sequences: $z_{2n-1} = x_{2n-1} + y_{2n-1} i$, $z_{2n} = x_{2n} + y_{2n} i$ $,n \in \mathbb{N}$

$$x_{n+1} = F\left(x_n, y_n\right) \quad , \quad y_{n+1} = G\left(x_n, y_n\right) \quad , n \in \mathbb{N} \tag{18}$$

$$F\left(x, y\right) = e^{-x}\left(\left(2+\sqrt{3}\right)\cos y + \sin y\right) \tag{19}$$

$$G\left(x, y\right) = e^{-x}\left(\cos y - \left(2+\sqrt{3}\right)\sin y\right) \tag{20}$$

$$\begin{cases} x_{2n+1} = F\left(F\left(x_{2n-1}, y_{2n-1}\right), G\left(x_{2n-1}, y_{2n-1}\right)\right) \\ y_{2n+1} = G\left(F\left(x_{2n-1}, y_{2n-1}\right), G\left(x_{2n-1}, y_{2n-1}\right)\right) \end{cases} \quad , n \in \mathbb{N} \tag{21}$$

$$\begin{cases} x_{2n+2} = F\left(F\left(x_{2n}, y_{2n}\right), G\left(x_{2n}, y_{2n}\right)\right) \\ y_{2n+2} = G\left(F\left(x_{2n}, y_{2n}\right), G\left(x_{2n}, y_{2n}\right)\right) \end{cases} \quad , n \in \mathbb{N} \tag{22}$$

Pi formula

$$\pi = 6\,\mathrm{Im}\left(w_1 + w_2\right) - 6\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Im}\left(\left(1 - w_1 w_2\right)^n\right) \tag{23}$$

References

1. M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions. Dover Publications, New York , 1965.